HARDY TYPE INEQUALITIES
RELATED TO CARNOT-CARATHÉODORY DISTANCE
ON THE HEISENBERG GROUP

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Abstract. Being motivated by a representation formula associated with the
Korányi-Folland nonisotropic gauge proved by Cohn and Lu, we prove an anal-
ogous representation formula related to the Carnot-Carathéodory distance on
the Heisenberg group. Using this formula, we obtain some Hardy inequalities
associated with the Carnot-Carathéodory distance on such groups.

1. Introduction

It is well known that Hardy’s inequality and its generalization in \( \mathbb{R}^N \) play an
important role in many areas of mathematics. In the case of the Heisenberg group
\( \mathbb{H}_n \), Garofalo and Lanconelli (cf. [9]) first proved the following Hardy inequality:

\[
\int_{\mathbb{H}_n} |\nabla_H u|^2 \geq \frac{(Q - 2)^2}{4} \int_{\mathbb{H}_n} \frac{u^2}{d^2} |\nabla_H d|^2, \quad u \in C_0^\infty(\mathbb{H}_n \setminus \{e\}),
\]

where \( e \) is the origin of \( \mathbb{H}_n \), \( d = (|z|^4 + t^2)^{\frac{1}{4}} \) is the Korányi-Folland nonisotropic
gauge induced by the fundamental solution and \( Q = 2n + 2 \) is the homogeneous
dimension of \( \mathbb{H}_n \) (see also [10]). P. Niu et al. ([11]) generalized inequality (1.1) to
the nonlinear cases. That is, for \( 1 < p < Q \), we have

\[
\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left( \frac{Q - p}{p} \right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d^p} |\nabla_H d|^p, \quad u \in C_0^\infty(\mathbb{H}_n \setminus \{e\}).
\]

To obtain (1.2), they used a Picone type identity on \( \mathbb{H}_n \). D’Ambrosio (cf. [3]
[4] [5]) extended (1.2) to some degenerate elliptic differential operators such as
the Heisenberg-Greiner operator, a Baouendi-Grushin type operator and the sub-
Laplacian on Carnot groups. More recently, Danielli, Garofalo and Phuc (cf. [6] [7])
consider various types of Hardy-Sobolev inequalities on a Carnot-Carathéodory
space. They use the fundamental solution of the corresponding \( p \)-Laplace operator
and generalize (1.2) to Carnot groups of arbitrary step.

The aim of this paper is to prove an analogous Hardy-type inequality (1.2)
on \( \mathbb{H}_n \), where the Korányi-Folland nonisotropic gauge is replaced by the Carnot-
Carathéodory distance \( d_{cc} \) on \( \mathbb{H}_n \) (we refer to [1] [11] for more information about
this distance). To our knowledge, little is known about Hardy inequalities related to such a distance. Since \(d_{cc}\) is not differentiable in the center of \(\mathbb{H}_n\), it seems that the methods used in [3, 5, 7, 12] do not work for such a distance. To do so, we prove a representation formula associated with \(d_{cc}\), and the idea is due to Cohn and Lu ([2]). The main result is the following theorem.

**Theorem 1.1.** Let \(1 < p < Q\) and \(u \in C_0^{\infty}(\mathbb{H}_n)\). Then

\[
\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left( \frac{Q - p}{p} \right)^p \int_{\mathbb{H}_n} |u|^p \frac{dp}{d_{cc}^p}.
\]

The Hardy type inequalities with weights imply the following Hardy-Rellich type inequalities (see e.g. [14] for analogous inequalities on \(\mathbb{R}^N\)).

**Theorem 1.2.** Let \(Z = \{(z, t) \in \mathbb{H}_n : z = 0\}\) be the center of \(\mathbb{H}_n\). It follows, for all \(u \in C_0^{\infty}(\mathbb{H}_n \setminus Z)\), that

\[
\int_{\mathbb{H}_n} |\Delta_H u|^2 \geq \left( \frac{Q(Q - 4)}{4} \right)^2 \int_{\mathbb{H}_n} u^2 \frac{d^2}{d_{cc}^2}, \quad n \geq 2,
\]

and

\[
\int_{\mathbb{H}_n} |\Delta_H u|^2 \geq \frac{Q^2}{4} \int_{\mathbb{H}_n} \frac{|\nabla_H u|^2}{d_{cc}^2}, \quad n \geq 3.
\]

Finally, we obtain some Hardy inequalities involving the distance from the boundary which generalize a result of D’Ambrosio (cf. [5]).

