DIVISIBILITY PROPERTIES OF COEFFICIENTS OF LEVEL $p$
MODULAR FUNCTIONS FOR GENUS ZERO PRIMES

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Abstract. Lehner’s 1949 results on the $j$-invariant showed high divisibility
of the function’s coefficients by the primes $p \in \{2, 3, 5, 7\}$. Expanding his
results, we examine a canonical basis for the space of level $p$ modular functions
holomorphic at the cusp 0. We show that the Fourier coefficients of these
functions are often highly divisible by these same primes.

1. Introduction and statement of results

A level $p$ modular function $f(\tau)$ is a holomorphic function on the complex upper
half-plane which satisfies

$$f \left( \frac{a\tau + b}{c\tau + d} \right) = f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$$

and is meromorphic at the cusps of $\Gamma_0(p)$. Equivalently, $f(\tau)$ is a weakly holomor-
phic modular form of weight 0 on $\Gamma_0(p)$. Such a function will necessarily have a
$q$-expansion of the form $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$, where $q = e^{2\pi i \tau}$.

Of particular interest in the study of modular forms is the classical $j$-invariant,
$j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$, which is a modular function of level 1. The
coefficients $c(n)$ of the $j$-function, like the Fourier coefficients of many other mod-
ular forms, are of independent arithmetic interest; for instance, they appear as
dimensions of a special graded representation of the Monster group.

In 1949 Lehner showed [7, 8] that

$$c(2^a 3^b 5^c 7^d n) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d},$$

proving that the coefficients $c(n)$ are often highly divisible by small primes. Similar
results have recently been proven for other modular functions in [6], and for modular
forms of level 1 and small weight in [4], [3]. It is natural to ask whether such
congruences hold for the Fourier coefficients of modular functions of higher level,
such as those studied by Ahlgren [11] in his work on Ramanujan’s $\theta$-operator.

Lehner’s results for $j(\tau)$ are in fact more general; in [8] he pointed out that for $p = 2, 3, 5, 7$, similar congruences hold for the coefficients of level $p$ modular
functions which have integral coefficients at both cusps and have poles of order less
than $p$ at the cusp at infinity.

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In this paper, for \( p \in \{2, 3, 5, 7\} \), we examine canonical bases for spaces of level \( p \) modular functions which are holomorphic at the cusp 0. To construct these bases, we introduce the level \( p \) modular function \( \psi^{(p)}(\tau) \), defined as

\[
\psi^{(p)}(\tau) = \left( \frac{\eta(\tau)}{\eta(p\tau)} \right)^{\frac{24}{p-1}} \text{ where } \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]

The integer \( \frac{24}{p-1} \) for \( p = 2, 3, 5, 7 \) will appear frequently, so we will denote it by \( \lambda^{(p)} \), or simply \( \lambda \) where no confusion arises. The function \( \psi^{(p)}(\tau) \) is a modular function of level \( p \) with a simple pole at \( \infty \) and a simple zero at 0. We will also use the modular function

\[
\phi^{(p)}(\tau) = (\psi^{(p)}(\tau))^{-1}.
\]

Following Ahlgren [1] and using the notation of Duke and Jenkins [5], for \( p = 2, 3, 5, 7 \) we construct a basis \( \{f_{0,m}^{(p)}(\tau)\}_{m=0}^{\infty} \) for the space of level \( p \) modular functions which are holomorphic at 0 as follows:

\[
f_{0,0}^{(p)}(\tau) = 1, f_{0,m}^{(p)}(\tau) = q^{-m} + O(1) = \psi^{(p)}(\tau)^m - Q(\psi^{(p)}(\tau)),
\]

where \( Q(x) \) is a polynomial of degree \( m - 1 \) with no constant term, chosen to eliminate all negative powers of \( q \) in \( \psi^{(p)}(\tau)^m \) except for \( q^{-m} \). Since \( \psi^{(p)}(\tau) \) vanishes at 0 and the polynomial \( Q \) has no constant term, we see that the functions \( f_{0,m}^{(p)} \) also vanish at 0 when \( m > 0 \). We write

\[
f_{0,m}^{(p)} = q^{-m} + \sum_{n=0}^{\infty} a_{0}^{(p)}(m,n) q^n
\]

so that for \( n \geq 0 \), the symbol \( a_{0}^{(p)}(m,n) \) denotes the coefficient of \( q^n \) in the \( m \)th basis element of level \( p \). Note that the function \( f_{0,m}^{(p)} \) corresponds to Ahlgren’s \( j_m^{(p)} \).

