

NEWTON POLYGONS, SUCCESSIVE MINIMA, AND DIFFERENT BOUNDS FOR DRINFELD MODULES OF RANK 2

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ABSTRACT. Let $K = \mathbb{F}_q(T)$. For a Drinfeld A -module ϕ of rank 2 defined over C_∞ , there are an associated exponential function e_ϕ and lattice Λ_ϕ in C_∞ given by uniformization over C_∞ . We explicitly determine the Newton polygons of e_ϕ and the successive minima of Λ_ϕ . When ϕ is defined over K_∞ , we give a refinement of Gardeyn's bounds for the action of wild inertia at ∞ on the torsion points of ϕ and a criterion for the lattice field to be unramified over K_∞ . If ϕ is in addition defined over K , we make explicit Gardeyn's bounds for the action of wild inertia at finite primes on the torsion points of ϕ , using results of Rosen, and this gives an explicit bound on the degree of the different divisor of division fields of ϕ over K .

1. INTRODUCTION

Let $K = \mathbb{F}_q(T)$, $A = \mathbb{F}_q[T]$, where q is a power of a prime p , and suppose that $v = (\frac{1}{T})$ is the place at infinity of K with associated normalized valuation function $v = v_\infty : K \rightarrow \mathbb{Z} \cup \{+\infty\}$. Let $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the completion of K at ∞ , and C_∞ be the completion of an algebraic closure of K_∞ . Denote also by v the extension of v from K to C_∞ . Let the absolute value associated to v be given by $|x| = q^{-v(x)}$.

For a specified homomorphism $\iota : A \rightarrow F$, where F is a field, a Drinfeld A -module ϕ over F is a homomorphism $\phi : A \rightarrow F\langle\tau\rangle$ such that for all $a \in A$, $\phi_a := \phi(a)$ has constant term $\iota(a)$, where $\tau : z \mapsto z^q$ is the q -th power Frobenius endomorphism and $F\langle\tau\rangle$ denotes the ring of twisted polynomials over F satisfying $\tau\alpha = \alpha^q\tau$ for all $\alpha \in F$. We require that the image of ϕ not be contained in F . It can be shown that there is an integer $r \geq 1$ called the rank of ϕ such that

$$\phi_a = \sum_{i=0}^{r \deg a} a_i(\phi, a)\tau^i$$

for all $a \in A$. Note that a Drinfeld A -module of rank r is completely determined by its value $\phi_T = \iota(T) + \sum_{i=1}^r a_i(\phi)\tau^i$.

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For a Drinfeld A -module ϕ of rank 2 over K , one knows by uniformization (cf. [10]) that there is an A -lattice $\Lambda_\phi = \Lambda_{\phi, \infty} \subseteq C_\infty$ of rank 2 and a surjective analytic function $e_\phi = e_{\phi, \infty} : C_\infty \rightarrow C_\infty$ with zero set equal to Λ_ϕ and such that

$$(1) \quad e_\phi(az) = \phi_a \circ e_\phi(z)$$

for all $a \in A$ and normalized so that the derivative $de_\phi(z)/dz$ of $e_\phi(z)$ is equal to 1. The function $e_\phi(z)$ is called *the exponential function associated to ϕ* . It is uniquely determined by the above properties and can be written in the form $e_\phi(z) = \sum_{i=0}^{\infty} c_i \tau^i(z)$, where $\tau(z) = z^q$, $c_i \in C_\infty$, and $c_0 = 1$.

Let ϕ be a Drinfeld A -module over a field F with respect to a specified homomorphism $\iota : A \rightarrow F$. For any $a \in A, a \neq 0$, we define the A -module of a -torsion points of ϕ as

$$\phi[a] = \{l \in \overline{F} \mid \phi_a(l) = 0\}$$

and let $F(\phi[a])$ be the field obtained by adjoining the a -torsion points of ϕ to F (here \overline{F} denotes a fixed algebraic closure of F).

There has been some interest in studying the field $K(\phi[a])$ generated over K by the a -torsion points of ϕ [2, 3, 4, 5, 11, 12, 13, 14, 17]. A natural object which arises in bounding the ramification over ∞ is the field $K_\infty(\Lambda_{\phi, \infty})$ which contains the field generated by the a -torsion points of ϕ over K_∞ . Since $\Lambda_{\phi, \infty}$ is the zero set of the analytic function $e_{\phi, \infty}(z)$, the different of $K_\infty(\Lambda_{\phi, \infty})/K_\infty$ can be bounded using information from the Newton polygon of $e_{\phi, \infty}(z)$ [4].

