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HARDY TYPE INEQUALITIES RELATED TO CARNOT-CARATHÉODORY DISTANCE ON THE HEISENBERG GROUP

QIAO-HUA YANG

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Abstract. Being motivated by a representation formula associated with the Korányi-Folland nonisotropic gauge proved by Cohn and Lu, we prove an analogous representation formula related to the Carnot-Carathéodory distance on the Heisenberg group. Using this formula, we obtain some Hardy inequalities associated with the Carnot-Carathéodory distance on such groups.

1. Introduction

It is well known that Hardy's inequality and its generalization in $\mathbb{R}^{\mathbb{N}}$ play an important role in many areas of mathematics. In the case of the Heisenberg group $\mathbb{H}_n$, Garofalo and Lanconelli (cf. [9]) first proved the following Hardy inequality:

$$\int_{\mathbb{H}_n} |\nabla_H u|^2 \geq \frac{(Q-2)^2}{4} \int_{\mathbb{H}_n} \frac{u^2}{d^2} |\nabla_H d|^2, \quad u \in C_0^\infty(\mathbb{H}_n \setminus \{e\}), \tag{1.1}$$

where e is the origin of $\mathbb{H}_n$, $d = (|z|^4 + t^2)^{\frac{1}{4}}$ is the Korányi-Folland nonisotropic gauge induced by the fundamental solution and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}_n$ (see also [10]). P. Niu et al. ([12]) generalized inequality (1.1) to the nonlinear cases. That is, for $1 < p < Q$, we have

$$\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d^p} |\nabla_H d|^p, \quad u \in C_0^\infty(\mathbb{H}_n \setminus \{e\}). \tag{1.2}$$

To obtain (1.2), they used a Picone type identity on $\mathbb{H}_n$. D'Ambrosio (cf. [3, 4, 5]) extended (1.2) to some degenerate elliptic differential operators such as the Heisenberg-Greiner operator, a Baouendi-Grushin type operator and the sub-Laplacian on Carnot groups. More recently, Danielli, Garofalo and Phuc (cf. [6, 7]) consider various types of Hardy-Sobolev inequalities on a Carnot-Carathéodory space. They use the fundamental solution of the corresponding p-Laplace operator and generalize (1.2) to Carnot groups of arbitrary step.

The aim of this paper is to prove an analogous Hardy-type inequality (1.2) on $\mathbb{H}_n$, where the Korányi-Folland nonisotropic gauge is replaced by the Carnot-Carathéodory distance d_{cc} on $\mathbb{H}_n$ (we refer to [1, 11] for more information about

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this distance). To our knowledge, little is known about Hardy inequalities related to such a distance. Since d_{cc} is not differentiable in the center of $\mathbb{H}_n$, it seems that the methods used in [3, 5, 7, 12] do not work for such a distance. To do so, we prove a representation formula associated with d_{cc}, and the idea is due to Cohn and Lu ([2]). The main result is the following theorem.

Theorem 1.1. *Let $1 < p < Q$ and $u \in C_0^\infty(\mathbb{H}_n)$. Then*

$$\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d_{cc}^p}. \tag{1.3}$$

The Hardy type inequalities with weights imply the following Hardy-Rellich type inequalities (see e.g. [14] for analogous inequalities on $\mathbb{R}^N$).

Theorem 1.2. *Let $\mathbb{Z} = \{(z,t) \in \mathbb{H}_n : z = 0\}$ be the center of $\mathbb{H}_n$. It follows, for all $u \in C_0^\infty(\mathbb{H}_n \setminus \mathbb{Z})$, that*

$$\int_{\mathbb{H}_n} |\Delta_H u|^2 \geq \left(\frac{Q(Q-4)}{4}\right)^2 \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4}, \quad n \geq 2, \tag{1.4}$$

and

$$\int_{\mathbb{H}_n} |\Delta_H u|^2 \geq \frac{Q^2}{4} \int_{\mathbb{H}_n} \frac{|\nabla_H u|^2}{d_{cc}^2}, \quad n \geq 3. \tag{1.5}$$

Finally, we obtain some Hardy inequalities involving the distance from the boundary which generalize a result of D'Ambrosio (cf. [5]).

Theorem 1.3. *Let $p > 1$ and $u \in C_0^\infty(B_{cc}(e,\rho))$, where $B_{cc}(e,\rho)$ is the Carnot-Carathéodory metric ball centered at the origin. Then*

$$\int_{B_{cc}(e,\rho)} |\nabla_H u|^p \geq \left(\frac{p-1}{p}\right)^p \int_{B_{cc}(e,\rho)} \frac{|u|^p}{(\rho - d_{cc})^p}. \tag{1.6}$$

2. Notation and preliminaries

Let $\mathbb{H}_n = (\mathbb{C}^n \times \mathbb{R}, \circ)$ be the the $(2n+1)$-dimensional Heisenberg group whose group structure is given by