**Theorem 1.3.** Let \(p > 1\) and \(u \in C_0^{\infty}(B_{cc}(e, \rho))\), where \(B_{cc}(e, \rho)\) is the Carnot-Carathéodory metric ball centered at the origin. Then

\[
\int_{B_{cc}(e, \rho)} |\nabla_H u|^p \geq \left( \frac{p - 1}{p} \right)^p \int_{B_{cc}(e, \rho)} \frac{|u|^p}{(\rho - d_{cc})^p}.
\]

2. **Notation and preliminaries**

Let \(\mathbb{H}_n = (\mathbb{C}^n \times \mathbb{R}, \circ)\) be the the \((2n + 1)\)-dimensional Heisenberg group whose group structure is given by

\[
(z, t) \circ (z', t') = (z + z', t + t' + 2\text{Im}(z, z')),
\]

where \(z = (z_1, \cdots, z_n)\), \(z' = (z_1', \cdots, z_n') \in \mathbb{C}^n\), \(z_j = x_j + iy_j\) \((x_j, y_j \in \mathbb{R})\) and \(\langle z, z' \rangle = \sum_{j=1}^{n} z_j \cdot \overline{z_j}\). The vector fields

\[
X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}
\]

\((j = 1, \cdots, n)\) are left invariant and generate the Lie algebra of \(\mathbb{H}_n\). The commutators of these vector fields satisfy

\[
[X_j, Y_j] = -4 \frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n
\]
with all other brackets equal to zero. We denote by $\mathbb{Z} = \{(z, t) \in \mathbb{H}_n : z = 0\}$ the center of $\mathbb{H}_n$. Then Kohn’s sub-Laplacian on $\mathbb{H}_n$ is
\[
\Delta_H = \sum_{j=1}^{n} (X_j^2 + Y_j^2) = \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) + 4|z|^2 \frac{\partial^2}{\partial t^2} + 4 \sum_{k=1}^{n} (y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}) \frac{\partial}{\partial t},
\]
and the horizontal gradient is the $(2n)$-dimensional vector given by
\[
\nabla_H = (X_1, \cdots, X_n, Y_1, \cdots, Y_n).
\]
We call a curve $\gamma : [a, b] \to \mathbb{H}_n$ a horizontal curve connecting two points $\xi, \eta \in \mathbb{H}_n$ if $\gamma(a) = \xi$, $\gamma(b) = \eta$ and $\gamma'(s) \in \text{span}\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ for all $s$. Then the Carnot-Carathéodory distance between $\xi, \eta$ is defined as
\[
d_{cc}(\xi, \eta) = \inf_{\gamma} \int_{a}^{b} \|\gamma'(s)\|ds,
\]
where the infimum is taken over all horizontal curves $\gamma$ connecting $\xi$ and $\eta$. It is known that any two points $\xi, \eta$ on $\mathbb{H}_n$ can be joined by a horizontal curve of finite length and then $d_{cc}$ is a metric on $\mathbb{H}_n$. An important feature of this distance function is that the distance and thus the associated metric balls are left-invariant. With this norm, we can define the metric ball centered at the origin and with radius $\rho$ associated with this metric by
\[
B_{cc}(e, \rho) = \{ \eta : d_{cc}(e, \eta) < \rho \}
\]
and the unit sphere $\Sigma = \partial B_{cc}(e, 1)$. For simplicity, we write $d_{cc}(\xi) = d_{cc}(e, \xi)$.

For each real number $\lambda > 0$, there is a dilation naturally associated with the group structure which is usually denoted as $\delta_{\lambda}(z, t) = (\lambda z, \lambda^2 t)$. The Jacobian determinant of $\delta_{\lambda}$ is $\lambda^Q$, where $Q = 2n + 2$ is the homogeneous dimension of $H_n$. For simplicity, we use the notation $\lambda(z, t) = (\lambda z, \lambda^2 t)$. The Carnot-Carathéodory distance $d_{cc}$ satisfies
\[
d_{cc}(\lambda(z, t)) = \lambda d_{cc}(z, t), \quad \lambda > 0.
\]
Given any $\xi = (z, t) \in \mathbb{H}_n$, set $z^* = \frac{z}{d_{cc}(z, t)}$, $t^* = \frac{t}{d_{cc}(z, t)^2}$, and $\xi^* = (z^*, t^*)$. The polar coordinates on $\mathbb{H}_n$ associated with $d_{cc}$ are the following (cf. [8], Proposition 1.15 or [1], via the coarea formula):
\[
\int_{\mathbb{H}_n} f(z, t)dxdt = \int_{0}^{\infty} \int_{\Sigma} f(\lambda(z^*, t^*))\lambda^{Q-1}d\sigma d\lambda
\]
for all $f \in L^1(\mathbb{H}_n)$.

Set
\[
\mu(\theta) = \frac{2\theta - \sin 2\theta}{2\sin^2 \theta} : (-\pi, \pi) \to \mathbb{R}.
\]
$\mu$ is a diffeomorphism of the interval $(-\pi, \pi)$ onto $\mathbb{R}$ (cf. [1]). We denote by $\mu^{-1}$ the inverse function of $\mu$. The Carnot-Carathéodory distance $d_{cc}$ satisfies (cf. [1])
\[
d_{cc}(z, t) = \begin{cases} 
\frac{\pi |t|}{\theta} & \text{if } \theta \neq 0, \\
\frac{\theta}{\sin \theta} \|z\| & \text{if } z = 0,
\end{cases}
\]
where
\[
\theta = \mu^{-1}\left(\frac{t}{\|z\|}\right) \quad \text{and} \quad \|z\|^2 = \sum_{j=1}^{n} \|z_j\|^2.
\]
Lemma 3.1. Let \( \mu \) and define \( \Phi : \Omega \rightarrow \mathbb{H}_n \) by \( \Phi(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) = (z, t) \), where