For an example of some of these functions, consider the case \( p = 2 \):

\[
\begin{align*}
f_{0,1}^{(2)}(\tau) &= \psi^{(2)}(\tau) \\
&= q^{-1} - 24 + 276q - 2048q^2 + 11202q^3 - 49152q^4 + \ldots, \\
f_{0,2}^{(2)}(\tau) &= \psi^{(2)}(\tau)^2 + 48 \psi^{(2)}(\tau) \\
&= q^{-2} - 24 - 4096q + 98580q^2 - 1228800q^3 + 10745856q^4 + \ldots, \\
f_{0,3}^{(2)}(\tau) &= \psi^{(2)}(\tau)^3 + 72 \psi^{(2)}(\tau)^2 + 900 \psi^{(2)}(\tau) \\
&= q^{-3} - 96 + 33606q - 1843200q^2 + 43434816q^3 - 648216576q^4 + \ldots.
\end{align*}
\]

The function \( f_{0,m}^{(p)} \) is a level \( p \) modular function that vanishes at 0 (if \( m \neq 0 \)) and has a pole of order \( m \) at \( \infty \). The conditions at the cusps determine this function uniquely; if two such functions exist, their difference is a holomorphic modular function, which must be a constant. Since both functions vanish at 0, this constant must be 0.

The functions composing these bases for \( p = 2, 3, 5, 7 \) have divisibility properties which bear a striking resemblance to the divisibility properties of \( j(\tau) \); in many cases they are identical. As an example of some of the divisibility properties we encounter with this basis, we experimentally examine the 2-adic valuation of the even-indexed coefficients of \( f_{0,m}^{(2)}(\tau) \) for \( m = 1, 3, 5, 7 \) in Table [4]. As the data in the
table suggest, the 2-divisibility which \(j(\tau)\) exhibits gives us a lower bound on the 2-divisibility of the odd-indexed \(p = 2\) basis elements.

Table 1. 2-adic valuation of \(a_0^{(2)}(m, n)\) compared to corresponding coefficients in \(j(\tau)\)

<table>
<thead>
<tr>
<th>(m)</th>
<th>(a_0^{(2)}(m, 2))</th>
<th>(a_0^{(2)}(m, 4))</th>
<th>(a_0^{(2)}(m, 6))</th>
<th>(a_0^{(2)}(m, 8))</th>
<th>(a_0^{(2)}(m, 10))</th>
<th>(a_0^{(2)}(m, 12))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>14</td>
<td>13</td>
<td>17</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>16</td>
<td>15</td>
<td>19</td>
<td>14</td>
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<td>5</td>
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<tr>
<td>7</td>
<td>14</td>
<td>17</td>
<td>16</td>
<td>20</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>min</td>
<td>11</td>
<td>14</td>
<td>13</td>
<td>17</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>

\(j(\tau)\)| 11 | 14 | 13 | 17 | 12 | 16 |

Note that these functions form a basis for \(M^\infty_0(p)\), the space of modular forms of weight 0 and level \(p\) with poles allowed only at the cusp at \(\infty\). A full basis for the space \(M^w_0(p)\) of weakly holomorphic modular forms of weight 0 and level \(p\) is generated by the \(f_0^{(p)}(\tau)\) and the functions \((\phi^{(p)}(\tau))^n\) for integers \(n \geq 1\).

Recall that the concluding remarks of Lehner’s second paper \cite{8} state that the coefficients of certain level \(p\) modular functions having a pole of order less than \(p\) at \(\infty\) have the same \(p\)-divisibility properties as the coefficients \(c(n)\) of \(j(\tau)\). More precisely, we have the following theorem.

