

FAMILIES OF QUASIMODULAR FORMS AND JACOBI FORMS: THE CRANK STATISTIC FOR PARTITIONS

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ABSTRACT. Families of quasimodular forms arise naturally in many situations, such as curve counting on Abelian surfaces and counting ramified covers of orbifolds. In many cases the family of quasimodular forms naturally arises as the coefficients of a Taylor expansion of a Jacobi form. In this paper we give examples of such expansions that arise in the study of partition statistics.

The crank partition statistic has gathered much interest recently. For instance, Atkin and Garvan showed that the generating functions for the moments of the crank statistic are quasimodular forms. The two-variable generating function for the crank partition statistic is a Jacobi form. Exploiting the structure inherent in the Jacobi theta function, we construct explicit expressions for the functions of Atkin and Garvan. Furthermore, this perspective opens the door for further investigation, including a study of the moments in arithmetic progressions. We conduct a thorough study of the crank statistic restricted to a residue class modulo 2.

1. INTRODUCTION

A well-known and useful fact is that the graded ring of holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$, denoted $M_*(\mathrm{SL}_2(\mathbb{Z}))$, is generated by the weight 4 and weight 6 Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The weight k Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$, for $k \geq 2$ even, is given by

$$(1.1) \quad E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

with $q := e^{2\pi i\tau}$, $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$, and B_k the k th Bernoulli number.

On the other hand, $E_2(\tau)$ is not a modular form, but a quasimodular form. Quasimodular forms are elements of the smallest algebra which contain the classical modular forms and which are closed under differentiation $\frac{1}{2\pi i} \frac{d}{d\tau} = \frac{1}{q} \frac{d}{dq}$.

Families of quasimodular forms arise naturally throughout mathematics. The following are three such examples:

- (i) Dijkgraaf [12] and Kaneko and Zagier [18] proved that for each $g > 1$ the generating function for the number of genus g and degree d covers of an elliptic curve with prescribed ramification is a quasimodular form.

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The family of generating functions indexed by the genus g is a family of quasimodular forms.

- (ii) For a lattice $L \subset \mathbb{C}$, let $\mathbb{T}^2 = \mathbb{C}/L$. Eskin and Okounkov proved that there is a two-parameter family of quasimodular forms counting ramified covers of the pillowcase orbifold $\mathbb{T}^2/\pm 1$ formed by taking the quotient by the automorphism $z \mapsto -z$. The family of quasimodular forms is indexed by a partition μ and a partition of an even number into odd parts ν . The generating function for the number of degree d covers with ramification data determined by μ and ν , $Z(\mu, \nu; q)$, is a quasimodular form.
- (iii) Andrews and Rose [4] proved there is a one-parameter family of quasimodular forms arising from a curve counting problem on Abelian surfaces. They show that for each genus g , the generating function for the number of hyperplane sections which are hyperelliptic curves of genus g with δ nodes on a generic polarized abelian surface is a quasimodular form.

For each of these families there is a Jacobi form that may be viewed as the generating function for the family of quasimodular forms. Precisely, each of these families arise as the coefficients of a Taylor expansion of a Jacobi form.

Jacobi forms may be understood as two variable automorphic forms satisfying an elliptic transformation and a modular transformation. In the foundational text on Jacobi forms, Eichler and Zagier show that a suitable “correction” to the Taylor coefficients with respect to the elliptic variable of a Jacobi form are modular forms (see Theorem 3.1 of [14]). In Eskin and Okounkov’s work [15], the Taylor expansion with respect to the elliptic variable yields arithmetically interesting generating functions. In both of these cases, the uncorrected Taylor coefficients are of interest. The uncorrected coefficients are, generally, quasimodular forms of mixed weight rather than modular forms of a fixed weight.

In this paper we give examples of Taylor expansions that arise in the study of partition statistics. In our context (see Section 2) the uncorrected Taylor coefficients arise. Our focus is to produce explicit expressions for the generating functions of the moments of these statistics and the moments restricted to arithmetic progressions. As a consequence of those results we deduce congruences for the coefficients of the moment generating functions. Furthermore, we exploit the structure of the full Jacobi form to give asymptotics for the moments themselves.

We give an explicit expression for the moment generating functions in Section 2. In Section 3 we give explicit results from which one may easily obtain such asymptotic and congruence results for the moments of the crank statistic restricted to arithmetic progressions modulo 2.