\[
\begin{align*}
\text{for } j = 1, 2, \cdots, n, \text{ then the range of } \Phi \text{ is } \mathbb{H}_n \text{ and the center } Z \text{ is just the set of points } \\
\Phi(B_1, \cdots, B_n, A_1, \cdots, A_n, \phi, \rho) \text{ with } \phi \rho = \pm 2\pi. \text{ Furthermore, if one fixes } \rho > 0, \text{ equation (2.3) parameterizes } \partial B_{cc}(e, \rho) \text{ (cf. [11]).}
\end{align*}
\]

From (2.3), we have

\[
\|z\| = \sqrt{2(1 - \cos \phi \rho)} = \sqrt{\frac{4\sin^2 \frac{\phi \rho}{2}}{\phi^2}} = \frac{2\sin \frac{\phi \rho}{2}}{\phi}
\]

and

\[
\mu(\theta) = \frac{t}{\|z\|^2} = \frac{\phi \rho - \sin \phi \rho}{2\sin^2 \frac{\phi \rho}{2}} = \mu(\frac{\phi \rho}{2}).
\]

Therefore,

\[
\theta = \frac{\phi \rho}{2} \text{ when } -2\pi < \phi \rho < 2\pi
\]

since \( \mu \) is a a diffeomorphism of the interval \((-\pi, \pi)\) onto \(\mathbb{R}\).

3. Main results

To prove the main result, we first need the following representation formula, and the basic idea of the proof is the same as Cohn and Lu’s (see [2]).

**Lemma 3.1.** Let \( R_2 > R_1 > 0 \) and \( f \in C^1(B_{cc}(e, R_2) \setminus B_{cc}(e, R_1)) \). Then

\[
\int_\Sigma f(R_2 \xi^*) d\sigma - \int_\Sigma f(R_1 \xi^*) d\sigma = \int_{B_{cc}(e, R_2) \setminus B_{cc}(e, R_1)} \langle \nabla_H f, \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^2} d\xi.
\]

**Proof.** Let \( \xi^* \) be a point on the sphere, that is, \( \xi^* = (z^*, t^*) \), where \( d_{cc}(z^*, t^*) = 1 \). We consider for \( 0 < R_1 < R_2 \) the following difference using the fundamental theorem of calculus:

\[
\int_\Sigma f(R_2 \xi^*) d\sigma - \int_\Sigma f(R_1 \xi^*) d\sigma = \int_\Sigma \int_{R_1}^{R_2} \frac{d}{d\rho} f(\rho \xi^*) d\rho d\sigma
\]

\[
= \int_\Sigma \int_{R_1}^{R_2} \left( \sum_{j=1}^n \frac{\partial f(\xi^*)}{\partial x_j} \frac{\partial x_j}{\partial \rho} + \sum_{j=1}^n \frac{\partial f(\xi^*)}{\partial y_j} \frac{\partial y_j}{\partial \rho} + \frac{\partial f(\xi^*)}{\partial t} \frac{\partial t}{\partial \rho} \right) d\rho d\sigma,
\]

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where $\xi = (x, t) = \rho \xi^*$. Using equation (2.3), we have

\begin{align*}
(3.3) \\
\sum_{j=1}^n \frac{\partial f(\xi)}{\partial x_j} \cdot \frac{\partial x_j}{\partial \rho} + \sum_{j=1}^n \frac{\partial f(\xi)}{\partial y_j} \cdot \frac{\partial y_j}{\partial \rho} + \frac{\partial f(\xi)}{\partial t} \cdot \frac{\partial t}{\partial \rho} \\
= \sum_{j=1}^n \frac{\partial f(\xi)}{\partial x_j} \cdot (A_j \sin \phi \rho + B_j \cos \phi \rho) + \sum_{j=1}^n \frac{\partial f(\xi)}{\partial y_j} \cdot (-B_j \sin \phi \rho + A_j \cos \phi \rho) \\
+ \frac{\partial f(\xi)}{\partial t} \cdot \frac{2 - 2 \cos \phi \rho}{\phi} \\
= \sum_{j=1}^n [X_j f \cdot (A_j \sin \phi \rho + B_j \cos \phi \rho) + Y_j f \cdot (-B_j \sin \phi \rho + A_j \cos \phi \rho)] + \frac{\partial f(\xi)}{\partial t} \\
\cdot \left\{ \frac{2 - 2 \cos \phi \rho}{\phi} - \sum_{j=1}^n [2y_j (A_j \sin \phi \rho + B_j \cos \phi \rho) - 2x_j (-B_j \sin \phi \rho + A_j \cos \phi \rho)] \right\}.
\end{align*}