In this paper, we explicitly determine the Newton polygon and slopes of $e_{\phi, \infty}(z)$ for a general Drinfeld A -module ϕ of rank 2 defined over K (in fact, over C_∞) determined by $\phi_T = T + a_1(\phi)\tau + a_2(\phi)\tau^2$. The different cases of Newton polygons which arise depend on $v(j(\phi))$, where $j(\phi)$ is the j -invariant of ϕ , defined by $j(\phi) = a_1(\phi)^{q+1}/a_2(\phi)$. Some applications of this determination of the Newton polygons are given.

We point out that the essential nature of the Newton polygon of e_ϕ depends only on the C_∞ -isomorphism class of ϕ . Suppose we have an isomorphism f from ϕ to ϕ' , where $\phi_T = T + a_1\tau + a_2\tau^2$, and $\phi'_T = T + a'_1\tau + a'_2\tau^2$. It follows that $f \circ \phi_a = \phi'_a \circ f$ for all $a \in A$, $f(z) = cz$ for some $c \in C_\infty^*$, and $a'_i = a_i/c^{q^i-1}$. Using Equation (1), we see by induction that $e_{\phi'}(z) = \sum_{i=0}^{\infty} c'_i \tau^i(z) = \sum_{i=0}^{\infty} \frac{c_i}{c^{q^i-1}} \tau^i(z)$. It follows that $f \circ e_\phi = e_{\phi'} \circ f$ and hence $\Lambda_{\phi'} = c\Lambda_\phi$. Thus, the slopes of the Newton polygon of $e_{\phi'}$ are simply the slopes of the Newton polygon of e_ϕ translated by $-v(c)$, with projected sides having the same lengths and coordinates.

We give explicit bounds on the ramification of $K_\infty(\Lambda_{\phi, \infty})/K_\infty$ at ∞ based on a method which slightly refines the one in [4]. The ingredient which is needed to make the bound explicit is based on work of Gekeler, which relates $v(j(z))$ and $v(z)$ for $z \in \mathcal{F}$, where $\mathcal{F} = \{z \in C_\infty : |z| = |z|_i = \inf_{x \in K_\infty} |z - x| \geq 1\}$. In the case of $v(j(\phi)) \geq -q$, the bound does not depend on ϕ .

Finally, using work of Rosen [15], we make explicit Gardeyn's bounds for the action of wild inertia at finite primes on the torsion points of ϕ . As a result, we obtain an explicit bound on the degrees of the different divisors of the division fields of ϕ over K .

2. NEWTON POLYGON OF THE EXPONENTIAL FUNCTION
 ASSOCIATED TO A TWIST OF ϕ

We say that $\Lambda \subseteq C_\infty$ is an A -lattice of rank r if $\Lambda = A\lambda_1 + \cdots + A\lambda_r$ with $\lambda_i \in C_\infty$ being K_∞ -linearly independent, and we refer to $\{\lambda_1, \dots, \lambda_r\}$ as an A -basis for Λ .

Let $B_\kappa = \{\lambda \in \Lambda : |\lambda| \leq \kappa\}$ for $\kappa \in \mathbb{R}$. We define the i th *successive minimum* ν_i to be the minimum of the set of κ such that B_κ contains i number of K -linearly independent elements for $i = 1, 2, \dots, r$, and (ν_1, \dots, ν_r) is called the *successive minima* of Λ . An ordered A -basis $(\lambda_1, \dots, \lambda_r)$ for Λ is called a *minimal A -basis* for Λ if $|\lambda_i| = \nu_i$ for $i = 1, 2, \dots, r$ (note: a minimal ordered A -basis for Λ always exists because of the discreteness of the valuations of elements in Λ).

Lemma 2.1. *If $\{\omega_1, \dots, \omega_n\}$ is an A -basis for an A -lattice Λ such that $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_n|$, then the following are equivalent:*

- $(\omega_1, \dots, \omega_n)$ is a minimal A -basis for Λ ;
- $|\sum_{i=1}^n a_i \omega_i| = \max\{|a_i \omega_i| : 1 \leq i \leq n\}$ for all $a_i \in A$.