2. MOMENTS OF THE CRANK PARTITION STATISTIC

Dyson [13] conjectured the existence of a statistic, the “crank”, that would provide a combinatorial explanation of Ramanujan’s congruences for the partition function modulo 5, 7, and 11. Garvan [16] found the crank statistic and, together with Andrews [2], presented the following definition. Let $o(\lambda)$ denote the number of ones in λ , and $\mu(\lambda)$ denote the number of parts strictly larger than $o(\lambda)$. The crank of λ is defined by

$$(2.1) \quad \text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Let $\mathcal{M}(m, n)$ be the number of partitions of n with crank m . The two-parameter generating function may be written as [2, 6]

$$C(x; q) := \sum_{\substack{m \in \mathbb{Z} \\ n \geq 0}} \mathcal{M}(m, n) x^m q^n = \prod_{n \geq 1} \frac{1 - q^n}{(1 - xq^n)(1 - x^{-1}q^n)}.$$

The study of the crank, and other partition statistics, has led to a better understanding of the partition function. For instance, Mahlburg [20] showed that when x is specialized to be a root of unity, $C(x; q)$ is, up to a power of q , a modular form. He then used the theory of modular forms to establish congruences for the crank statistic, resulting in infinitely many combinatorial congruences for the partition function.

Atkin and Garvan [5] studied the distribution of the crank statistic by considering its moments. For a nonnegative integer k , define the k th crank moment as

$$(2.2) \quad M_k(n) := \sum_{m \in \mathbb{Z}} m^k \mathcal{M}(m, n).$$

They showed that for odd k $M_k(n) = 0$ and for each $\ell \in \mathbb{N}$ the generating function

$$(2.3) \quad C_{2\ell}(q) := \sum_{n \geq 0} M_{2\ell}(n) q^n$$

is a quasimodular form. Atkin and Garvan construct this family of quasimodular forms recursively.

We show that this family of generating functions naturally appears as the Taylor coefficients of a Jacobi form. Let $x := e^{2\pi i u}$ and $q := e^{2\pi i \tau}$. Abusing notation, $q^{-\frac{1}{24}} \sin(\pi u)^{-1} C(u; \tau) = q^{-\frac{1}{24}} \sin(\pi u)^{-1} C(x; q)$ is a weight $\frac{1}{2}$ index $-\frac{1}{2}$ Jacobi form on $\mathrm{SL}_2(\mathbb{Z})$ with multiplier. For Z a complex number let

$$(2.4) \quad \mathcal{C}(Z; q) := \sum_{k=0}^{\infty} C_k(q) \frac{Z^k}{k!}$$

be the exponential generating functions for the crank moment generating functions. Rearranging the order of summation, we see that

$$\mathcal{C}(Z; q) = \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} \mathcal{M}(m, n) q^n \sum_{k \geq 0} \frac{(Zm)^k}{k!} = \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} \mathcal{M}(m, n) (e^Z)^m q^n = C(e^Z; q).$$

Hence the Taylor expansion with respect to u of the Jacobi form $C(u; \tau)$ encodes the crank moments.

In this paper, we focus on concrete instances and describe the Taylor coefficients, and hence moment generating functions, explicitly. Define

$$(2.5) \quad \Phi_{k-1}(\tau) := \frac{B_k}{2k} (1 - E_k(\tau)) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Theorem 2.1. *For $\ell \geq 1$ we have*

$$C_{2\ell}(q) = \frac{1}{(q)_{\infty}} \sum_{1 \leq k \leq \ell} \sum_{\substack{i_1 + \dots + i_k = \ell \\ i_1, i_2, \dots, i_k > 0}} \frac{2^k (2\ell)!}{k! (2i_1)! \dots (2i_k)!} \Phi_{2i_1-1}(\tau) \Phi_{2i_2-1}(\tau) \dots \Phi_{2i_k-1}(\tau),$$

with $(q)_{\infty} := (q; q)_{\infty}$ and $(x; q)_{\infty} := \prod_{n \geq 0} (1 - xq^n)$. Hence $(q)_{\infty} C_{2\ell}(q)$ is a mixed weight quasimodular form on $\mathrm{SL}_2(\mathbb{Z})$ with highest weight 2ℓ .

Remark. Atkin and Garvan [5] used a recurrence relation to deduce Theorem 2.1. However, the constants appearing in Theorem 2.1 are not given.