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\mathrm{Im}\langle z, z'\rangle),$$

where $z = (z_1, \cdots, z_n)$, $z' = (z_1', \cdots, z_n') \in \mathbb{C}^n$, $z_j = x_j + iy_j$ $(x_j, y_j \in \mathbb{R})$ and $\langle z, z'\rangle = \sum_{j=1}^n z_j \cdot \overline{z_j}$. The vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

$(j = 1, \cdots, n)$ are left invariant and generate the Lie algebra of $\mathbb{H}_n$. The commutators of these vector fields satisfy

$$[X_j, Y_j] = -4\frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n$$

with all other brackets equal to zero. We denote by $\mathbb{Z} = \{(z,t) \in \mathbb{H}_n : z = 0\}$ the center of $\mathbb{H}_n$. Then Kohn's sub-Laplacian on $\mathbb{H}_n$ is

$$\Delta_H = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) + 4|z|^2\frac{\partial^2}{\partial t^2} + 4\sum_{k=1}^{n}(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j})\frac{\partial}{\partial t},$$

and the horizontal gradient is the $(2n)$-dimensional vector given by

$$\nabla_H = (X_1, \cdots, X_n, Y_1, \cdots, Y_n).$$

We call a curve $\gamma : [a,b] \to \mathbb{H}_n$ a horizontal curve connecting two points $\xi, \eta \in \mathbb{H}_n$ if $\gamma(a) = \xi$, $\gamma(b) = \eta$ and $\gamma'(s) \in \operatorname{span}\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ for all s. Then the Carnot-Carathéodory distance between ξ, η is defined as

$$d_{cc}(\xi,\eta) = \inf_{\gamma}\int_a^b \|\gamma'(s)\|ds,$$

where the infimum is taken over all horizontal curves γ connecting ξ and η. It is known that any two points ξ, η on $\mathbb{H}_n$ can be joined by a horizontal curve of finite length and then d_{cc} is a metric on $\mathbb{H}_n$. An important feature of this distance function is that the distance and thus the associated metric balls are left-invariant. With this norm, we can define the metric ball centered at the origin and with radius ρ associated with this metric by

$$B_{cc}(e,\rho) = \{\eta : d_{cc}(e,\eta) < \rho\}$$

and the unit sphere $\Sigma = \partial B_{cc}(e,1)$. For simplicity, we write $d_{cc}(\xi) = d_{cc}(e,\xi)$.

For each real number $\lambda > 0$, there is a dilation naturally associated with the group structure which is usually denoted as $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q = 2n+2$ is the homogeneous dimension of H_n. For simplicity, we use the notation $\lambda(z,t) = (\lambda z, \lambda^2 t)$. The Carnot-Carathéodory distance d_{cc} satisfies

$$d_{cc}(\lambda(z,t)) = \lambda d_{cc}(z,t), \quad \lambda > 0.$$

Given any $\xi = (z,t) \in \mathbb{H}_n$, set $z^* = \frac{z}{d_{cc}(z,t)}$, $t^* = \frac{t}{d_{cc}(z,t)^2}$ and $\xi^* = (z^*, t^*)$. The polar coordinates on $\mathbb{H}_n$ associated with d_{cc} are the following (cf. [8], Proposition 1.15 or [11], via the coarea formula):

$$\int_{\mathbb{H}_n} f(z,t)dxdt = \int_0^\infty \int_\Sigma f(\lambda(z^*,t^*))\lambda^{Q-1}d\sigma d\lambda$$

for all $f \in L^1(\mathbb{H}_n)$.

Set

$$\mu(\theta) = \frac{2\theta - \sin 2\theta}{2\sin^2\theta} : (-\pi,\pi) \to \mathbb{R}.$$

μ is a diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$ (cf. [1]). We denote by μ^{-1} the inverse function of μ. The Carnot-Carathéodory distance d_{cc} satisfies (cf. [1])

$$d_{cc}(z,t) = \begin{cases} \pi|t|, & z = 0, \\ \frac{\theta}{\sin\theta}\|z\|, & z \neq 0, \end{cases} \tag{2.1}$$

where

$$\theta = \mu^{-1}\left(\frac{t}{\|z\|^2}\right) \quad \text{and} \quad \|z\|^2 = \sum_{j=1}^{n}\|z_j\|^2. \tag{2.2}$$

On the other hand, if we set

$$\Omega = \{(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) : \sum_{j=1}^{n}(A_j^2 + B_j^2) = 1, -2\pi \le \phi\rho \le 2\pi, \rho \ge 0\}$$

and define $\Phi : \Omega \to \mathbb{H}_n$ by $\Phi(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) = (z, t)$, where

$$\begin{cases} x_j(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) = \dfrac{A_j(1-\cos\phi\rho) + B_j \sin\phi\rho}{\phi} \\ y_j(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) = \dfrac{-B_j(1-\cos\phi\rho) + A_j \sin\phi\rho}{\phi} \\ t(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) = 2\dfrac{\phi\rho - \sin\phi\rho}{\phi^2} \end{cases} \tag{2.3}$$

$(j = 1, 2, \cdots, n)$, then the range of Φ is $\mathbb{H}_n$ and the center $\mathbb{Z}$ is just the set of points $\Phi(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho)$ with $\phi\rho = \pm 2\pi$. Furthermore, if one fixes $\rho > 0$, equation (2.3) parameterizes $\partial B_{cc}(e, \rho)$ (cf. [11]).