Theorem 1 (Lehner). Let \(p \in \{2, 3, 5, 7\}\) and let \(f(\tau)\) be a modular function on \(\Gamma_0(p)\) having a pole at \(\infty\) of order \(< p\) and \(q\)-series of the form

\[
f(\tau) = \sum_{n=n_0}^{\infty} a(n)q^n,
\]

\[
f(-1/p\tau) = \sum_{n=n_0}^{\infty} b(n)q^n,
\]

where \(a(n), b(n) \in \mathbb{Z}\). Then the coefficients \(a(n)\) satisfy the following congruence properties:

\[
a(2^n a^n) \equiv 0 \pmod{2^{3a+8}} \quad \text{if } p = 2,
\]

\[
a(3^n a^n) \equiv 0 \pmod{3^{2a+3}} \quad \text{if } p = 3,
\]

\[
a(5^n a^n) \equiv 0 \pmod{5^{a+1}} \quad \text{if } p = 5,
\]

\[
a(7^n a^n) \equiv 0 \pmod{7^a} \quad \text{if } p = 7.
\]

Note that Lehner’s original statement of this theorem mistakenly states that a function on \(\Gamma_0(p)\) inherits the \(p\)-divisibility property for every prime in \(\{2, 3, 5, 7\}\), not just the prime matching the level.

A necessary condition in the statement of Lehner’s theorem is that the function must have an integral \(q\)-expansion at 0. This condition is quite strong; in fact, neither the function \(\phi^{(p)}(\tau)\) nor any of its powers satisfy it, although the functions \(f_0^{(p)}(\tau)\) do.
Further, Lehner’s theorem assumes that the order of the pole at \( \infty \) must be less than \( p \). In this paper, we remove this restriction on the order of the pole to show that every function in the \( f_{0,m}^{(p)} \) basis has divisibility properties similar to those in Theorem 1. Specifically, we prove the following theorem.

**Theorem 2.** Let \( p \in \{2, 3, 5, 7\} \), and let

\[
n_{0,m}(\tau) = q^{-m} + \sum_{n=0}^{\infty} a_0^{(p)}(m, n) q^n
\]

be an element of the basis described above, with \( m = p^\alpha m' \) and \((m', p) = 1\). Then, for \( \beta > \alpha \),

\[
\begin{align*}
    a_0^{(2)}(2^\alpha m', 2^\beta n) &\equiv 0 \pmod{2^{3(\beta - \alpha) + 8}} \quad \text{if } p = 2, \\
    a_0^{(3)}(3^\alpha m', 3^\beta n) &\equiv 0 \pmod{3^{2(\beta - \alpha) + 3}} \quad \text{if } p = 3, \\
    a_0^{(5)}(5^\alpha m', 5^\beta n) &\equiv 0 \pmod{5^{(\beta - \alpha) + 1}} \quad \text{if } p = 5, \\
    a_0^{(7)}(7^\alpha m', 7^\beta n) &\equiv 0 \pmod{7^{(\beta - \alpha)}} \quad \text{if } p = 7.
\end{align*}
\]

Note that for basis elements \( n_{0,m}^{(p)} \) with \((m, p) = 1\), these divisibility properties match those in Theorem 1; in fact, Lehner’s proof is easily extended to prove the congruences in these cases. For basis elements with \( m = p^\alpha m' \) and \( \alpha \geq 1 \), the divisibility is “shifted”. This shifting occurs in the \((\beta - \alpha)\) factor in the exponent of the modulus.

For the coefficients \( a_0^{(p)}(p^\alpha m', p^\beta n) \) with \( \alpha > \beta \), computations suggest that similar congruences should hold. Additionally, it appears that powers of the function \( \phi^{(p)}(\tau) \) have Fourier coefficients with slightly weaker divisibility properties, despite the fact that their Fourier coefficients at 0 are not integral. It would be interesting to more fully understand these congruences.

### 2. Preliminary lemmas and definitions

In this section, we provide the necessary definitions and background for the proof of the main theorem.

For a prime \( p \) we define the level \( p \) Hecke operator \( U_p \) by

\[
U_p f(\tau) = \frac{1}{p} \sum_{\ell=0}^{p-1} f\left(\frac{\tau + \ell}{p}\right),
\]

using the notation \( U_p^n f = U_p U_p \cdots U_p f \) for repeated applications of \( U_p \). When \( f \) has the Fourier expansion \( f(\tau) = \sum_{n=n_0}^{\infty} a(n) q^n \), this operator takes the form

\[
U_p f(\tau) = \sum_{n=n_0}^{\infty} a(pn) q^n,
\]

essentially “pulling out” all of the coefficients of \( f \) whose index is divisible by \( p \). This operator preserves modularity: if \( f \) is a level \( p \) modular function, then \( U_p f \) is also a level \( p \) modular function.