3. MOMENTS IN ARITHMETIC PROGRESSIONS

This perspective opens the door for further exploration of the distribution of the crank. For example, we may investigate the crank moments in arithmetic progressions, such as

$$\sum_{m \equiv A \pmod{B}} m^k \mathcal{M}(m, n)$$

for any nonnegative integers A and B . The case $k = 0$ was considered by Mahlburg [20]. The cases for $k = 0$ and $B = 2, 3, 4$ have been studied explicitly by a number of authors, namely Andrews and Lewis [3] and Choi, Kang, and Lovejoy [11]. To understand these moments it is enough to have an explicit description of the twisted crank moments

$$(3.1) \quad M_k(\zeta, n) := \sum_{m \in \mathbb{Z}} \zeta^m m^k \mathcal{M}(m, n),$$

with ζ a root of unity. Define the twisted crank moment generating function by

$$(3.2) \quad C_k(\zeta, q) := \sum_{n \geq 0} M_k(\zeta, n) q^n.$$

For concreteness, we consider the case when $\zeta = -1$ and give the following explicit description of these moment generating functions.

Theorem 3.1. *For $k \geq 1$ define*

$$(3.3) \quad F_{2k}(\tau) := 2^{2k} \Phi_{2k-1}(2\tau) - \Phi_{2k-1}(\tau).$$

For $\ell \geq 1$ we have

$$C_{2\ell}(-1, q) = (q)_\infty (-q)_\infty^2 \sum_{\substack{1 \leq k \leq \ell \\ i_1, i_2, \dots, i_k > 0}} \sum_{i_1 + \dots + i_k = \ell} \frac{2^k (2\ell)!}{k! (2i_1)! \dots (2i_k)!} F_{2i_1}(\tau) F_{2i_2}(\tau) \cdots F_{2i_k}(\tau).$$

Hence $\frac{C_{2\ell}(-1, q)}{(q)_\infty (-q)_\infty^2}$ is a mixed weight quasimodular form on $\Gamma_0(2)$ with highest weight 2ℓ .

We give two applications of this perspective. First, since quasimodular forms are p -adic modular forms in the sense of Serre, for instance $E_2(\tau) \equiv E_{p+1}(\tau) \pmod{p}$, it is routine to deduce infinitely many congruences for the moments (for instance, see [21]).

Corollary 3.2. *For every ℓ there exist infinitely many primes p and infinitely many nonnested arithmetic progressions $\{An + B : n \in \mathbb{N}\}$ such that*

$$M_{2\ell}(-1, An + B) \equiv 0 \pmod{p}$$

for all $n \in \mathbb{N}$.

Second, we give precise asymptotics for the twisted crank moments $M_{2\ell}(-1, n)$ as $n \rightarrow \infty$. The asymptotics of $C_{2\ell}(\tau)$ as τ approaches rational numbers yields asymptotic information about $M_{2\ell}(-1, n)$. Rather than computing the asymptotics after Taylor expanding the Jacobi form, we compute the asymptotics and then Taylor expand. This allows a unified approach toward understanding the asymptotics of $M_{2\ell}(-1, n)$ which would otherwise be out of reach. This idea was first used by

the author with Bringmann and Mahlburg [9] to produce precise asymptotics for the moments of the crank and rank partition statistics.

Define

$$(3.4) \quad \alpha(a, b, c) := \frac{(2(a+b+c))!(-1)^{b+c}}{(2a)!(2b)!c!4^{a+b+c}} E_{2b},$$

where E_{2b} is the Euler number given by $\sum_n E_n \frac{x^n}{n!} = \frac{1}{\cosh(x)}$, and for $k \geq 1$ define the Kloosterman sum

$$(3.5) \quad A_k(n) := \sqrt{\frac{k}{24}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{x}\right) e\left(\frac{x}{12k}\right),$$

where the sum runs over residue classes modulo $24k$.

Theorem 3.3. *As $n \rightarrow \infty$ we have*

$$M_{2\ell}(-1, n) = \pi \sum_{1 \leq k \leq \sqrt{n}/2} \frac{(-1)^{k+\lfloor \frac{k+1}{2} \rfloor} A_{2k} \left(n - \frac{k(1+(-1)^k)}{4} \right)}{k} \sum_{a+b+c=\ell} (2k)^c \alpha(a, b, c) \\ \times (24n-1)^{b+\frac{\epsilon}{2}-\frac{1}{4}} I_{\frac{1}{2}-2b-c} \left(\frac{\pi\sqrt{24n-1}}{12k} \right) + O(n^{2\ell+\epsilon}),$$

where I_ν is the modified Bessel function of order ν .