From (2.3), we have

$$\|z\| = \sqrt{\frac{2(1-\cos\phi\rho)}{\phi^2}} = \sqrt{\frac{4\sin^2\frac{\phi\rho}{2}}{\phi^2}} = \frac{2\sin\frac{\phi\rho}{2}}{\phi} \tag{2.4}$$

and

$$\mu(\theta) = \frac{t}{\|z\|^2} = \frac{\phi\rho - \sin\phi\rho}{2\sin^2\frac{\phi\rho}{2}} = \mu(\frac{\phi\rho}{2}).$$

Therefore,

$$\theta = \frac{\phi\rho}{2} \quad \text{when} \quad -2\pi < \phi\rho < 2\pi \tag{2.5}$$

since μ is a a diffeomorphism of the interval $(-\pi, \pi)$ onto $\mathbb{R}$.

3. Main results

To prove the main result, we first need the following representation formula, and the basic idea of the proof is the same as Cohn and Lu's (see [2]).

Lemma 3.1. *Let $R_2 > R_1 > 0$ and $f \in C^1(B_{cc}(e, R_2) \setminus B_{cc}(e, R_1))$. Then*

$$\int_\Sigma f(R_2\xi^*)d\sigma - \int_\Sigma f(R_1\xi^*)d\sigma = \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \langle \nabla_H f, \nabla_H d_{cc}\rangle \cdot \frac{1}{d_{cc}^{Q-1}} d\xi. \tag{3.1}$$

Proof. Let ξ^* be a point on the sphere, that is, $\xi^* = (z^*, t^*)$, where $d_{cc}(z^*, t^*) = 1$. We consider for $0 < R_1 < R_2$ the following difference using the fundamental theorem of calculus:

$$\begin{aligned} &\int_\Sigma f(R_2\xi^*)d\sigma - \int_\Sigma f(R_1\xi^*)d\sigma = \int_\Sigma \int_{R_1}^{R_2} \frac{d}{d\rho} f(\rho\xi^*)d\rho d\sigma \\ &= \int_\Sigma \int_{R_1}^{R_2} \left(\sum_{j=1}^{n} \frac{\partial f(\xi)}{\partial x_j} \cdot \frac{\partial x_j}{\partial \rho} + \sum_{j=1}^{n} \frac{\partial f(\xi)}{\partial y_j} \cdot \frac{\partial y_j}{\partial \rho} + \frac{\partial f(\xi)}{\partial t} \cdot \frac{\partial t}{\partial \rho} \right) d\rho d\sigma, \end{aligned} \tag{3.2}$$

where $\xi = (x,t) = \rho\xi^*$. Using equation (2.3), we have

(3.3)
$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial f(\xi)}{\partial x_j} \cdot \frac{\partial x_j}{\partial \rho} + \sum_{j=1}^{n} \frac{\partial f(\xi)}{\partial y_j} \cdot \frac{\partial y_j}{\partial \rho} + \frac{\partial f(\xi)}{\partial t} \cdot \frac{\partial t}{\partial \rho} \\
&= \sum_{j=1}^{n} \frac{\partial f(\xi)}{\partial x_j} \cdot (A_j \sin\phi\rho + B_j \cos\phi\rho) + \sum_{j=1}^{n} \frac{\partial f(\xi)}{\partial y_j} \cdot (-B_j \sin\phi\rho + A_j \cos\phi\rho) \\
&+ \frac{\partial f(\xi)}{\partial t} \cdot \frac{2 - 2\cos\phi\rho}{\phi} \\
&= \sum_{j=1}^{n} [X_j f \cdot (A_j \sin\phi\rho + B_j \cos\phi\rho) + Y_j f \cdot (-B_j \sin\phi\rho + A_j \cos\phi\rho)] + \frac{\partial f(\xi)}{\partial t} \\
&\quad \cdot \left\{ \frac{2-2\cos\phi\rho}{\phi} - \sum_{j=1}^{n} [2y_j(A_j \sin\phi\rho + B_j \cos\phi\rho) - 2x_j(-B_j \sin\phi\rho + A_j \cos\phi\rho)] \right\}.
\end{aligned}$$