For the primes \( p = 2, 3, 5, 7 \) the topological genus of \( \Gamma_0(p) \backslash \mathcal{H} \) is zero, so the field of level \( p \) modular functions is generated by a single modular function called a Hauptmodul. For the primes under consideration, one such function is \( \psi^{(p)}(\tau) \). Note that the modular function \( \phi^{(p)}(\tau) = \psi^{(p)}(\tau)^{-1} = q + O(q^2) \) is also a Hauptmodul.
Lemma 5. Let the substitution $g = \phi$ by $\phi$.

Proof. The following lemma gives relations for $\psi(p)(\tau)$ and $\phi(p)(\tau)$ at 0 and makes clear that powers of $\phi(p)$ do not satisfy Lehner’s integrality condition.

Lemma 3. The functions $\psi(p)(\tau)$ and $\phi(p)(\tau)$ satisfy the relations

\begin{align*}
(2.1) & \quad \psi(p)(-1/p\tau) = p^{\lambda/2}\phi(p)(\tau), \\
(2.2) & \quad \phi(p)(-1/p\tau) = p^{-\lambda/2}\psi(p)(\tau).
\end{align*}

Proof. The functional equation for $\eta(\tau)$ is $\eta(-1/\tau) = \sqrt{-i}\eta(\tau)$. Using this, we find that

$$
\psi(p)\left(\frac{-1}{p\tau}\right) = \left(\frac{\eta(-1/(p\tau))}{\eta(-1/\tau)}\right)^\lambda = \left(\frac{\sqrt{-i}p\eta(p\tau)}{\sqrt{-i}\eta(\tau)}\right)^\lambda = (\sqrt{p})^\lambda \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^\lambda = p^{\lambda/2}\phi(p)(\tau).
$$

The second statement follows after replacing $\tau$ by $-1/(p\tau)$ in the first statement. \qed

We next state a well-known lemma which gives a formula for determining the behavior of a modular function at 0 after $U_p$ has been applied. A proof can be found in [2, p. 83].

Lemma 4. Let $p$ be prime and let $f(\tau)$ be a level $p$ modular function. Then

$$
(2.3) \quad p(U_p f)(-1/(p\tau)) = p(U_p f)(p\tau) + f(-1/(p^2\tau)) - f(\tau).
$$

Lehner’s original papers included the following lemma and its proof, which gives two important equations. The first gives a formula for $U_p\phi(p)$ as a polynomial with integral coefficients in $\phi(p)$; the second gives an algebraic relation which is satisfied by $\phi(p)(\tau/p)$.

Lemma 5. Let $p \in \{2, 3, 5, 7\}$. Then there exist integers $b^{(p)}_j$ such that

(a) \quad $U_p\phi(p)(\tau) = p \sum_{j=1}^{p} b^{(p)}_j \phi(p)(\tau)^j$.

Further, let $h^{(p)}(\tau) = p^{\lambda/2}\phi(p)(\tau/p)$. Then

(b) \quad $(h^{(p)}(\tau))^p + \sum_{j=1}^{p} (-1)^j g_j(\tau)(h^{(p)}(\tau))^{p-j} = 0,$

where $g_j(\tau) = (-1)^{j+1} p^{\lambda/2+2} \sum_{\ell=j}^{p} b^{(p)}_\ell \phi(p)(\tau)^{\ell-j+1}$.