Proof. See [18, Lemma 4.2]. □

By uniformization, we may regard the coefficients $c_i = c_i(z)$ of e_ϕ as functions on the upper half-plane Ω , where $\Omega := C_\infty - K_\infty$, and ϕ is the Drinfeld A -module associated to the A -lattice $\Lambda = A + Az$. These functions are dubbed the *para-Eisenstein series* by Gekeler [7] and are studied in [6, 8].

Let

$$\mathcal{F} = \{z \in C_\infty : |z| = |z|_i \geq 1\},$$

$$\mathcal{F}_k = \{z \in C_\infty : |z| = |z|_i = q^k\},$$

where $k \geq 0$ and $|z|_i = \inf_{x \in K_\infty} |z - x|$.

The subset \mathcal{F} of Ω is a kind of fundamental domain for Ω under the action of $\Gamma = \text{GL}_2(A)$ in the following sense.

Proposition 2.2. *Each element $z \in \Omega$ is Γ -equivalent to an element of \mathcal{F} .*

Proof. See [6, Corollary 6.7]. □

Further properties of \mathcal{F} can be found in [6].

Theorem 2.1. *For $z \in \mathcal{F}$ we have the following:*

- $\log_q |j(z)| \leq q \iff z \in \mathcal{F}_0$.
- Suppose $k \geq 1$. Then $\log_q |j(z)| = q^{k+1} \iff z \in \mathcal{F}_k$.
- Suppose $m \geq 1$. Then $m - 1 < -v(z) < m$ if and only if $q^m < -v(j(z)) < q^{m+1}$. Furthermore, we have the following linear interpolation property:

$$-v(z) = \frac{-v(j(z)) - q^m}{q^m(q - 1)} + m - 1.$$

Proof. See [9, Corollary 3.11, Remark 2.4]. □

Lemma 2.3. *If $z \in \mathcal{F}$, then $(1, z)$ is a minimal A -basis for the A -lattice $\Lambda = A + Az$.*

Proof. Let $z \in \mathcal{F}$. Assume that $(1, z)$ is not a A -minimal basis for $A + Az$. Since $|z| \geq 1$, by Lemma 2.1, there exist $a, b \in A$ such that

$$|az + b| < \max\{|az|, |b|\} \text{ and } |az| = |b|.$$

It is clear that a must be nonzero. Thus $|z + b/a| < |z|$ and $|z| = |b/a|$. Since z is in \mathcal{F} , we have $|z| = |z|_i = \inf_{x \in K_\infty} |z - x|$. Therefore, $|z| \leq |z + b/a|$. This contradicts the inequality above. \square

Let ϕ be a Drinfeld A -module of rank 2 over C_∞ . Let $\Lambda = \Lambda_\phi$ be the A -lattice associated to ϕ by uniformization, which is the zero set of the exponential function e_ϕ . We know by the fundamental domain property of \mathcal{F} (cf. Proposition 2.2) that ϕ is isomorphic over C_∞ to another Drinfeld A -module ϕ' such that its exponential function $e_{\phi'}$ has associated A -lattice $\Lambda' = A + Az$, where $z \in \mathcal{F}$.

It suffices to determine the Newton polygon of $e_{\phi'}$ as the Newton polygon of e_ϕ can be deduced from that of $e_{\phi'}$ (see Section 3 for further details).

By Lemma 2.3, $(1, z)$ is a minimal A -basis for $\Lambda' = A + Az$.

2.1. Case 1: $-v(j(\phi)) \leq q$. By Theorem 2.1, this corresponds to the situation $z \in \mathcal{F}_0$. The following argument can be found in [6, p. 513]. Since $|z| = 1$, we have that $|az + b| = \max\{|a|, |b|\}$ holds for any a, b in A by Lemma 2.1. This implies that Λ' has precisely $q^{2(i+1)}$ elements of valuation $\geq -i$, and for each $i \geq 0$, there are $q^{2(i+1)} - q^{2i}$ elements of valuation $-i$.

Hence, the Newton polygon of $e_{\phi'}$ has one segment of length $q^{2i}(q^2 - 1)$ of slope i for each $i \geq 0$.