Again using (2.3), we have

$$\begin{aligned}
&\sum_{j=1}^{n} [2y_j(A_j \sin\phi\rho + B_j \cos\phi\rho) - 2x_j(-B_j \sin\phi\rho + A_j \cos\phi\rho)] \\
&\quad = 2\sum_{j=1}^{n} \frac{(-B_j(1-\cos\phi\rho) + A_j \sin\phi\rho)(A_j \sin\phi\rho + B_j \cos\phi\rho)}{\phi} \\
&\quad - 2\sum_{j=1}^{n} \frac{(A_j(1-\cos\phi\rho) + B_j \sin\phi\rho)(-B_j \sin\phi\rho + A_j \cos\phi\rho)}{\phi} \\
&\quad = 2\sum_{j=1}^{n} \frac{(A_j^2 + B_j^2)(1-\cos\phi\rho)}{\phi} = 2\frac{1-\cos\phi\rho}{\phi}.
\end{aligned} \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
&\int_\Sigma f(R_2\xi^*)d\sigma - \int_\Sigma f(R_1\xi^*)d\sigma \\
&= \int_\Sigma \int_{R_1}^{R_2} [X_j f \cdot (A_j \sin\phi\rho + B_j \cos\phi\rho) + Y_j f \cdot (-B_j \sin\phi\rho + A_j \cos\phi\rho)]\, d\rho d\sigma.
\end{aligned}$$

Rewriting the last expression into a solid integral using the polar coordinates over $\mathbb{H}_n$, we get

$$\begin{aligned}
&\int_\Sigma f(R_2\xi^*)d\sigma - \int_\Sigma f(R_1\xi^*)d\sigma \\
&= \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \sum_{j=1}^{n} \frac{(A_j \sin\phi\rho + B_j \cos\phi\rho)X_j f + (-B_j \sin\phi\rho + A_j \cos\phi\rho)Y_j f}{d_{cc}(\xi)^{Q-1}} d\xi.
\end{aligned}$$

To finish our proof, it is enough to show that

$$X_j d_{cc}(\xi) = A_j \sin\phi\rho + B_j \cos\phi\rho, \quad Y_j d_{cc}(\xi) = -B_j \sin\phi\rho + A_j \cos\phi\rho$$

$(j = 1, \cdots, n)$ in $\mathbb{H}_n \setminus \mathbb{Z}$. This is done by the following Lemma 3.2. The proof of Lemma 3.1 is now completed. $\square$

Lemma 3.2. *It follows that, for* $\xi = (z,t) \in \mathbb{H}_n \setminus \mathbb{Z}$,

$$X_j d_{cc}(\xi) = A_j \sin\phi\rho + B_j \cos\phi\rho, \quad Y_j d_{cc}(\xi) = -B_j \sin\phi\rho + A_j \cos\phi\rho$$

$(j = 1, \cdots, n)$.

Proof. Recall that if $z \neq 0$, then

$$d_{cc}(\xi) = d_{cc}(z,t) = \frac{\theta}{\sin\theta}\|z\|,$$

where $\theta = \mu^{-1}(t/\|z\|^2)$. A simple calculation shows that, for $j = 1, \cdots, n$,

$$\mu'(\theta) = \frac{2\sin\theta - 2\theta\cos\theta}{\sin^3\theta}, \quad \frac{\partial\theta}{\partial x_j} = -\frac{2tx_j}{\|z\|^4} \cdot \frac{1}{\mu'(\theta)} = -\frac{tx_j}{\|z\|^4} \cdot \frac{\sin^3\theta}{\sin\theta - \theta\cos\theta},$$

$$\frac{\partial\theta}{\partial y_j} = -\frac{ty_j}{\|z\|^4} \cdot \frac{\sin^3\theta}{\sin\theta - \theta\cos\theta}, \qquad \frac{\partial\theta}{\partial t} = \frac{1}{\|z\|^2} \cdot \frac{\sin^3\theta}{2\sin\theta - 2\theta\cos\theta}.$$

Therefore, for $\xi = (z,t) \in \mathbb{H}_n \setminus \mathbb{Z}$,

$$\begin{aligned}
\frac{\partial d_{cc}(\xi)}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\frac{\theta}{\sin\theta}\|z\|\right) = \frac{x_j}{\|z\|} \cdot \frac{\theta}{\sin\theta} + \|z\| \cdot \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \cdot \frac{\partial\theta}{\partial x_j} \\
&= \frac{x_j}{\|z\|} \cdot \frac{\theta}{\sin\theta} - \frac{tx_j}{\|z\|^3} \cdot \sin\theta = \frac{x_j}{\|z\|} \cdot \frac{\theta}{\sin\theta} - \frac{x_j}{\|z\|} \cdot \mu(\theta)\sin\theta \\
&= \frac{x_j}{\|z\|} \cdot \cos\theta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial d_{cc}(\xi)}{\partial y_j} &= \frac{\partial}{\partial y_j}\left(\frac{\theta}{\sin\theta}\|z\|\right) = \frac{y_j}{\|z\|} \cdot \frac{\theta}{\sin\theta} + \|z\| \cdot \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \cdot \frac{\partial\theta}{\partial y_j} \\
&= \frac{y_j}{\|z\|} \cdot \cos\theta,
\end{aligned}$$

$$\frac{\partial d_{cc}(\xi)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\theta}{\sin\theta}\|z\|\right) = \|z\| \cdot \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \cdot \frac{\partial\theta}{\partial t} = \frac{1}{\|z\|} \cdot \frac{\sin\theta}{2}.$$

Thus

$$X_j d_{cc}(\xi) = \left(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\right) d_{cc}(\xi) = \frac{x_j}{\|z\|} \cdot \cos\theta + \frac{y_j}{\|z\|} \cdot \sin\theta.$$