Proof. (a) Since $\phi$ vanishes at $\infty$, $U_p\phi$ also vanishes at $\infty$; we will now consider its behavior at 0. Using (2.3) and replacing $\tau$ by $p\tau$ in (2.1) we obtain

$$
U_p\phi(-1/(p\tau)) = U_p\phi(p\tau) + p^{-1}\phi(-1/(p^2\tau)) - p^{-1}\phi(\tau) = U_p\phi(p\tau) + p^{-1}\psi(p\tau) - p^{-1}\phi(\tau) = O(q^p) + p^{-\lambda/2-1}q^{-p} + O(1) - p^{-1}q + O(q^2),
$$

$$
p^{\lambda/2+1}U_p\phi(-1/(p\tau)) = q^{-p} + O(1).
$$
The right side of this equation is a level \( p \) modular function with integer coefficients, so we may write it as a polynomial in \( \psi(\tau) \) with integer coefficients. The polynomial will not have a constant term since the left side vanishes at 0. Therefore,

\[
p^{\lambda/2+1}U_p\phi(-1/(p\tau)) = \sum_{j=1}^{p} c_j\psi(\tau)^j.
\]

Now, replacing \( \tau \) by \(-1/(p\tau)\), we find that

\[
p^{\lambda/2+1}U_p\phi(\tau) = \sum_{j=1}^{p} c_j p^{\lambda j/2} \phi(\tau)^j.
\]

After cancelling the \( p^{\lambda/2+1} \), we find that

\[
U_p\phi(\tau) = \sum_{j=1}^{p} c'_j \phi(\tau)^j.
\]

We now use Lemma 3 with the knowledge that \( \psi(\tau) = \phi(\tau) \) to obtain

\[
p^2 \sum_{j=1}^{p} b_{j}^{(p)} \phi(-1/(p\tau))^j = p^2 \sum_{j=1}^{p} b_{j}^{(p)} \phi(p\tau)^j + \phi(-1/(p^2\tau)) - \phi(\tau).
\]

We now use Lemma 2 with the knowledge that \( \psi(\tau) = \phi(\tau)^{-1} \) to obtain

\[
p^2 \sum_{j=1}^{p} b_{j}^{(p)} p^{-\lambda j/2} \phi(\tau)^{-j} - p^2 \sum_{j=1}^{p} b_{j}^{(p)} \phi(p\tau)^j + \phi(\tau) - p^{-\lambda/2} \phi(p\tau)^{-1} = 0.
\]

After replacing \( \tau \) by \( \tau/p \) and multiplying by \( p^{\lambda/2} \), we obtain

\[
p^{\lambda/2+2} \sum_{j=1}^{p} b_{j}^{(p)} (h(\tau)^{-j} - \phi(\tau)^j) + h(\tau) - \phi(\tau)^{-1} = 0.
\]

Table 2. Values of \( b_{j}^{(p)} \)

<table>
<thead>
<tr>
<th>( j )</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 \cdot 2^2</td>
<td>10 \cdot 3^1</td>
<td>63 \cdot 5^0</td>
<td>82 \cdot 7^0</td>
</tr>
<tr>
<td>2</td>
<td>2^{10}</td>
<td>4 \cdot 3^6</td>
<td>52 \cdot 5^3</td>
<td>176 \cdot 7^2</td>
</tr>
<tr>
<td>3</td>
<td>3^{10}</td>
<td>63 \cdot 5^5</td>
<td>845 \cdot 7^3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6 \cdot 5^8</td>
<td>272 \cdot 7^5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5^{10}</td>
<td>46 \cdot 7^9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>\n</td>
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<tr>
<td>7</td>
<td>\n</td>
<td>\n</td>
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<td></td>
</tr>
</tbody>
</table>

(b) We again apply (2.3) to \( \phi(\tau) \), this time using what we know from (a):

\[
pU_p\phi(-1/(p\tau)) = pU_p\phi(p\tau) + \phi(-1/(p^2\tau)) - \phi(\tau),
\]

\[
p^2 \sum_{j=1}^{p} b_{j}^{(p)} \phi(-1/(p\tau))^j = p^2 \sum_{j=1}^{p} b_{j}^{(p)} \phi(p\tau)^j + \phi(-1/(p^2\tau)) - \phi(\tau).
\]

We now use Lemma 3 with the knowledge that \( \psi(\tau) = \phi(\tau) \) to obtain

\[
p^2 \sum_{j=1}^{p} b_{j}^{(p)} p^{-\lambda j/2} \phi(\tau)^{-j} - p^2 \sum_{j=1}^{p} b_{j}^{(p)} \phi(p\tau)^j + \phi(\tau) - p^{-\lambda/2} \phi(p\tau)^{-1} = 0.
\]
We now divide by $h^{-1} - \phi$. Note two facts:

$$h^{-j} - \phi^j = (h^{-1} - \phi) \sum_{\ell=0}^{j-1} h^{-\ell} \phi^{j-\ell-1},$$

$$\frac{h - \phi^{-1}}{h^{-1} - \phi} = \frac{h(h\phi - 1)}{\phi(1 - h\phi)} = -\frac{h}{\phi}.$$