2.2. Case 2 (i): $q < -v(j(\phi)) \neq q^{m+1}$ for any $m \geq 1$. By Theorem 2.1, this corresponds to the situation $q^{m-1} < |z| < q^m$ for some $m \geq 1$. Let $|z| = q^\kappa$, where $m - 1 < \kappa < m$. Furthermore, we know that $0 < \kappa = -v(z) = \frac{-v(j(\phi)) - q^m}{q^m(q-1)} + m - 1$.

Now, by Lemma 2.1,

$$(2) \quad |az + b| = \max\{q^\kappa |a|, |b|\}$$

holds for any a, b in A . In this situation, we see that there are two types of non-zero elements of Λ' : those with valuation in $-\mathbb{Z}_{\geq 0}$ and those with valuation in $-(\kappa + \mathbb{Z}_{\geq 0})$. Thus, the possible slopes of segments of the Newton polygon of $e_{\phi'}$, in order, are

$$0, 1, \dots, m - 1, \kappa, m, \kappa + 1, m + 1, \kappa + 2, m + 2, \dots$$

We now have to count the number of elements of Λ' of each possible valuation using (2). For $0 \leq i \leq m - 1$, there are $q^{i+1} - q^i$ elements of valuation $-i$. For $j \geq 0$, there are $q^{m+2j+1} - q^{m+2j}$ elements of valuation $-(\kappa + j)$. For $j \geq 0$, there are $q^{m+2j+2} - q^{m+2j+1}$ elements of valuation $-(m + j)$.

Hence, the Newton polygon of $e_{\phi'}$ has one segment of length $q^{i+1} - q^i$ of slope i for each $0 \leq i \leq m - 1$ and one segment of length $q^{m+2j+1} - q^{m+2j}$ of slope $\kappa + j$, followed by one segment of length $q^{m+2j+2} - q^{m+2j+1}$ of slope $m + j$, for each $j \geq 0$.

2.3. Case 2 (ii): $-v(j(\phi)) = q^{m+1}$ for some $m \geq 1$. By Theorem 2.1, this corresponds to the situation $|z| = q^m$. Thus, as $|az + b| = \max\{q^m |a|, |b|\}$ holds for any a, b in A , there are q^{i+1} elements of Λ' of valuation $\geq -i$ if $0 \leq i \leq m - 1$ and q^{2i-m+2} elements of Λ' of valuation $\geq -i$ if $i \geq m$. In particular, for $0 \leq i \leq m - 1$, there are $q^{i+1} - q^i$ elements of valuation $-i$, and for $i \geq m$, there are $q^{2i-m+2} - q^{2i-m}$ elements of valuation $-i$.

Hence, the Newton polygon of $e_{\phi'}$ has one segment of length $q^i(q - 1)$ of slope i for each $0 \leq i < m$ and one segment of length $q^{2i-m}(q^2 - 1)$ of slope i for each $i \geq m$.

3. NEWTON POLYGON OF e_ϕ
AND VALUATIONS OF SUCCESSIVE MINIMA

Recall that ϕ is a given Drinfeld A -module of rank 2 over C_∞ with associated exponential function e_ϕ and A -lattice $\Lambda = \Lambda_\phi$.

The calculation in Section 2 determined the Newton polygon of $e_{\phi'}$, where ϕ' was a Drinfeld A -module isomorphic to ϕ over C_∞ such that its associated A -lattice is of the form $\Lambda' = A + Az$ with $z \in \mathcal{F}$.

Since $\Lambda' = c\Lambda$ for some $c \in C_\infty^*$, we know that the slopes of the Newton polygon of e_ϕ are obtained by translating the slopes of the Newton polygon of $e_{\phi'}$ by $v(c)$.

Let $\phi_T = T + a_1\tau + a_2\tau^2$, $\phi'_T = T + a'_1\tau + a'_2\tau^2$, $e_\phi = \sum_i c_i\tau^i$, and $e_{\phi'} = \sum_i c'_i\tau^i$. We have that $a'_i = a_i/c^{q^i-1}$, $c'_i = c_i/c^{q^i-1}$, and set $c_0 = c'_0 = 1$ as a normalization.

We know that the first two vertices of the Newton polygon of e_ϕ are either $(1, 0)$, $(q, v(c_1))$ in Case 2 or $(1, 0)$, $(q^2, v(c_2))$ in Case 1. In all cases, the slope of the first segment of the Newton polygon of $e_{\phi'}$ was 0.