Again using (2.3), we have

\begin{align*}
\sum_{j=1}^n [2y_j (A_j \sin \phi \rho + B_j \cos \phi \rho) - 2x_j (-B_j \sin \phi \rho + A_j \cos \phi \rho)] \\
= 2 \sum_{j=1}^n \frac{(-B_j (1 - \cos \phi \rho) + A_j \sin \phi \rho)(A_j \sin \phi \rho + B_j \cos \phi \rho)}{\phi} \\
- 2 \sum_{j=1}^n \frac{(A_j (1 - \cos \phi \rho) + B_j \sin \phi \rho)(-B_j \sin \phi \rho + A_j \cos \phi \rho)}{\phi} \\
= 2 \sum_{j=1}^n \frac{(A_j^2 + B_j^2)(1 - \cos \phi \rho)}{\phi} = 2 \frac{1 - \cos \phi \rho}{\phi}.
\end{align*}

Combining (3.2), (3.3) and (3.4), we obtain

\begin{align*}
\int_{\Sigma} f(R_2 \xi^*) d\sigma - \int_{\Sigma} f(R_1 \xi^*) d\sigma \\
= \int_{\Sigma} \int_{R_2} [X_j f \cdot (A_j \sin \phi \rho + B_j \cos \phi \rho) + Y_j f \cdot (-B_j \sin \phi \rho + A_j \cos \phi \rho)] dp \, d\sigma.
\end{align*}

Rewriting the last expression into a solid integral using the polar coordinates over $\mathbb{H}_n$, we get

\begin{align*}
\int_{\Sigma} f(R_2 \xi^*) d\sigma - \int_{\Sigma} f(R_1 \xi^*) d\sigma \\
= \int_{B_{cc}(e,R_2) \backslash B_{cc}(e,R_1)} \sum_{j=1}^n \frac{(A_j \sin \phi \rho + B_j \cos \phi \rho) X_j f + (-B_j \sin \phi \rho + A_j \cos \phi \rho) Y_j f}{d_{cc}(\xi)^{Q-1}} d\xi.
\end{align*}

To finish our proof, it is enough to show that

\begin{align*}
X_j d_{cc}(\xi) &= A_j \sin \phi \rho + B_j \cos \phi \rho, \\
Y_j d_{cc}(\xi) &= -B_j \sin \phi \rho + A_j \cos \phi \rho \\
(j = 1, \ldots, n) \text{ in } \mathbb{H}_n \setminus \mathbb{Z}. 
\end{align*}

This is done by the following Lemma 3.2. The proof of Lemma 3.1 is now completed. \[\square\]
Lemma 3.2. It follows that, for \( \xi = (z, t) \in \mathbb{H}_n \setminus \mathbb{Z} \),
\[
X_j d_{cc}(\xi) = A_j \sin \phi \rho + B_j \cos \phi \rho, \quad Y_j d_{cc}(\xi) = -B_j \sin \phi \rho + A_j \cos \phi \rho
\]
\((j = 1, \cdots, n)\).