This theorem and the fact that as $y \rightarrow \infty$ we have

$$I_\nu(y) \sim \frac{1}{\sqrt{2\pi y}} e^y$$

readily give the leading order asymptotics for the twisted crank moment.

Corollary 3.4. *In the notation above,*

$$M_{2\ell}(-1, n) \sim (-1)^n |E_{2\ell}| 2^{\ell-1} 3^\ell n^{\ell-\frac{1}{2}} e^{\pi\sqrt{\frac{n}{6}}}.$$

This should be compared with

$$M_{2\ell}(n) \sim 2^{3\ell-2} 3^{\ell-\frac{1}{2}} (1-2^{1-2\ell}) |B_{2\ell}| \cdot n^{\ell-1} e^{\pi\sqrt{\frac{2n}{3}}},$$

which was first obtained by Bringmann, Mahlburg, and the author in [8]. Hence, the twisted moment is of exponentially smaller order of magnitude. Also, we see that as $n \rightarrow \infty$ the sign of $M_{2\ell}(-1, n)$ depends on the parity of n , which suggests that this is true for all n . Andrews and Lewis [3] proved that this is the case for $\ell = 0$, namely $(-1)^n M_0(-1, n) > 0$.

Section 4 contains some preliminary results for the Jacobi theta function. Section 5 uses these results to establish Theorems 2.1 and 3.1. The final section gives some discussion of the rank partition statistic.

4. JACOBI'S THETA FUNCTION

In this section we prove some results for Jacobi's theta function

$$(4.1) \quad \vartheta(u; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} e^{\pi i \nu^2 \tau + 2\pi i \nu u} = -2 \sin(\pi u) q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1-xq^n)(1-x^{-1}q^n),$$

where the second equality is Jacobi's triple product identity. The Dedekind eta function is

$$\eta(\tau) := \sum_{n \in \mathbb{Z}} \left(\frac{12}{n} \right) q^{n^2/24} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Proposition 4.1. *With $Z = 2\pi i u$ and F_ℓ as defined in (3.3) for even ℓ and 0 otherwise, we have*

$$\vartheta(u; \tau) = -2 \sin(\pi u) \eta^3(\tau) \exp \left(-2 \sum_{\substack{\ell \text{ even} \\ \ell > 0}} \frac{Z^\ell}{\ell!} \Phi_{\ell-1}(\tau) \right)$$

and

$$\vartheta \left(u + \frac{1}{2}; \tau \right) = -2 \cos(\pi u) \frac{\eta(2\tau)^2}{\eta(\tau)} \exp \left(-2 \sum_{\substack{\ell > 0 \\ \ell \text{ even}}} \frac{Z^\ell}{\ell!} F_\ell(\tau) \right).$$

Proof. We have

$$\begin{aligned} & \log \left(-\frac{\vartheta(u; \tau)}{2 \sin(\pi u) \eta^3(\tau)} \right) \\ &= \log \left(\prod_{n=1}^{\infty} (1 - xq^n)(1 - x^{-1}q^n)(1 - q^n)^{-2} \right) \\ &= -\sum_{n \geq 1} \sum_{r \geq 1} (x^r + x^{-r} - 2) \frac{1}{r} q^{nr} = -2 \sum_{\ell \text{ even}, \ell > 0} \frac{Z^\ell}{\ell!} \sum_{n, r \geq 1} r^{\ell-1} q^{nr} \\ &= -2 \sum_{\ell \text{ even}, \ell > 0} \frac{Z^\ell}{\ell!} \Phi_{\ell-1}(\tau) \end{aligned}$$

Note that $\vartheta(\frac{1}{2}; \tau) = -2q^{\frac{1}{8}} \prod_{n \geq 1} (1 - q^n)(1 + q^n)^2 = -2 \frac{\eta(2\tau)^2}{\eta(\tau)}$. The second result follows from a calculation similar to the previous one and

$$\begin{aligned} \sum_{n \geq 1} \sum_{r \geq 1} (-1)^r r^{\ell-1} q^{nr} &= 2 \sum_{n \geq 1} \sum_{r \geq 1, r \text{ even}} r^{\ell-1} q^{nr} - \sum_{n \geq 1} \sum_{r \geq 1} r^{\ell-1} q^{nr} \\ &= 2^\ell \sum_{n, r \geq 1} r^{\ell-1} q^{2nr} - \sum_{n, r \geq 1} r^{\ell-1} q^{nr} = 2^\ell \Phi_{\ell-1}(2\tau) - \Phi_{\ell-1}(\tau). \end{aligned}$$

□

5. THE CRANK STATISTIC

By the definition of $C(u; \tau)$ and the triple product formula,

$$(5.1) \quad C(u; \tau) = -\frac{2 \sin(\pi u) q^{\frac{1}{24}} \eta^2(\tau)}{\vartheta(u; \tau)}.$$

The following lemma is useful.