So (2.4) becomes

$$p^{\lambda/2 + 2} \sum_{j=1}^{p} b_j^{(p)} \sum_{\ell=0}^{j-1} h^{-\ell} \phi^{j-\ell-1} - \phi^{-1} h = 0,$$

which, after multiplying by $\phi h^{p-1}$, becomes

$$p^{\lambda/2 + 2} \sum_{j=1}^{p} b_j^{(p)} \sum_{\ell=0}^{j-1} h^{p-\ell-1} \phi^{j-\ell} - h^p = 0.$$

We now change the order of summation and rearrange terms to obtain the desired formula:

$$h(\tau)^p = \sum_{j=1}^{p} \left( p^{\lambda/2 + 2} \sum_{\ell=j}^{p} b_j^{(p)} \phi(\tau)^{\ell-j+1} \right) h(\tau)^{p-j}. \quad \Box$$

The next lemma states that when you apply $U_p$ to a certain type of polynomial in $\phi_p$, you get a similar polynomial back which has picked up a power of $p$. The details of this lemma are found in both [7] and [8], scattered throughout the proofs of the main theorems. For our purposes, it will be more useful in the following form.

**Lemma 6.** Let $p \in \{2, 3, 5, 7\}$ and let $R^{(p)}$ denote the set of polynomials in $\phi^{(p)}$ of the form

$$d_1 \phi^{(p)}(\tau) + \sum_{n=2}^{N} d_n \gamma \phi^{(p)}(\tau)^n,$$

where

$$\gamma = \begin{cases} 8(n-1) & \text{if } p = 2, \\ 4(n-1) & \text{if } p = 3, \\ n & \text{if } p = 5, \\ n & \text{if } p = 7. \end{cases}$$

Then $U_p$ maps $R^{(p)}$ to $p^\delta R^{(p)}$ where $\delta = 3, 2, 1, 1$ for $p = 2, 3, 5, 7$, respectively. That is, applying $U_p$ to a polynomial of the above form yields a polynomial of the same form with an extra factor of $p^\delta$.

**Proof.** Consider the function

$$d_1 U_p \phi(\tau) + \sum_{n=2}^{r} d_n \gamma U_p \phi(\tau)^n.$$

For the first term, Lemma [5(a)] shows that $U_p \phi(\tau) \in p^\delta R_p$ since, by inspection, the $b_j^{(p)}$ integers are divisible by sufficiently high powers of $p$. 
For the remaining terms, we will prove that
\[ p^\gamma U_p \phi^n = p^\delta r, \]
where \( r \in R_p \). The result will immediately follow.

By the definition of \( U_p \) we have
\[ U_p \phi^n = p^{-1} \sum_{\ell=0}^{p-1} \phi \left( \frac{\tau + \ell}{p} \right)^n = p^{-1 - \lambda/2} \sum_{\ell=0}^{p-1} h_\ell(\tau)^n, \]
where \( h_\ell(\tau) = p^{\lambda/2} \phi \left( \frac{\tau + \ell}{p} \right) \) is related to \( h \) from Lemma 5(b). Let \( S_n \) be the sum of the \( n \)th powers of the \( h_\ell \) so that
\[ S_n = \sum_{\ell=0}^{p-1} h_\ell^n. \]

Define the polynomial \( F(x) = \sum_{j=0}^{p} (-1)^j g_j(\tau)x^{p-j} \), where the \( g_j(\tau) \) are as in Lemma 5. In the same lemma, if we replace \( \tau \) with \( \tau + \ell \), the \( g_j(\tau) \) are unaffected since \( \phi(\tau + 1) = \phi(\tau) \). Therefore, that lemma tells us that the \( p \) roots of the polynomial \( F(x) \) are precisely the \( h_\ell \). Using Newton’s formula for the \( n \)th power sum of the roots of a polynomial, we obtain
\[ S_n = \sum_{\ell=0}^{p-1} h_\ell^n = \sum_{j=1}^{n} (-1)^{j+1} g_j S_{n-j}, \]
where \( g_j = 0 \) for \( j > p \) and \( S_0 = n \).