Using equations (2.3) and (2.4), we get

$$\begin{aligned}
X_j d_{cc}(\xi) &= \frac{\phi}{2\sin\frac{\phi\rho}{2}}\Bigg(\frac{A_j(1-\cos\phi\rho) + B_j\sin\phi\rho}{\phi}\cos\theta \\
&\qquad\qquad + \frac{-B_j(1-\cos\phi\rho) + A_j\sin\phi\rho}{\phi}\sin\theta\Bigg) \\
&= \left(A_j\sin\frac{\phi\rho}{2} + B_j\cos\frac{\phi\rho}{2}\right)\cos\theta + \left(-B_j\sin\frac{\phi\rho}{2} + A_j\cos\frac{\phi\rho}{2}\right)\sin\theta \\
&= A_j\sin\left(\frac{\phi\rho}{2} + \theta\right) + B_j\cos\left(\frac{\phi\rho}{2} + \theta\right).
\end{aligned}$$

Therefore we obtain, by (2.5),

$$X_j d_{cc}(\xi) = A_j\sin\phi\rho + B_j\cos\phi\rho.$$

On the other hand, we have, again using equations (2.3), (2.4) and (2.5),

$$\begin{aligned}
Y_j d_{cc}(\xi) &= \left(\frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}\right) d_{cc}(\xi) = \frac{y_j}{\|z\|} \cdot \cos\theta - \frac{x_j}{\|z\|} \cdot \sin\theta \\
&= \frac{\phi}{2\sin\frac{\phi\rho}{2}} \Big(\frac{-B_j(1-\cos\phi\rho) + A_j \sin\phi\rho}{\phi} \cos\theta \\
&\qquad\qquad - \frac{A_j(1-\cos\phi\rho) + B_j \sin\phi\rho}{\phi} \sin\theta \Big) \\
&= \left(-B_j \sin\frac{\phi\rho}{2} + A_j \cos\frac{\phi\rho}{2}\right)\cos\theta - \left(A_j \sin\frac{\phi\rho}{2} + B_j \cos\frac{\phi\rho}{2}\right)\sin\theta \\
&= A_j \cos\left(\frac{\phi\rho}{2} + \theta\right) - B_j \sin\left(\frac{\phi\rho}{2} + \theta\right) \\
&= A_j \cos\phi\rho - B_j \sin\phi\rho.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

Remark 3.3. Putting $f = u \in C_0^\infty(\mathbb{H}_n)$ in Lemma 3.1 and letting $R_2 \to \infty$ and $R_1 \to 0+$, we have

$$|\Sigma| u(e) = -\int_{\mathbb{H}_n} \langle \nabla_H u, \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^{Q-1}} d\xi,$$

where $|\Sigma| = \int_\Sigma d\sigma$ (see [2] for an analogous representation formula associated with the Korányi-Folland nonisotropic gauge). We note the representation formula in Lemma 3.1 is also valid for $f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z})$. In fact, since $\xi \in \mathbb{Z}$ iff $\lambda\xi \in \mathbb{Z}$ for all $\lambda > 0$, one has, for all $f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z})$, as in the proof of Lemma 3.1,

$$\begin{aligned}
&\int_{\Sigma\setminus(\Sigma\cap\mathbb{Z})} f(R_2\xi^*) d\sigma - \int_{\Sigma\setminus(\Sigma\cap\mathbb{Z})} f(R_1\xi^*) d\sigma \\
&= \int_{(B_{cc}(e,R_2)\setminus\mathbb{Z})\setminus(B_{cc}(e,R_1)\setminus\mathbb{Z})} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi,
\end{aligned}$$

where $R_2 > R_1 > 0$. Note that $\Sigma \cap \mathbb{Z} = \{(\mathbf{0}, 1/\pi), (\mathbf{0}, -1/\pi)\}$. So

$$\int_{\Sigma\setminus(\Sigma\cap\mathbb{Z})} f(R_k\xi^*) d\sigma = \int_\Sigma f(R_k\xi^*) d\sigma < \infty, \quad k = 1, 2,$$

since $f \in C(\mathbb{H}_n)$. These imply that

$$\begin{aligned}
&\left| \int_{(B_{cc}(e,R_2)\setminus\mathbb{Z})\setminus(B_{cc}(e,R_1)\setminus\mathbb{Z})} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi \right| \\
&= \left| \int_{\Sigma\setminus(\Sigma\cap\mathbb{Z})} f(R_2\xi^*) d\sigma - \int_{\Sigma\setminus(\Sigma\cap\mathbb{Z})} f(R_1\xi^*) d\sigma \right| \\
&= \left| \int_\Sigma f(R_2\xi^*) d\sigma - \int_\Sigma f(R_1\xi^*) d\sigma \right| < \infty.
\end{aligned}$$

Therefore,

$$\int_{(B_{cc}(e,R_2)\setminus\mathbb{Z})\setminus(B_{cc}(e,R_1)\setminus\mathbb{Z})} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi = \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi$$

since $\mathbb{Z}$ is a measure-zero set. This means that the representation formula in Lemma 3.1 is valid for $f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z})$.