Proof. Recall that if \( z \neq 0 \), then
\[
d_{cc}(\xi) = d_{cc}(z, t) = \frac{\theta}{\sin \theta} \|z\|,
\]
where \( \theta = \mu^{-1}(t/\|z\|^2) \). A simple calculation shows that, for \( j = 1, \cdots, n \),
\[
\mu'(\theta) = \frac{2 \sin \theta - 2 \theta \cos \theta}{\sin^3 \theta}, \quad \frac{\partial \theta}{\partial x_j} = \frac{2tx_j}{\|z\|^4} \cdot \frac{1}{\mu'(\theta)} = -\frac{tx_j}{\|z\|^4} \cdot \frac{\sin \theta}{\sin \theta - \theta \cos \theta},
\]
\[
\frac{\partial \theta}{\partial y_j} = \frac{ty_j}{\|z\|^4} \cdot \frac{\sin \theta}{\sin \theta - \theta \cos \theta} \cdot \frac{\partial \theta}{\partial t} = \frac{1}{\|z\|^2} \cdot \frac{\sin \theta}{\sin \theta - \theta \cos \theta}.
\]
Therefore, for \( \xi = (z, t) \in \mathbb{H}_n \setminus \mathbb{Z} \),
\[
\frac{\partial d_{cc}(\xi)}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\theta}{\sin \theta} \|z\| \right) = \frac{x_j}{\|z\|} \cdot \frac{\theta}{\sin \theta} + \|z\| \cdot \frac{\sin \theta - \theta \cos \theta}{\sin^2 \theta} \cdot \frac{\partial \theta}{\partial x_j} = \frac{x_j}{\|z\|} \cdot \frac{\theta}{\sin \theta} - \frac{tx_j}{\|z\|^3} \cdot \frac{\sin \theta}{\|z\|} - \frac{x_j}{\|z\|} \cdot \frac{\theta}{\sin \theta} - \frac{x_j}{\|z\|} \cdot \mu(\theta) \sin \theta
\]
\[
\frac{\partial d_{cc}(\xi)}{\partial y_j} = \frac{\partial}{\partial y_j} \left( \frac{\theta}{\sin \theta} \|z\| \right) = \frac{y_j}{\|z\|} \cdot \frac{\theta}{\sin \theta},
\]
\[
\frac{\partial d_{cc}(\xi)}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\theta}{\sin \theta} \|z\| \right) = \|z\| \cdot \frac{\sin \theta - \theta \cos \theta}{\sin^2 \theta} \cdot \frac{\partial \theta}{\partial t} = \frac{1}{\|z\|} \cdot \frac{\sin \theta}{-2}.
\]
Thus
\[
X_j d_{cc}(\xi) = \left( \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \right) d_{cc}(\xi) = \frac{x_j}{\|z\|} \cdot \cos \theta + \frac{y_j}{\|z\|} \cdot \sin \theta.
\]
Using equations (2.3) and (2.4), we get
\[
X_j d_{cc}(\xi) = \frac{\phi}{2 \sin \frac{\phi \rho}{2}} \left( \frac{A_j (1 - \cos \phi \rho) + B_j \sin \phi \rho}{\phi} \cos \theta \right.
\]
\[
\left. + \frac{-B_j (1 - \cos \phi \rho) + A_j \sin \phi \rho \sin \theta}{\phi} \right)
\]
\[
= \left( A_j \sin \frac{\phi \rho}{2} + B_j \cos \frac{\phi \rho}{2} \right) \cos \theta + \left( -B_j \sin \frac{\phi \rho}{2} + A_j \cos \frac{\phi \rho}{2} \right) \sin \theta
\]
\[
= A_j \sin \left( \frac{\phi \rho}{2} + \theta \right) + B_j \cos \left( \frac{\phi \rho}{2} + \theta \right).
\]
Therefore we obtain, by (2.5),
\[
X_j d_{cc}(\xi) = A_j \sin \phi \rho + B_j \cos \phi \rho.
\]
On the other hand, we have, again using equations (2.3), (2.4) and (2.5),
\[
Y_j d_{cc}(\xi) = \left( \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \right) d_{cc}(\xi) = \frac{y_j}{\|z\|} \cdot \cos \theta - \frac{x_j}{\|z\|} \cdot \sin \theta
\]
\[
= \frac{\phi}{2 \sin \frac{\phi \rho}{2}} \left( -B_j (1 - \cos \phi \rho) + A_j \sin \phi \rho \cos \theta \right)
\]
\[
- \frac{A_j (1 - \cos \phi \rho) + B_j \sin \phi \rho}{\phi} \sin \theta
\]
\[
= \left( -B_j \sin \frac{\phi \rho}{2} + A_j \cos \frac{\phi \rho}{2} \right) \cos \theta - \left( A_j \sin \frac{\phi \rho}{2} + B_j \cos \frac{\phi \rho}{2} \right) \sin \theta
\]
\[
= A_j \cos \left( \frac{\phi \rho}{2} + \theta \right) - B_j \sin \left( \frac{\phi \rho}{2} + \theta \right)
\]
\[
= A_j \cos \phi \rho - B_j \sin \phi \rho.
\]
This completes the proof of Lemma 3.2. \qed

Remark 3.3. Putting \( f = u \in C_0^\infty(\mathbb{H}_n) \) in Lemma 3.1 and letting \( R_2 \to \infty \) and \( R_1 \to 0^+ \), we have
\[
|\Sigma| u(e) = -\int_{\mathbb{H}_n} \langle \nabla_H u, \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^{Q-1}} d\xi,
\]
where \( |\Sigma| = \int_\Sigma d\sigma \) (see [2] for an analogous representation formula associated with the Korányi–Folland nonisotropic gauge). We note the representation formula in Lemma 3.1 is also valid for \( f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z}) \). In fact, since \( \xi \in \mathbb{Z} \) iff \( \lambda \xi \in \mathbb{Z} \) for all \( \lambda > 0 \), one has, for all \( f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z}) \), as in the proof of Lemma 3.1,
\[
\int_{\Sigma \setminus (\Sigma \cap \mathbb{Z})} f(R_2 \xi^*) d\sigma - \int_{\Sigma \setminus (\Sigma \cap \mathbb{Z})} f(R_1 \xi^*) d\sigma
\]
\[
= \int_{(B_{cc}(e,R_2) \setminus \mathbb{Z}) \setminus (B_{cc}(e,R_1) \setminus \mathbb{Z})} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi,
\]
where \( R_2 > R_1 > 0 \). Note that \( \Sigma \cap \mathbb{Z} = \{(0,1/\pi), (0,-1/\pi)\} \). So
\[
\int_{\Sigma \setminus (\Sigma \cap \mathbb{Z})} f(R_k \xi^*) d\sigma = \int_{\Sigma} f(R_k \xi^*) d\sigma < \infty, \quad k = 1, 2,
\]
since \( f \in C(\mathbb{H}_n) \). These imply that
\[
\left| \int_{(B_{cc}(e,R_2) \setminus \mathbb{Z}) \setminus (B_{cc}(e,R_1) \setminus \mathbb{Z})} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi \right|
\]
\[
= \int_{\Sigma \setminus (\Sigma \cap \mathbb{Z})} f(R_2 \xi^*) d\sigma - \int_{\Sigma \setminus (\Sigma \cap \mathbb{Z})} f(R_1 \xi^*) d\sigma
\]
\[
= \int_{\Sigma} f(R_2 \xi^*) d\sigma - \int_{\Sigma} f(R_1 \xi^*) d\sigma < \infty.
\]
Therefore,
\[
\int_{(B_{cc}(e,R_2) \setminus \mathbb{Z}) \setminus (B_{cc}(e,R_1) \setminus \mathbb{Z})} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi = \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} \frac{\langle \nabla_H f, \nabla_H d_{cc} \rangle}{d_{cc}^{Q-1}} d\xi
\]
since \( \mathbb{Z} \) is a measure-zero set. This means that the representation formula in Lemma 3.1 is valid for \( f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus \mathbb{Z}) \).
Proof of Theorem 1.1 Let $\epsilon > 0$. Then $0 \leq u_\epsilon := (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p \in C^\infty_0(\mathbb{H}_n)$. In fact, $u_\epsilon$ has the same support as $u$. Putting $f = u_\epsilon d_{cce}^{-p}$ in Lemma 3.1 and letting $R_2 \to \infty$ and $R_1 \to 0^+$, we get, by Remark 3.3, since $d_{cc} \in C(\mathbb{H}_n) \cap C^\infty(\mathbb{H}_n \setminus Z)$ and $d_{cc}(e) = 0$,