In Case 2, we have that $v(c_1) = v(a_1) + q$ since $c_1 = \frac{a_1}{T^q - T}$, so the slope of the first segment of the Newton polygon of e_ϕ is $s_1 = \frac{v(a_1)+q}{q-1}$. The slopes of the remaining segments are those of $e_{\phi'}$ translated by s_1 .

On the other hand, in Case 1, we have that $v(c_2) = v(a_2) + q^2$; this is because $c_2 = \frac{a_1c_1^q + a_2}{T^{q^2} - T}$ and $v(a_2) < v(a_1c_1^q)$ in this case. Hence, the slope of the first segment of the Newton polygon of e_ϕ is $s_1 = \frac{v(a_2)+q^2}{q^2-1}$. The slopes of the remaining segments are those of $e_{\phi'}$ translated by s_1 .

Putting everything together, we obtain the following theorem.

Theorem 3.1. *Let ϕ be a Drinfeld A -module of rank 2 over C_∞ given by $\phi_T = T + a_1\tau + a_2\tau^2$. Let e_ϕ be its associated exponential function. Let m be the least positive integer such that $-v(j(\phi)) \leq q^{m+1}$. Let $s_1 = \frac{v(a_2)+q^2}{q^2-1}$ in Case 1 and $s_1 = \frac{v(a_1)+q}{q-1}$ in Case 2. Then the Newton polygon of e_ϕ is determined as follows.*

Case 1: $-v(j(\phi)) \leq q$.

The Newton polygon of e_ϕ has one segment of length $q^{2i}(q^2 - 1)$ of slope $i + s_1$ for each $i \geq 0$.

Case 2 (i): $q < -v(j(\phi)) \neq q^{m+1}$.

Let $\kappa = \frac{-v(j(\phi)) - q^m}{q^m(q-1)} + m - 1$. The Newton polygon of e_ϕ has one segment of length $q^{i+1} - q^i$ of slope $i + s_1$ for each $0 \leq i \leq m-1$ and one segment of length $q^{m+2j+1} - q^{m+2j}$ of slope $\kappa + j + s_1$, followed by one segment of length $q^{m+2j+2} - q^{m+2j+1}$ of slope $m + j + s_1$, for each $j \geq 0$.

Case 2 (ii): $q < -v(j(\phi)) = q^{m+1}$.

The Newton polygon of e_ϕ has one segment of length $q^i(q-1)$ of slope $i + s_1$ for each $0 \leq i < m$ and one segment of length $q^{2i-m}(q^2 - 1)$ of slope $i + s_1$ for each $i \geq m$.

Corollary 3.1. *Assume the hypotheses and notation of Theorem 3.1. Let Λ be the A -lattice associated to ϕ by uniformization, and suppose (λ_1, λ_2) is a minimal A -basis for Λ . Then $v(\lambda_1) = -s_1 = v(\lambda_2)$ if $-v(j(\phi)) \leq q$, and $v(\lambda_1) = -s_1, v(\lambda_2) = -s_1 - \kappa$ if $q < -v(j(\phi)) \leq q^{m+1}$.*

Proof. Recall from the proof of Theorem 3.1 that $\Lambda' = A + Az = c\Lambda$ for some $z \in \mathcal{F}$, where $c \in C_\infty^*$, and $(1, z)$ is a minimal A -basis for Λ' . Furthermore, it was shown that $v(c) = s_1$.

Now, $(\lambda'_1, \lambda'_2) = (\frac{1}{c}, \frac{1}{c}z)$ is a choice of minimal A -basis for Λ since $(1, z)$ is a minimal A -basis for $\Lambda' = c\Lambda$. Therefore, $v(\lambda'_1) = -v(c) = -s_1$, and $v(\lambda'_2) = -v(c) + v(z) = -s_1$ in Case 1, and $-s_1 - \kappa$ in Case 2. \square

Corollary 3.2. *Assume the hypotheses and notation of Theorem 3.1 and further that ϕ is defined over K_∞ . Let Λ be the A -lattice associated to ϕ by uniformization.*

If $-v(j(\phi)) \leq q$, then $K_\infty(\Lambda)/K_\infty$ is unramified if and only if $v(a_2) \equiv -1 \pmod{q^2 - 1}$.