Lemma 5.1. *As a formal power series, let $\sum_{r \geq 0} \frac{Z^r}{r!} c_r = \exp \left(\sum_{\ell > 0} \frac{Z^\ell}{\ell!} a_\ell \right)$. Then*

$$c_r = \sum_{0 \leq s \leq r} \frac{r!}{s!} \sum_{\substack{i_1 + \dots + i_s = r \\ i_j > 0}} \frac{1}{i_1! i_2! \dots i_s!} a_{i_1} a_{i_2} \dots a_{i_s}.$$

Proof of Theorem 2.1. Applying Proposition 4.1 and Lemma 5.1 to (5.1) yields the result. \square

Proof of Theorem 3.1. We have the following Taylor expansion:

$$\mathcal{C}(Z + \pi i; q) = \sum_{\ell \geq 0} \frac{Z^\ell}{\ell!} \sum_{n \geq 1} \left(\sum_m (-1)^m m^\ell M(m, n) \right) q^n.$$

Note that by symmetry $\sum_m (-1)^m m^\ell M(m, n)$ is zero unless ℓ is even. Applying Proposition 4.1 and Lemma 5.1, we readily deduce the explicit formula for the twisted moment generating function.

By 3.3.8 of [15] the algebra of quasimodular forms on $\Gamma_0(2)$ is generated by $E_2(\tau)$, $E_2(2\tau)$, and $E_4(2\tau)$. Thus $(q)_\infty^{-1}(-q)_\infty^{-2}C_{2\ell}(-1, q)$ is a quasimodular form on $\Gamma_0(2)$. \square

5.1. Crank asymptotics. In this subsection we use the method from [9] to prove Theorem 3.3. We have

$$\mathcal{C}(2\pi i u + \pi i; q) = \frac{2 \cos(\pi u) e^{\frac{i\pi\tau}{12}} \eta^2(\tau)}{\vartheta(u + \frac{1}{2}; \tau)}.$$

To state the modular transformation properties of the twisted crank generating functions we denote the inverse of a modulo b by $[a]_b$. When $2 \mid b$ we may assume $[a]_b = [a]_{2b}$. We do this implicitly in what follows.

Proposition 5.2. *For $2 \mid k$ we have*

$$\begin{aligned} & C_{2\ell} \left(-1, e^{\frac{2\pi i}{k}(h+iz)} \right) \\ &= i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h-[-h]_k)} \chi^{-1}(h, [-h]_k, k) \cdot (-1)^{\frac{k}{2}} (-1)^{\frac{k+1-h-[-h]_k}{2}} e^{2\pi i \frac{-h-\frac{k}{2}[-h]_k}{4}} \\ & \quad \times e^{\frac{\pi}{12k}(\frac{1}{z}-z)} \sum_{a+b+c=\ell} k^c \alpha(a, b, c) z^{-\frac{1}{2}-2b-c} + O \left(z^{-\frac{1}{2}-2\ell} e^{-\frac{\pi}{k} \operatorname{Re}(\frac{1}{z})} \right) \end{aligned}$$

for some $\alpha > 0$ independent of k , and $\chi^{-1}(h, [-h]_k, k)$ is a multiplier system defined in (5.4). Furthermore, for $2 \nmid k$ we have

$$C_{2\ell} \left(-1, e^{\frac{2\pi i}{k}(h+iz)} \right) \ll z^{-2\ell-\frac{1}{2}} e^{-\frac{\pi}{6k} \operatorname{Re}(\frac{1}{z})}.$$

Lemma 5.3. *For appropriate a and b and $\ell \in \mathbb{N}$ we have*

$$\vartheta(a + \ell b; b) = (-1)^\ell e^{-\pi i \ell^2 b - 2\pi i \ell a} \vartheta(a; b).$$

Proof. The proof follows from induction on $\vartheta(a + b; b) = -e^{-\pi i b - 2\pi i a} \vartheta(a; b)$. \square