$$
\int_{\mathbb{H}_n} \langle \nabla_H u_\epsilon, \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^{p-1}} + (Q - p) \int_{\mathbb{H}_n} \frac{u_\epsilon}{d_{cc}^p} = 0.
$$

Here we use the fact that $|\nabla_H d_{cc}(\xi)| = 1$ when $\xi \in \mathbb{H}_n \setminus Z$ (cf. [11]). By Hölder’s inequality:

$$(Q - p) \int_{\mathbb{H}_n} \frac{u_\epsilon}{d_{cc}^p} = -p \int_{\mathbb{H}_n} (|u|^2 + \epsilon^2)^{(p-2)/2} u \langle \nabla_H u, \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^{p-1}} \leq p \int_{\mathbb{H}_n} (|u|^2 + \epsilon^2)^{(p-2)/2} \frac{|u|}{d_{cc}^{p-1}} |\nabla_H u| \leq p \left( \int_{\mathbb{H}_n} |\nabla_H f|^p \right)^{p-1} \left( \int_{\mathbb{H}_n} |\nabla_H u|^p \right)^{\frac{1}{p}}.
$$

By dominated convergence, letting $\epsilon \to 0^+$, canceling and raising both sides to the power $p$, we get (1.3). The proof of Theorem 1.1 is complete. \(\square\)

Remark 3.4. Let $V^{1,p}(\mathbb{H}_n)$ denote the space

$$
V^{1,p}(\mathbb{H}_n) = \{ f \in C(\mathbb{H}_n) \cap C^1(\mathbb{H}_n \setminus Z) : \int_{\mathbb{H}_n} |u|^p d_{cc}^p < +\infty, \int_{\mathbb{H}_n} |\nabla_H f|^p < +\infty \}.
$$

In this remark, we shall show that the following Hardy inequalities hold for all $f \in V^{1,p}(\mathbb{H}_n)$:

$$
\int_{\mathbb{H}_n} |\nabla_H f|^p \geq \left( \frac{Q - p}{p} \right)^p \int_{\mathbb{H}_n} |f|^p d_{cc}^p.
$$

Furthermore, the constant $\left( \frac{Q - p}{p} \right)^p$ is sharp in the sense of

$$
\left( \frac{Q - p}{p} \right)^p = \inf_{f \in V^{1,p}(\mathbb{H}_n) \setminus \{0\}} \frac{\int_{\mathbb{H}_n} |\nabla_H f|^p}{\int_{\mathbb{H}_n} |f|^p d_{cc}^p}.
$$