Proof of Theorem 1.1. Let $\epsilon > 0$. Then $0 \le u_\epsilon := (|u|^2+\epsilon^2)^{p/2} - \epsilon^p \in C_0^\infty(\mathbb{H}_n)$. In fact, u_ϵ has the same support as u. Putting $f = u_\epsilon d_{cc}^{Q-p}$ in Lemma 3.1 and letting $R_2 \to \infty$ and $R_1 \to 0+$, we get, by Remark 3.3, since $d_{cc} \in C(\mathbb{H}_n) \cap C^\infty(\mathbb{H}_n \setminus \mathbb{Z})$ and $d_{cc}(e) = 0$,

$$\int_{\mathbb{H}_n} \langle \nabla_H u_\epsilon, \nabla_H d_{cc}\rangle \cdot \frac{1}{d_{cc}^{p-1}} + (Q-p)\int_{\mathbb{H}_n} \frac{u_\epsilon}{d_{cc}^p} = 0.$$

Here we use the fact that $|\nabla_H d_{cc}(\xi)| = 1$ when $\xi \in \mathbb{H}_n \setminus \mathbb{Z}$ (cf. [11]). By Hölder's inequality:

$$\begin{aligned}(Q-p)\int_{\mathbb{H}_n} \frac{u_\epsilon}{d_{cc}^p} &= -p\int_{\mathbb{H}_n} (|u|^2+\epsilon^2)^{(p-2)/2} u\langle \nabla_H u, \nabla_H d_{cc}\rangle \cdot \frac{1}{d_{cc}^{p-1}} \\ &\le p\int_{\mathbb{H}_n} \frac{(|u|^2+\epsilon^2)^{(p-2)/2}|u|\cdot|\nabla_H u|}{d_{cc}^{p-1}} \\ &\le p\left(\int_{\mathbb{H}_n} \frac{(|u|^2+\epsilon^2)^{p(p-2)/2(p-1)}|u|^{\frac{p}{p-1}}}{d_{cc}^p}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{H}_n} |\nabla_H u|^p\right)^{\frac{1}{p}}.\end{aligned}$$

By dominated convergence, letting $\epsilon \to 0+$, canceling and raising both sides to the power p, we get (1.3). The proof of Theorem 1.1 is complete. □

Remark 3.4. Let $V^{1,p}(\mathbb{H}_n)$ denote the space

$$V^{1,p}(\mathbb{H}_n) = \{f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z}) : \int_{\mathbb{H}_n} \frac{|u|^p}{d_{cc}^p} < +\infty, \int_{\mathbb{H}_n} |\nabla_H f|^p < +\infty\}.$$

In this remark, we shall show that the following Hardy inequalities hold for all $f \in V^{1,p}(\mathbb{H}_n)$:

$$\int_{\mathbb{H}_n} |\nabla_H f|^p \ge \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{H}_n} \frac{|f|^p}{d_{cc}^p}.$$

Furthermore, the constant $\left(\frac{Q-p}{p}\right)^p$ is sharp in the sense of

$$\left(\frac{Q-p}{p}\right)^p = \inf_{f \in V^{1,p}(\mathbb{H}_n)\setminus\{0\}} \frac{\int_{\mathbb{H}_n} |\nabla_H f|^p}{\int_{\mathbb{H}_n} \frac{|f|^p}{d_{cc}^p}}.$$

Step 1. We shall show that if $f \in V^{1,p}(\mathbb{H}_n)$, then

$$\lim_{R\to+\infty} R^{Q-p}\int_\Sigma |f(R\xi^*)|^p d\sigma = 0. \tag{3.5}$$

We note from Lemma 3.1 that

$$\begin{aligned}
&\left| R_2^{Q-p} \int_\Sigma |f(R_2\xi^*)|^p d\sigma - R_1^{Q-p} \int_\Sigma |f(R_1\xi^*)|^p d\sigma \right| \\
&= \left| \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \langle \nabla_H(|f|^p d_{cc}^{Q-p}), \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^{Q-1}} d\xi \right| \\
&\le p \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \frac{|f|^{p-1}|\nabla_H f|}{d_{cc}^{p-1}} d\xi + (Q-p) \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \frac{|f|^p}{d_{cc}^p} d\xi \\
&\le p \left(\int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \frac{|f|^p}{d_{cc}^p} \right)^{\frac{p-1}{p}} \left(\int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} |\nabla_H f|^p \right)^{\frac{1}{p}} \\
&\quad + (Q-p) \int_{B_{cc}(e,R_2)\setminus B_{cc}(e,R_1)} \frac{|f|^p}{d_{cc}^p} d\xi.
\end{aligned}$$

Thus,

$$\left| R_2^{Q-p} \int_\Sigma |f(R_2\xi^*)|^p d\sigma - R_1^{Q-p} \int_\Sigma |f(R_1\xi^*)|^p d\sigma \right| \to 0$$

as $R_1, R_2 \to +\infty$. This shows that $R^{Q-p} \int_\Sigma |f(R\xi^*)|^p d\sigma$ converges to a finite constant $f_\infty \ge 0$ as $R \to +\infty$. One must have $f_\infty = 0$ for

$$\int_{\mathbb{H}_n} \frac{|f|^p}{d_{cc}^p} d\xi = \int_0^\infty \int_\Sigma |f(\lambda\xi^*)|^p \lambda^{Q-p-1} d\sigma d\lambda = \int_0^\infty \frac{\lambda^{Q-p} \int_\Sigma |f(\lambda\xi^*)|^p d\sigma}{\lambda} d\lambda < +\infty.$$

Therefore, (3.5) holds.