Proof of Proposition 5.2. We use the modular transformations for the Dedekind η -function and the Jacobi ϑ -function, namely

$$(5.2) \quad \eta \left(\frac{1}{k}(h + iz) \right) = \sqrt{\frac{i}{z}} \chi(h, [-h]_k, k) \eta \left(\frac{1}{k} \left([-h]_k + \frac{i}{z} \right) \right)$$

$$(5.3) \quad \vartheta \left(u; \frac{1}{k}(h + iz) \right) = \sqrt{\frac{i}{z}} \chi^3(h, [-h]_k, k) e^{-\frac{\pi k u^2}{z}} \vartheta \left(\frac{i u}{z}; \frac{1}{k} \left([-h]_k + \frac{i}{z} \right) \right),$$

where

$$(5.4) \quad \chi(h, [-h]_k, k) = i^{-\frac{1}{2}} \omega_{h,k}^{-1} e^{-\frac{\pi i}{12k}([-h]_k - h)}$$

and $[-h]_k$ is the multiplicative inverse of $-h$ modulo k and $\omega_{h,k} := \exp(\pi i s(h, k))$. Here, $s(h, k)$ is the usual Dedekind sum. See [19] and [9] and the references therein for more on these transformation formulae. Hence

$$(5.5) \quad \begin{aligned} & \mathcal{C}(2\pi i u + \pi i; e^{\frac{2\pi i}{k}(h+iz)}) \\ &= 2 \cos(\pi u) \chi^{-1}(h, [-h]_k, k) \sqrt{\frac{i}{z}} \frac{\eta^2\left(\frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right)^2 e^{\frac{\pi k(u+\frac{1}{2})^2}{z}}}{\vartheta\left(\frac{i(u+1/2)}{z}; \frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right)} e^{\frac{\pi i}{12k}(h+iz)}. \end{aligned}$$

Set $\tau = \frac{1}{k}\left([-h]_k + \frac{i}{z}\right)$ and write

$$\vartheta\left(\frac{i(u+1/2)}{z}; \tau\right) = \vartheta\left(\frac{i u}{z} - \frac{[-h]_k}{2} + \frac{k}{2}\tau; \tau\right).$$

In the case $2 \mid k$ set $\ell = \frac{k}{2}$. Applying Lemma 5.3 we have

$$\vartheta\left(\frac{i u}{z} - \frac{[-h]_k}{2} + \ell\tau; \tau\right) = (-1)^\ell e^{-\pi i \ell^2 \tau - 2\pi i \ell\left(\frac{i u}{z} - \frac{[-h]_k}{2}\right)} \vartheta\left(\frac{i u}{z} - \frac{[-h]_k}{2}; \tau\right).$$

To calculate the asymptotic we use the triple product identity and abuse notation by setting $q = e^{2\pi i \tau}$ and $x = e^{-\frac{2\pi u}{z} - \pi i[-h]_k}$. Then we have

$$\begin{aligned} & \vartheta\left(\frac{i u}{z} - \frac{[-h]_k}{2}; \tau\right)^{-1} \\ &= -\frac{q^{-\frac{1}{8}}}{2 \sin\left(\frac{i u \pi}{z} - \frac{\pi[-h]_k}{2}\right)} \prod_{n \geq 1} (1 - q^n)^{-1} (1 - x q^n)^{-1} (1 - x^{-1} q^n)^{-1} \\ &= -\frac{q^{-\frac{1}{8}}}{2 \left(\frac{[-h]_k}{4}\right) \cosh\left(\frac{u \pi}{z}\right)} + \sum_{r \geq 0} a_r(z) \frac{u^r}{r!}, \end{aligned}$$

with $a_r(z) \ll |z|^{-r} e^{-\frac{7\pi}{4k} \operatorname{Re}\left(\frac{1}{z}\right)}$ and $(:)$ the Kronecker symbol.