If $q^m < -v(j(\phi)) \leq q^{m+1}$ for $m \geq 1$, then $K_\infty(\Lambda)/K_\infty$ is unramified if and only if $v(a_1) \equiv -1 \pmod{q - 1}$ and $v(j(\phi)) \equiv -q^m \pmod{q^m(q - 1)}$.

Remark 3.3. Let $z \in \mathcal{F}_k$, where $k \geq 0$, and let ϕ be the Drinfeld A -module associated to the A -lattice $A + Az$, where $\phi_T = T + a_1(z)\tau + a_2(z)\tau^2$. The proof of Theorem 3.1 can also be used to give a different proof of the special case $k \leq 0$ of [8, Theorem 2.13], which determines the valuations $v(a_i(z))$ in terms of k (and $v(j(z))$ if $k = 0$). The idea is to take $\phi = \phi'$ in the proof of Theorem 3.1 so that $c = 1$, but we also know that $v(c) = s_1$, which shows that $v(a_2(z)) = -q^2$ in Case 1 and $v(a_1(z)) = -q$ in Case 2. Note, however, that both proofs require Gekeler's theory as a fundamental ingredient.

The case $k = 0$ of [8, Theorem 2.13], together with results in [1], can be used to determine the valuations $v(c_i(z))$ in some new cases not covered in [6].

4. GARDEYN'S BOUNDS FOR WILD RAMIFICATION AT ∞

Let ϕ be a Drinfeld A -module of rank 2 defined over K_∞ , e_ϕ its associated exponential function, and $\Lambda_\phi = \Lambda_{\phi, \infty}$ its associated A -lattice in C_∞ .

In the following theorem, we give explicit bounds on the ramification of $K_\infty(\Lambda_\phi)/K_\infty$ based on a slight refinement of the method in [4], which we present in the specific case of rank 2. We point out that the upper bound of the following theorem is not optimal as Corollary 3.2 shows.

Theorem 4.1. *Let ϕ be a Drinfeld A -module of rank 2 over K_∞ and let $\mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty)$ be the different of $K_\infty(\Lambda_\phi)/K_\infty$. Let m be the least positive integer such that $-v(j(\phi)) \leq q^{m+1}$. Then*

$$v(\mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty)) \leq \begin{cases} 1 & \text{if } -v(j(\phi)) \leq q, \\ 1 + \kappa (q^{\kappa+1} - 1) & \text{if } q < -v(j(\phi)) \leq q^{m+1}, \end{cases}$$

where $\kappa = \frac{-v(j(\phi)) - q^m}{q^m(q-1)} + m - 1$.

Proof. Put $\Lambda = \Lambda_\phi$, and let (λ_1, λ_2) be a minimal A -basis for Λ such that $z = \lambda_2/\lambda_1 \in \mathcal{F}$.

For any $s > 0$, there exist $d \in C_\infty^*$ and δ such that $v(d) = -v(\lambda_2) + \delta$, where $0 \leq \delta < \frac{1}{q^s - 1}$, and the ramification index of $K'_\infty = K_\infty(d)$ divides $q^s - 1$.

Let $\Lambda^0 = A\lambda_1^0 + A\lambda_2^0$, where $\lambda_i^0 = d\lambda_i$. Then $(\lambda_1^0, \lambda_2^0)$ is a minimal A -basis for Λ^0 since (λ_1, λ_2) is a minimal A -basis for Λ , and $K'_\infty(\Lambda^0) = K'_\infty(\Lambda)$. Let G_{Λ^0} be the Galois group of $K'_\infty(\Lambda^0)/K'_\infty$, and K_∞^0 be the maximal tamely ramified subextension of $K'_\infty(\Lambda^0)/K'_\infty$, corresponding to the Sylow p -subgroup P_{Λ^0} of G_{Λ^0} (recall that $p = \text{char}(K)$). For $\sigma \in G_{\Lambda^0}$, $\sigma\lambda_1^0 = \alpha\lambda_1^0$, where $\alpha \in \mathbb{F}_q^*$, as $|\sigma\lambda_1^0| = |\lambda_1^0|$. It follows that $\lambda_1^0 \in K_\infty^0$ and $K'_\infty(\Lambda^0) = K_\infty^0(\lambda_2^0)$.