Step 1. We shall show that if $f \in V^{1,p}(\mathbb{H}_n)$, then

$$
(3.5) \quad \lim_{R \to +\infty} R^{Q-2} \int_{\Sigma} |f(R\xi^*)|^p d\sigma = 0.
$$
We note from Lemma 3.1 that
\[
\left| R_2^{Q-p} \int_\Sigma |f(R_2 \xi^*)|^p d\sigma - R_1^{Q-p} \int_\Sigma |f(R_1 \xi^*)|^p d\sigma \right|
\]
\[
= \left| \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} \langle \nabla_H (|f|^p \frac{Q^{Q-p}}{d_{cc}}, \nabla_H d_{cc} \rangle \cdot \frac{1}{d_{cc}^{Q-p}} d\xi \right|
\]
\[
\leq p \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} \frac{|f|^{p-1} |\nabla_H f|}{d_{cc}} d\xi + (Q-p) \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} \frac{|f|^p}{d_{cc}} d\xi
\]
\[
\leq p \left( \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} \frac{|f|^p}{d_{cc}} d\xi \right)^{\frac{p-1}{p}} \left( \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} |\nabla_H f|^p \right)^{\frac{1}{p}}
\]
\[
+ (Q-p) \int_{B_{cc}(e,R_2) \setminus B_{cc}(e,R_1)} \frac{|f|^p}{d_{cc}} d\xi.
\]
Thus,
\[
\left| R_2^{Q-p} \int_\Sigma |f(R_2 \xi^*)|^p d\sigma - R_1^{Q-p} \int_\Sigma |f(R_1 \xi^*)|^p d\sigma \right| \to 0
\]
as \( R_1, R_2 \to +\infty \). This shows that \( R_2^{Q-p} \int_\Sigma |f(R \xi^*)|^p d\sigma \) converges to a finite constant \( f_\infty \geq 0 \) as \( R \to +\infty \). One must have \( f_\infty = 0 \) for
\[
\int_{\mathbb{H}_n} \frac{|f|^p}{d_{cc}} d\xi = \int_{0}^{\infty} \int_{\Sigma} |f(\lambda \xi^*)|^p \lambda^{Q-p-1} d\sigma d\lambda = \int_{0}^{\infty} \lambda^{Q-p} \int_\Sigma |f(\lambda \xi^*)|^p d\sigma d\lambda < +\infty.
\]
Therefore, (3.5) holds.

**Step 2.** Replace \( f \) by \( d_{cc}^{Q-p} |f|^p \) in Lemma 3.1 and let \( R_2 \to \infty \) and \( R_1 \to 0+ \). Using the result of Step 1 and following the proof of Theorem 1.1, one can obtain the Hardy inequalities as expected.

We now show that the constant \( \left( \frac{Q-p}{p} \right)^p \) is sharp. The proof is similar to that of Lemma 2.3 in [13]. Let
\[
f_\varepsilon(z,t) = \begin{cases} 1, & d_{cc} \leq 1; \\ \frac{1}{d_{cc}^{-(Q-p)/p+\varepsilon}}, & d_{cc} > 1. \end{cases}
\]
We note that this family of functions can be approximated by the functions in \( V^{1,p}(\mathbb{H}_n) \). In addition,
\[
\int_{\mathbb{H}_n} \frac{|f_\varepsilon|^p}{d_{cc}} d\xi = \frac{\|\Sigma\|}{Q-p} + \frac{\|\Sigma\|}{\varepsilon p}
\]
and
\[
\int_{\mathbb{H}_n} |\nabla_H f_\varepsilon|^p d\xi = \left( -\frac{Q-p}{p} - \varepsilon \right)^p \frac{\|\Sigma\|}{\varepsilon p}.
\]
Passing to the limit as \( \varepsilon \to 0 \), one can see that the constant \( \left( \frac{Q-p}{p} \right)^p \) is sharp. We note that since the Carnot-Carathéodory distance is not differentiable in \( \mathbb{Z} \), it is unknown whether \( f_\varepsilon \), the family of functions above, can be approximated by the functions in \( C_0^\infty(\mathbb{H}_n) \). So, we do not know whether the constant \( \left( \frac{Q-p-\alpha}{p} \right)^p \) is sharp for all \( f \in C_0^\infty(\mathbb{H}_n) \) in Theorem 1.1.
Remark 3.5. Let $\alpha < Q - p$. Put $f = u^p d_{cc}^{Q-p-\alpha}$ in Lemma 3.1 and let $R_2 \to \infty$ and $R_1 \to 0+$. We have

\[
(Q - p - \alpha) \int_{\mathbb{H}_n} \frac{u}{d_{cc}^{\alpha+\alpha}} = -p \int_{\mathbb{H}_n} (|u|^2 + \epsilon^2)^{(p-2)/2} u (\nabla_H u, \nabla d_{cc}) \cdot \frac{1}{d_{cc}^{\alpha+\alpha-1}} \leq p \int_{\mathbb{H}_n} (|u|^2 + \epsilon^2)^{(p-2)/2} |u| \cdot |\nabla_H u| \leq p \left( \int_{\mathbb{H}_n} (|u|^2 + \epsilon^2)^{(p-2)/(p-1)} |u|^\frac{p}{p-1} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{H}_n} |\nabla_H u|^p \right)^{\frac{1}{p}}.
\]

As in the proof of Theorem 1.1, we obtain the following Hardy-type inequalities with weights:

\[
\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left( \frac{Q - p - \alpha}{p} \right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d_{cc}^{p+\alpha}}, \quad \alpha < Q - p.
\]

Proof of Theorem 1.2 Notice that

\[-\int_{\mathbb{H}_n} u\Delta_H u \cdot \frac{1}{d_{cc}^2} = -\int_{\mathbb{H}_n} \Delta_H u^2 \cdot \frac{1}{d_{cc}^2} + \int_{\mathbb{H}_n} |\nabla_H u|^2.
\]