We now proceed case-by-case. The \( p = 2 \) case illustrates the method, so we will only include the intermediate steps in the \( p = 3, 5, 7 \) cases.

Case 1. \( p = 2 \). Then, using (2.6), equation (2.5) is equivalent to
\[ 2^{8(n-1)} \left( 2^{-1-12n} S_n \right) = 2^3 r \quad \text{or} \quad S_n = 2^{4n+12} r. \]

We now use (2.7) to calculate \( S_1 \) and \( S_2 \):
\[ S_1 = g(1), \]
\[ S_2 = g_1 S_1 - 2g_2 = g_1^2 - 2g_2. \]

From Lemma 5 we can compute the values of the \( g_j \). Using the \( b_j \) values from the table in that lemma, we have
\[ g_1 = 2^{14}(b_1\phi_2 + b_2\phi_2^2) = 2^{16}(3\phi_2 + 2^8\phi_2^2), \]
\[ g_2 = -2^{14}b_2\phi_2 = -2^{24}\phi_2. \]

We can now see that
\[ S_1 = g_1 = 2^{16}(3\phi_2 + 2^8\phi_2^2), \]
\[ S_2 = 2^{32}(3\phi_2 + 2^8\phi_2^2)^2 + 2^{25}\phi_2 = 2^{20}(5\phi_2 + 2^{12}3^2\phi_2^2 + 2^213\phi_3 + 2^{28}\phi_4). \]

Thus (2.8) is satisfied for \( n = 1, 2 \). We proceed by induction. Assume (2.8) is satisfied for all integers \( < n \). We show that it is satisfied for \( n \). For ease of
computation, we introduce the set
\[ R^* = 2^8 R^{(2)} = \left\{ \sum_{i=1}^{m} d_i 2^i \phi_i^i \mid d_i \in \mathbb{Z}, m \in \mathbb{Z}^+ \right\}, \]
which, the reader will notice, is a ring without 1. From (2.7) we obtain
\[ S_n = g_1 S_{n-1} - g_2 S_{n-2} = 2^8 r_1^* \cdot 2^{4n} r_2^* + 2^{16} r_3^* \cdot 2^{4(n-1)} r_4^* = 2^{4n+8} r_5^* = 2^{4n+16} r, \]
where \( r_i^* \in R^* \) and \( r \in R^{(2)}. \)

**Case 2.** \( p = 3. \) We want to show that
\[ (2.9) \quad S_n = 3^{2n+7} r, \]
where \( r \in R^{(3)}. \) We compute the \( g_j \) and \( S_n \) as follows, using the \( b_j \) from the table:
\[ g_1 = 3^9 (3^9 \phi_3^3 + 3^5 4 \phi_3^2 + 10 \phi_3), \quad g_2 = 3^{14} (-3^4 \phi_3^2 - 4 \phi_3), \quad g_3 = 3^{18} \phi_3, \]
\[ S_1 = g_1, \quad S_2 = g_1^2 - 2g_2, \quad S_3 = g_1^3 - 3g_1g_2 + 3g_3. \]
From this, we obtain
\[ S_1 = 3^9 (3^9 \phi_3^3 + 3^5 4 \phi_3^2 + 10 \phi_3), \]
\[ S_2 = 3^{14} (8 \phi_3 + 3^3 34 \phi_3^2 + 3^9 80 \phi_3 + 3^3 168 \phi_3^4 + 3^1 188 \phi_3^5 + 3^3 25 \phi_3^6), \]
\[ S_3 = 3^{19} (\phi_3 + 3^5 40 \phi_3^2 + 3^8 1174 \phi_3^3 + 3^{15} 136 \phi_3^4 + 3^{18} 581 \phi_3^5 + 3^{25} 16 \phi_3^6 \]
\[ + 3^{27} 58 \phi_3^7 + 3^{32} 4 \phi_3^8 + 3^{35} \phi_3^9) \]
which proves (2.9) for \( n = 1, 2, 3. \) For the inductive step, let \( R^* \) be the ring without 1 given by \( 3^4 R^{(3)} \) so that
\[ S_n = g_1 S_{n-1} - g_2 S_{n-2} + g_3 S_{n-3} \]
\[ = 3^5 r_1^* 3^{2n+1} r_2^* + 3^{10} r_3^* 3^{2n-1} r_4^* + 3^{14} r_5^* 3^{2n-3} r_6^* \]
\[ = 3^{2n+6} r_7^* \]
\[ = 3^{2n+10} r, \]
where \( r_i^* \in R^* \) and \( r \in R^{(3)}. \)