Now, $v(\lambda_2^0) = \delta \geq 0$. Using [16, III, Cor. 2, p. 66], we have that

$$\mathcal{D}(K_\infty^0(\lambda_2^0)/K_\infty^0) \mid \prod_{\sigma \in P_{\Lambda^0}, \sigma \neq 1} (\sigma \lambda_2^0 - \lambda_2^0).$$

For $\sigma \in P_{\Lambda^0}$ with $\sigma \neq 1$, we have that $\sigma \lambda_2^0 = \beta \lambda_1^0 + \lambda_2^0$, where β is nonzero in A and satisfies $|\beta| \leq |\lambda_2/\lambda_1|$ by Lemma 2.1. It follows that $\#P_{\Lambda^0} \leq q|\lambda_2|/|\lambda_1|$. Finally, $|\sigma \lambda_2^0 - \lambda_2^0| = |\beta \lambda_1^0| \geq |\lambda_1^0|$, so $v(\sigma \lambda_2^0 - \lambda_2^0) \leq v(\lambda_1^0)$, and hence

$$\begin{aligned} v(\mathcal{D}(K_\infty^0(\lambda_2^0)/K_\infty^0)) &\leq (q|\lambda_2/\lambda_1| - 1)v(\lambda_1^0) \\ &= (q|\lambda_2/\lambda_1| - 1)(v(\lambda_1/\lambda_2) + \delta). \end{aligned}$$

The extension K_∞^0/K_∞ is tamely ramified, so we obtain that

$$v(\mathcal{D}(K_\infty(\Lambda)/K_\infty)) \leq 1 + v(\mathcal{D}(K_\infty^0(\lambda_2^0)/K_\infty^0)).$$

From Corollary 3.1, if $-v(j(\phi)) \leq q$, then $|z| = 1$ and $\delta = 0$, and if $q < -v(j(\phi)) \leq q^{m+1}$, then $|z| = q^\kappa$. Thus, we have

$$v(\mathcal{D}(K_\infty(\Lambda)/K_\infty)) \leq \begin{cases} 1 & \text{if } -v(j(\phi)) \leq q, \\ 1 + (\kappa + \delta)(q^{\kappa+1} - 1) & \text{if } q < -v(j(\phi)) \leq q^{m+1}, \end{cases}$$

where $0 \leq \delta < \frac{1}{q^s - 1}$. Taking $s \rightarrow \infty$ gives the desired bound. \square

Remark 4.1. For Theorem 4.1, using [4] directly would instead yield a bound of 1 if $-v(j(\phi)) \leq q$, and $1 + 2\kappa q^{\kappa+1}$ if $q < -v(j(\phi)) \leq q^{m+1}$.

We notice that in the range of $v(j(\phi)) \geq -q$, the bound on the different of $K_\infty(\Lambda_\phi)/K_\infty$ does not depend on ϕ .

5. GARDEYN'S BOUNDS FOR WILD RAMIFICATION AT FINITE PRIMES \mathfrak{p}

Let \mathfrak{p} be a finite prime of K , $K_{\mathfrak{p}}$ be the completion at \mathfrak{p} , and denote by $v_{\mathfrak{p}}$ its associated valuation. It is well known that ϕ has potentially Tate (resp. potentially good) reduction over $K_{\mathfrak{p}}$ if $v_{\mathfrak{p}}(j(\phi)) < 0$ (resp. $v_{\mathfrak{p}}(j(\phi)) \geq 0$) and that the stable reduction occurs over a finite tamely ramified extension of $K_{\mathfrak{p}}$ [15, Lemma 5.2].

By [4], we have that

$$v_{\mathfrak{p}}(\mathcal{D}(K_{\mathfrak{p}}(\phi[a])/K_{\mathfrak{p}})) \leq \begin{cases} 2v_{\mathfrak{p}}(a) & \text{if } \phi \text{ has good reduction over } K_{\mathfrak{p}}, \\ 2v_{\mathfrak{p}}(a) + 1 & \text{if } \phi \text{ has good reduction over } K'_{\mathfrak{p}}, \\ 2v_{\mathfrak{p}}(a) + 1 - 2v_{\mathfrak{p}}(\lambda_1) & \text{if } \phi \text{ has Tate reduction over } K'_{\mathfrak{p}}, \end{cases}$$

where $K'_{\mathfrak{p}}$ is a finite tamely ramified extension of $K_{\mathfrak{p}}$, and λ_1 is defined as follows.