Combining this with (5.5) we have

$$\begin{aligned} & \mathcal{C}\left(2\pi i u + \pi i; e^{\frac{2\pi i}{k}(h+iz)}\right) \\ &= -\frac{\cos(\pi u)}{\cosh\left(\frac{\pi u}{z}\right)} e^{-\frac{\pi i}{4k}\left([-h]_k + \frac{i}{z}\right)} \sqrt{\frac{i}{z}} e^{\frac{\pi i}{12k}(h+iz)} e^{\frac{\pi i}{6k}\left([-h]_k + \frac{i}{z}\right)} \chi^{-1}(h, [-h]_k, k) \\ & \quad \times e^{\frac{\pi k}{z}\left(u^2 + u + \frac{1}{4}\right)} (-1)^{\frac{k}{2}} e^{\frac{\pi i}{k}\left(\frac{k}{2}\right)^2\left([-h]_k + \frac{i}{z}\right)} + \pi i k \left(\frac{i u}{z} - \frac{[-h]_k}{2}\right) \left(\frac{[-h]_k}{2}\right) + \sum_{r=0}^{\infty} a_r(z) \frac{u^r}{r!} \\ &= -i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h-[-h]_k)} \chi^{-1}(h, [-h]_k, k) (-1)^{\frac{k}{2}} i^{-1} \left(\frac{-h}{4}\right) e^{-\frac{\pi i}{4} k[-h]_k} z^{-\frac{1}{2}} e^{\frac{\pi i}{12k}\left(\frac{1}{z} - z\right)} \\ & \quad \times \frac{\cos(\pi u) e^{\frac{\pi k u^2}{z}}}{\cosh\left(\frac{\pi u}{z}\right)} + \sum_{r=0}^{\infty} a_r(z) \frac{u^r}{r!}, \end{aligned}$$

where we have used the fact that $-h \equiv [-h]_k \pmod{4}$ with

$$a_r(z) \ll z^{-\frac{1}{2}-r} e^{-\frac{\alpha}{k} \operatorname{Re}\left(\frac{1}{z}\right)}$$

for some $\alpha > 0$. For later reference, using $1 - h[-h]_k \equiv 0 \pmod{2}$, we note that

$$(5.6) \quad (-1)^{\frac{k}{2}} i^{-1} \left(\frac{-h}{4}\right) e^{-\frac{\pi i}{4} k[-h]_k} = (-1)^{\frac{k+1-h[-h]_k}{2}} e^{2\pi i \frac{-h-\frac{k}{2}[-h]_k}{4}}.$$

Taylor expanding those functions depending on u , namely,

$$\cos(\pi u) = \sum_{a \geq 0} \frac{(2\pi i u)^{2a}}{(2a)! 2^{2a}} \quad \text{and} \quad \frac{1}{\cosh\left(\frac{\pi u}{z}\right)} = \sum_{b \geq 0} E_{2b} \frac{(2\pi i u)^{2b} (-1)^b}{z^{2b} (2b)! 2^{2b}},$$

where E_{2b} is the Euler number, we have

$$\frac{\cos(\pi u)}{\cosh\left(\frac{\pi u}{z}\right)} e^{\frac{\pi k u^2}{z}} = \sum_{\ell \geq 0} \frac{(2\pi i u)^{2\ell}}{(2\ell)!} \sum_{a+b+c=\ell} k^c \alpha(a, b, c) z^{-2b-c}.$$

Thus, we finish the proof in the case $2 \mid k$.

In the case when $2 \nmid k$ we write $k = 2\ell + 1$ with $\ell \geq 0$. Then

$$\vartheta\left(\frac{i u}{z} + \frac{i}{2z}; \tau\right) = (-1)^\ell e^{-\pi i(\ell^2 + \ell)\tau - 2\pi i \ell \left(\frac{i u}{z} - \frac{[-h]_k}{2}\right)} \vartheta\left(\frac{i u}{z} - \frac{[-h]_k}{2} + \frac{\tau}{2}; \tau\right).$$

Again, with $x = e^{2\pi i \left(\frac{i u}{z} - \frac{[-h]_k}{2}\right)}$ and $q = e^{2\pi i \tau}$,

$$\begin{aligned} & \vartheta\left(\frac{i u}{z} + \frac{i}{2z}; \tau\right)^{-1} \\ &= (-1)^{\ell+1} q^{\frac{\ell^2 + \ell}{2} + \frac{1}{8}} x^{\frac{2\ell+1}{2}} \prod_{n=1}^{\infty} (1 - q^n)^{-1} (1 - x q^{n-\frac{1}{2}})^{-1} (1 - x^{-1} q^{n-\frac{1}{2}})^{-1}. \end{aligned}$$

Combining this with (5.5) and Taylor expanding in u , as above, we obtain the result. \square

We will apply the circle method to obtain the asymptotic in Theorem 3.3. In the circle method one first computes the asymptotics of the function toward each of its singularities. These asymptotics are then combined to produce the asymptotics of the Fourier coefficients of the function. In our case we have a collection of functions indexed by an integer ℓ . The idea of the proof is to first compute the asymptotics for the generating function of the moment generating functions and then read off the asymptotics for each individual moment generating function. This is possible because the generating function of the moment generating functions is a Jacobi form. This idea was first used in [9].