Step 2. Replace f by $d_{cc}^{Q-p}|f|^p$ in Lemma 3.1 and let $R_2 \to \infty$ and $R_1 \to 0+$. Using the result of Step 1 and following the proof of Theorem 1.1, one can obtain the Hardy inequalities as expected.

We now show that the constant $\left(\frac{Q-p}{p}\right)^p$ is sharp. The proof is similar to that of Lemma 2.3 in [13]. Let

$$f_\varepsilon(z,t) = \begin{cases} 1, & d_{cc} \le 1; \\ d_{cc}^{-(Q-p)/p-\varepsilon}, & d_{cc} > 1. \end{cases}$$

We note that this family of functions can be approximated by the functions in $V^{1,p}(\mathbb{H}_n)$. In addition,

$$\int_{\mathbb{H}_n} \frac{|f_\varepsilon|^p}{d_{cc}^p} d\xi = \frac{|\Sigma|}{Q-p} + \frac{|\Sigma|}{\varepsilon p}$$

and

$$\int_{\mathbb{H}_n} |\nabla_H f_\varepsilon|^p d\xi = \left(-\frac{Q-p}{p} - \varepsilon \right)^p \cdot \frac{|\Sigma|}{\varepsilon p}.$$

Passing to the limit as $\varepsilon \to 0$, one can see that the constant $\left(\frac{Q-p}{p}\right)^p$ is sharp. We note that since the Carnot-Carathéodory distance is not differentiable in $\mathbb{Z}$, it is unknown whether f_ε, the family of functions above, can be approximated by the functions in $C_0^\infty(\mathbb{H}_n)$. So, we do not know whether the constant $\left(\frac{Q-p-\alpha}{p}\right)^p$ is sharp for all $f \in C_0^\infty(\mathbb{H}_n)$ in Theorem 1.1.

Remark 3.5. Let $\alpha < Q-p$. Put $f = u_\epsilon d_{cc}^{Q-p-\alpha}$ in Lemma 3.1 and let $R_2 \to \infty$ and $R_1 \to 0+$. We have

$$\begin{aligned}(Q-p-\alpha)\int_{\mathbb{H}_n} \frac{u_\epsilon}{d_{cc}^{p+\alpha}} &= -p\int_{\mathbb{H}_n} (|u|^2+\epsilon^2)^{(p-2)/2} u\langle \nabla_H u, \nabla_H d_{cc}\rangle \cdot \frac{1}{d_{cc}^{p+\alpha-1}} \\ &\le p\int_{\mathbb{H}_n} \frac{(|u|^2+\epsilon^2)^{(p-2)/2}|u|\cdot|\nabla_H u|}{d_{cc}^{p+\alpha-1}} \\ &\le p\left(\int_{\mathbb{H}_n} \frac{(|u|^2+\epsilon^2)^{p(p-2)/2(p-1)}|u|^{\frac{p}{p-1}}}{d_{cc}^{p+\alpha}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{H}_n} \frac{|\nabla_H u|^p}{d_{cc}^{\alpha}}\right)^{\frac{1}{p}}.\end{aligned}$$

As in the proof of Theorem 1.1, we obtain the following Hardy-type inequalities with weights:

$$\int_{\mathbb{H}_n} \frac{|\nabla_H u|^p}{d_{cc}^{\alpha}} \ge \left(\frac{Q-p-\alpha}{p}\right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d_{cc}^{p+\alpha}}, \quad \alpha < Q-p. \tag{3.6}$$

Proof of Theorem 1.2. Notice that

$$-\int_{\mathbb{H}_n} u\Delta_H u \cdot \frac{1}{d_{cc}^2} = -\frac{1}{2}\int_{\mathbb{H}_n} \Delta_H u^2 \cdot \frac{1}{d_{cc}^2} + \int_{\mathbb{H}_n} \frac{|\nabla_H u|^2}{d_{cc}^2}.$$

Through integration by parts, since $u \in C_0^\infty(\mathbb{H}_n \setminus \mathbb{Z})$ and d_{cc} is C^1 in $\mathbb{H}_n \setminus \mathbb{Z}$ (cf. [11]),

$$-\frac{1}{2}\int_{\mathbb{H}_n} \Delta_H u^2 \cdot \frac{1}{d_{cc}^2} = -\int_{\mathbb{H}_n} \langle \nabla_H u^2, \nabla_H d_{cc}\rangle \cdot \frac{1}{d_{cc}^3} d\xi.$$