Through integration by parts, since $u \in C_0^\infty(\mathbb{H}_n \setminus \mathcal{Z})$ and $d_{cc}$ is $C^1$ in $\mathbb{H}_n \setminus \mathcal{Z}$ (cf. [11]),

\[-\frac{1}{2} \int_{\mathbb{H}_n} \Delta_H u^2 \cdot \frac{1}{d_{cc}^2} = \int_{\mathbb{H}_n} \langle \nabla_H u^2, \nabla d_{cc} \rangle \cdot \frac{1}{d_{cc}^3} d\xi.
\]

Putting $f = u^2 d_{cc}^{Q-4}$ in Lemma 3.1 and letting $R_2 \to \infty$ and $R_1 \to 0+$, we have

\[-\int_{\mathbb{H}_n} \langle \nabla_H u^2, \nabla d_{cc} \rangle \cdot \frac{1}{d_{cc}^3} d\xi = (Q - 4) \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} d\xi.
\]

Thus, for all $\varepsilon > 0$,

\[
-\int_{\mathbb{H}_n} u\Delta_H u \cdot \frac{1}{d_{cc}^2} = \int_{\mathbb{H}_n} |\nabla_H u|^2 \left( \int_{\mathbb{H}_n} \frac{|u|^2}{d_{cc}^4} \right)^\frac{1}{2} \leq \left( \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} \right)^\frac{1}{2} \left( \int_{\mathbb{H}_n} |\Delta_H u|^2 \right)^\frac{1}{2}
\]

(3.7)

\[
\leq \varepsilon \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} + \frac{1}{2\varepsilon} \int_{\mathbb{H}_n} |\Delta_H u|^2.
\]

Choosing $\varepsilon = \frac{Q(Q-4)}{4}$ in (3.7), we have

\[
\int_{\mathbb{H}_n} |\Delta_H u|^2 + \frac{Q(Q - 4)^2(Q - 8)}{16} \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4} \geq \frac{Q(Q - 4)}{2} \int_{\mathbb{H}_n} |\nabla_H u|^2.
\]

Since (see 3.6)

\[
\int_{\mathbb{H}_n} |\nabla_H u|^2 \geq \frac{(Q - 4)^2}{4} \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4}, \quad n \geq 2,
\]

we obtain, by (3.8),

\[
\int_{\mathbb{H}_n} |\Delta_H u|^2 \geq \left( \frac{Q(Q - 4)}{4} \right)^2 \int_{\mathbb{H}_n} \frac{u^2}{d_{cc}^4}.
\]
When $Q \geq 8$, i.e. $n \geq 3$, again using (3.7) and (3.8), we obtain
\[
\frac{Q(Q - 4)}{2} \int_{\mathbb{H}_n} \frac{1}{d_{cc}^2} |\nabla u|^2 \leq \int_{\mathbb{H}_n} |\Delta u|^2 + \frac{Q(Q - 8)}{4} \int_{\mathbb{H}_n} \frac{1}{d_{cc}^2} |\nabla u|^2.
\]

The desired result follows. □

Proof of Theorem 1.3 As in the proof of Theorem 1.1, putting $f = \frac{u_{\epsilon} d_{cc}^{Q-1}}{(\rho - d_{cc})^{p-1}}$ in Lemma 3.1 and letting $R_2 = \rho$ and $R_1 \to 0+$, we get
\[
(Q - 1) \int_{B_{cc}(e, \rho)} \frac{u_{\epsilon}}{d_{cc}^{p-1}} + (p - 1) \int_{B_{cc}(e, \rho)} \frac{u_{\epsilon}}{(\rho - d_{cc})^{p-1}}
= -p \int_{B_{cc}(e, \rho)} (|u|^2 + \epsilon^2)^{(p-2)/2} u \langle \nabla u, \nabla d_{cc} \rangle \cdot \frac{1}{(\rho - d_{cc})^{p-1}}
\leq p \int_{B_{cc}(e, \rho)} \frac{(|u|^2 + \epsilon^2)^{(p-2)/2} |u|}{(\rho - d_{cc})^{p-1}}
\leq p \left( \int_{B_{cc}(e, \rho)} \frac{(|u|^2 + \epsilon^2)^{(p-2)/2} |u|}{(\rho - d_{cc})^{p-1}} \right)^{p-1} \left( \int_{B_{cc}(e, \rho)} |\nabla u|^p \right)^{\frac{1}{p}}.
\]

Letting $\epsilon \to 0+$, we have, by dominated convergence,
\[
\left( \int_{B_{cc}(e, \rho)} \frac{|u|^p}{(\rho - d_{cc})^{p}} \right)^{\frac{p-1}{p}} \left( \int_{B_{cc}(e, \rho)} |\nabla u|^p \right)^{\frac{1}{p}}
\leq \frac{p - 1}{p} \int_{B_{cc}(e, \rho)} \frac{|u|^p}{(\rho - d_{cc})^{p}}
+ \frac{Q - 1}{p} \int_{B_{cc}(e, \rho)} \frac{|u|^p}{d_{cc}(\rho - d_{cc})^{p-1}}
\geq \frac{p - 1}{p} \int_{B_{cc}(e, \rho)} \frac{|u|^p}{(\rho - d_{cc})^{p}}.
\]

Canceling and raising both sides to the power $p$, we get (1.6). □

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References


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