We follow [9] closely and give the following general result, which yields the proof of Theorem 3.3. Assume that

$$F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right) = \sum_n c_{r,\ell}(n) e^{\frac{2\pi i n}{k}(h+iz)}$$

is a holomorphic function of z satisfying

$$\begin{aligned} F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right) &= -i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h - [-h]_k)} \chi^{-1}(h, [-h]_k, k) \cdot (-1)^{\frac{k+1-h-[-h]_k}{2}} e^{2\pi i \left(-h - \frac{k}{4} \frac{[-h]_k}{k}\right)} \\ &\quad \times e^{\frac{\pi}{12k}(\frac{1}{z} - z)} \sum_{a+b+c=\ell} (rk)^c \alpha(a, b, c) z^{-\frac{1}{2} - 2b - c} + E_{r,\ell,k}(z)(z), \end{aligned}$$

and for $2 \mid k$

$$(5.7) \quad F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right) \ll E_{r,\ell,k}(z)$$

with $E_{r,\ell,k}(z) \ll_{r,\ell} z^{-\frac{1}{2} - 2\ell} e^{-\frac{\beta}{k} \operatorname{Re}\left(\frac{1}{z}\right)}$ for some $\beta > 0$ with $\alpha(a, b, c)$ defined as in (3.4).

Theorem 5.4. *With $F_{r,\ell}$ and $c_{r,\ell}$ as above we have*

$$c_{r,\ell}(n) = \pi \sum_{1 \leq k \leq \sqrt{n}/2} \frac{(-1)^{k+\lfloor \frac{k+1}{2} \rfloor} A_{2k} \left(n - \frac{k(1+(-1)^k)}{4} \right)}{k} \sum_{a+b+c=\ell} (2kr)^c \alpha(a, b, c) \\ \times (24n-1)^{b+\frac{c}{2}-\frac{1}{4}} I_{\frac{1}{2}-2b-c} \left(\frac{\pi\sqrt{24n-1}}{12k} \right) + O(n^{2\ell+\epsilon}),$$

where $A_k(n)$ is the Kloosterman sum defined in (3.5) and $\epsilon > 0$.

The proof is nearly identical to that of Theorem 4.1 of [9], so we do not include it here. One slight difference is that we must appeal to

$$(5.8) \quad (-1)^{c+\lfloor \frac{c+1}{2} \rfloor} A_{2c} \left(n - \frac{c(1+(-1)^c)}{4} \right) \\ = \sum_{h \pmod{2c}} \omega_{h,2c} \cdot (-1)^{\frac{2c+1-h[-h]_{2c}}{2}} e \left(\frac{-h-c[-h]_{2c}}{4} - \frac{nh}{2c} \right),$$

where

$$\omega_{h,k} = i^{-\frac{1}{2}} \chi(h, [-h]_k, k)^{-1} e^{-\frac{\pi i}{12k}([-h]_k - h)}.$$

Equation (5.8) follows from

$$(-1)^{\lfloor \frac{c+1}{2} \rfloor} A_{2c} \left(n - \frac{c(1+(-1)^c)}{4} \right) = \sum_{d \pmod{2c}} \omega_{-d,2c} \cdot (-1)^{\frac{2c+1+ad}{2}} e \left(\frac{a-3dc}{4} + \frac{nd}{2c} \right)$$

where $ad \equiv 1 \pmod{2c}$ which can be found in [10].

6. THE RANK STATISTIC

A second partition statistic is the rank statistic of Dyson [13]. Zwegers [24] proved that the two variable generating functions, $R(x; q)$, for the rank is a mock Jacobi form (see also Zagier's discussion in [23]). Thus, its Taylor coefficients are quasimock modular forms [7]. As a result, the automorphic properties of the rank moment generating functions are more complicated than those for the crank. An extensive study, relying on the so-called "Rank-Crank PDE" of Atkin and Garvan [5], was completed by Bringmann, Garvan and Mahlburg [7]. However, the perspective of Taylor coefficients of Jacobi forms simplifies some elements of that study and makes further study accessible. For instance, it would be of interest to understand these forms explicitly.

The mock theta function $f(q) = R(-1; q)$ has attracted much attention (see, for instance, the work of Bringmann and Ono [10]). It should be compared with the twisted crank discussed in this paper. The perspective of Taylor coefficients makes a full comparison of the twisted crank and rank moments, along the lines of that given in [9], possible. It would be of additional interest to have a combinatorial or q -series understanding of these twisted moments, similar to that given by Andrews [1] and Garvan [17], for the rank and crank moments.

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