Putting $f = u^2 d_{cc}^{Q-4}$ in Lemma 3.1 and letting $R_2 \to \infty$ and $R_1 \to 0+$, we have

$$-\int_{\mathbb{H}_n} \langle \nabla_H u^2, \nabla_H d_{cc}\rangle \cdot \frac{1}{d_{cc}^3} d\xi = (Q-4)\int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} d\xi.$$

Thus, for all $\varepsilon > 0$,

$$\begin{aligned}-\int_{\mathbb{H}_n} u\Delta_H u \cdot \frac{1}{d_{cc}^2} &= \int_{\mathbb{H}_n} \frac{|\nabla_H u|^2}{d_{cc}^2} + (Q-4)\int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} d\xi \\ &\le \left(\int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4}\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}_n} |\Delta_H u|^2\right)^{\frac{1}{2}} \\ &\le \frac{\varepsilon}{2}\int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} + \frac{1}{2\varepsilon}\int_{\mathbb{H}_n} |\Delta_H u|^2.\end{aligned} \tag{3.7}$$

Choosing $\varepsilon = \frac{Q(Q-4)}{4}$ in (3.7), we have

$$\int_{\mathbb{H}_n} |\Delta_H u|^2 + \frac{Q(Q-4)^2(Q-8)}{16}\int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} \ge \frac{Q(Q-4)}{2}\int_{\mathbb{H}_n} \frac{|\nabla_H u|^2}{d_{cc}^2}. \tag{3.8}$$

Since (see (3.6))

$$\int_{\mathbb{H}_n} \frac{|\nabla_H u|^2}{d_{cc}^2} \ge \frac{(Q-4)^2}{4}\int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4}, \quad n \ge 2, \tag{3.9}$$

we obtain, by (3.8),

$$\int_{\mathbb{H}_n} |\Delta_H u|^2 \ge \left(\frac{Q(Q-4)}{4}\right)^2 \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4}.$$

When $Q \geq 8$, i.e. $n \geq 3$, again using (3.7) and (3.8), we obtain

$$\frac{Q(Q-4)}{2}\int_{\mathbb{H}_n}\frac{|\nabla_H u|^2}{d_{cc}^2} \leq \int_{\mathbb{H}_n}|\Delta_H u|^2 + \frac{Q(Q-8)}{4}\int_{\mathbb{H}_n}\frac{|\nabla_H u|^2}{d_{cc}^2}.$$

The desired result follows. □

Proof of Theorem 1.3. As in the proof of Theorem 1.1, putting $f = \frac{u_\epsilon d_{cc}^{Q-1}}{(\rho - d_{cc})^{p-1}}$ in Lemma 3.1 and letting $R_2 = \rho$ and $R_1 \to 0+$, we get

$$\begin{aligned}
&(Q-1)\int_{B_{cc}(e,\rho)}\frac{u_\epsilon}{d_{cc}(\rho-d_{cc})^{p-1}} + (p-1)\int_{B_{cc}(e,\rho)}\frac{u_\epsilon}{(\rho-d_{cc})^p}\\
&\quad = -p\int_{B_{cc}(e,\rho)}(|u|^2+\epsilon^2)^{(p-2)/2}u\langle\nabla_H u, \nabla_H d_{cc}\rangle\cdot\frac{1}{(\rho-d_{cc})^{p-1}}\\
&\quad \leq p\int_{B_{cc}(e,\rho)}\frac{(|u|^2+\epsilon^2)^{(p-2)/2}|u|\cdot|\nabla_H u|}{(\rho-d_{cc})^{p-1}}\\
&\quad \leq p\left(\int_{B_{cc}(e,\rho)}\frac{(|u|^2+\epsilon^2)^{p(p-2)/2(p-1)}|u|^{\frac{p}{p-1}}}{(\rho-d_{cc})^p}\right)^{\frac{p-1}{p}}\left(\int_{B_{cc}(e,\rho)}|\nabla_H u|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Letting $\epsilon \to 0+$, we have, by dominated convergence,

$$\begin{aligned}
\left(\int_{B_{cc}(e,\rho)}\frac{|u|^p}{(\rho-d_{cc})^p}\right)^{\frac{p-1}{p}}&\left(\int_{B_{cc}(e,\rho)}|\nabla_H u|^p\right)^{\frac{1}{p}}\\
&\geq \frac{p-1}{p}\int_{B_{cc}(e,\rho)}\frac{|u|^p}{(\rho-d_{cc})^p}\\
&\quad + \frac{Q-1}{p}\int_{B_{cc}(e,\rho)}\frac{|u|^p}{d_{cc}(\rho-d_{cc})^{p-1}}\\
&\geq \frac{p-1}{p}\int_{B_{cc}(e,\rho)}\frac{|u|^p}{(\rho-d_{cc})^p}.
\end{aligned}$$

Canceling and raising both sides to the power p, we get (1.6). □

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School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, People's Republic of China

E-mail address: qaohyang2465@yahoo.com.cn