In the case that ϕ has Tate reduction over $K'_{\mathfrak{p}}$, we obtain from uniformization that there is a Drinfeld A -module ψ of rank 1 and a surjective exponential function $e_{\phi, \mathfrak{p}} : C_{\mathfrak{p}} \rightarrow C_{\mathfrak{p}}$ such that $e_{\phi, \mathfrak{p}} \circ \psi_a = \phi_a \circ e_{\phi, \mathfrak{p}}$ for all $a \in A$, where $C_{\mathfrak{p}}$ is the completion of an algebraic closure of $K_{\mathfrak{p}}$.

The zeroes of $e_{\phi, \mathfrak{p}}$ form an A -lattice $\Lambda_{\mathfrak{p}} = \Lambda_{\phi, \mathfrak{p}}$ of rank 1 in $C_{\mathfrak{p}}$, so suppose $\Lambda_{\mathfrak{p}} = A\lambda_1$. Note that it is necessarily the case that $v(\lambda_1) < 0$ and (λ_1) is a minimal A -basis for $\Lambda_{\mathfrak{p}}$.

From [15, Lemma 5.3], we have that $v_{\mathfrak{p}}(\lambda_1) = \frac{1}{q-1}v_{\mathfrak{p}}(j(\phi))$.

Combining the above estimations yields the following explicit upper bound for the different $\mathcal{D}(K_{\mathfrak{p}}(\phi[a])/K_{\mathfrak{p}})$.

Theorem 5.1. *Let ϕ be a Drinfeld A -module of rank 2 over $K_{\mathfrak{p}}$ and let $\mathcal{D}(K_{\mathfrak{p}}(\phi[a])/K_{\mathfrak{p}})$ be the different of $K_{\mathfrak{p}}(\phi[a])/K_{\mathfrak{p}}$. Then*

$$v_{\mathfrak{p}}(\mathcal{D}(K_{\mathfrak{p}}(\phi[a])/K_{\mathfrak{p}})) \leq \begin{cases} 2v_{\mathfrak{p}}(a) & \text{if } \phi \text{ has good reduction over } \\ & K_{\mathfrak{p}}, \\ 2v_{\mathfrak{p}}(a) + 1 & \text{if } v_{\mathfrak{p}}(j(\phi)) \geq 0 \text{ and } \phi \text{ has} \\ & \text{bad reduction over } K_{\mathfrak{p}}, \\ 2v_{\mathfrak{p}}(a) + 1 - \frac{2}{q-1}v_{\mathfrak{p}}(j(\phi)) & \text{if } v_{\mathfrak{p}}(j(\phi)) < 0. \end{cases}$$

For a finite extension L/K , let $\mathfrak{D}(L/K)$ be the different divisor of L/K and define the degree with respect to K of $\mathfrak{D}(L/K)$ as

$$\deg_K \mathfrak{D}(L/K) = \sum_v \max \{v(\mathcal{D}(L_w/K_v)) : w \mid v\} \deg_K v,$$

where v ranges through all normalized places of K , w through all places of L lying over each v , and $\mathcal{D}(L_w/K_v)$ is the different of L_w/K_v . It can be shown that

$$\deg_L \mathfrak{D}(L/K) \leq n' \deg_K \mathfrak{D}(L/K),$$

where n' is the geometric extension degree of L/K . Since $K_{\infty}(\phi[a]) \subseteq K_{\infty}(\Lambda_{\phi, \infty})$, we obtain:

Theorem 5.2. *Let ϕ be a Drinfeld A -module of rank 2 over K and $\mathfrak{D}(K(\phi[a])/K)$ be the different divisor of $K(\phi[a])/K$. Then*

$$\begin{aligned} \deg_K \mathfrak{D}(K(\phi[a])/K) &\leq 2 \deg_K a + \deg_K \eta + \frac{2}{q-1} \deg_K \delta \\ &\quad + v_{\infty}(\mathcal{D}(K_{\infty}(\Lambda_{\phi, \infty})/K_{\infty})), \end{aligned}$$

where δ is the (monic) denominator of $j(\phi)$ as represented by a fraction in reduced form, and η is the product of finite primes \mathfrak{p} such that ϕ has bad reduction over $K_{\mathfrak{p}}$.

This combined with Theorem 4.1 gives an explicit bound on $\deg_K \mathfrak{D}(K(\phi[a])/K)$ in terms of $j(\phi)$, the primes of bad reduction of ϕ , and a .

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