**Case 3.** \( p = 5. \) We want
\[ (2.10) \quad S_n = 5^{2n+2} r, \]
where \( r \in R^{(5)}. \) Computing the \( S_n \) we find
\[ S_1 = 5^5 r_1, \quad S_2 = 5^8 r_2, \quad S_3 = 5^{10} r_3, \]
\[ S_4 = 5^{13} r_4, \quad S_5 = 5^{16} r_5 \]
for some \( r_1, \ldots, r_5 \in R^{(5)}. \) This proves (2.10) for \( n = 1, \ldots, 5. \) For the inductive step, let \( R^* \) be the ring without 1 given by \( 5 R^{(5)} \) so that
\[ S_n = g_1 S_{n-1} - g_2 S_{n-2} + g_3 S_{n-3} - g_4 S_{n-4} + g_5 S_{n-5} \]
\[ = 5^4 r_1^* 3^{2n-1} r_2^* - \ldots + 5^{14} r_9^* 3^{2n-9} r_{10}^* \]
\[ = 5^{2n+3} r_{11}^* \]
\[ = 5^{2n+4} r, \]
where \( r_i^* \in R^* \) and \( r \in R^{(5)}. \)
Case 4. $p = 7$. We want

\begin{equation}
S_n = 7^{n+2}r,
\end{equation}

where $r \in R^{(7)}$. Computing the $S_n$ we find

\[
S_1 = 7^4r_1, \quad S_2 = 7^6r_2, \quad S_3 = 7^7r_3, \quad S_4 = 7^9r_4,
\]

\[
S_5 = 7^{11}r_5, \quad S_6 = 7^{13}r_6, \quad S_7 = 7^{15}r_7
\]

for some $r_1, \ldots, r_7 \in R^{(7)}$. This proves (2.11) for $n = 1, \ldots, 7$. For the inductive step, let $R^*$ be the ring without 1 given by $7R^{(7)}$ so that

\[
S_n = \sum_{i=1}^{7} (-1)^{i+1} g_i S_{n-i}
\]

\[
= 7^3r_1^* 7^n r_2^* - \ldots + 7^{13} r_{13}^* 7^{n-6} r_{14}^*
\]

\[
= 7^{n+3}r_{15}^*
\]

\[
= 7^{n+4}r,
\]

where $r_i^* \in R^*$ and $r \in R^{(7)}$. \hfill \Box

3. Proof of the theorem

To remind the reader of the main result of the paper, we include it here.

**Theorem.** Let $p \in \{2, 3, 5, 7\}$ and let $f_{0,m}(\tau) = q^{-m} + \sum a_{0}^{(p)}(m, n)q^n$ be an element of the basis described above, with $m = p^\alpha m'$ and $(m', p) = 1$. Then, for $\beta > \alpha$,

\[
a_{0}^{(2)}(2^\alpha m', 2^{\beta}n) \equiv 0 \pmod{2^{3(\beta-\alpha)+8}} \quad \text{if } p = 2,
\]

\[
a_{0}^{(3)}(3^\alpha m', 3^{\beta}n) \equiv 0 \pmod{3^{2(\beta-\alpha)+3}} \quad \text{if } p = 3,
\]

\[
a_{0}^{(5)}(5^\alpha m', 5^{\beta}n) \equiv 0 \pmod{5^{(\beta-\alpha)+1}} \quad \text{if } p = 5,
\]

\[
a_{0}^{(7)}(7^\alpha m', 7^{\beta}n) \equiv 0 \pmod{7^{(\beta-\alpha)}} \quad \text{if } p = 7.
\]

The proof is in three cases. The first illustrates the method for the simplest basis elements, namely those with $(m, p) = 1$. The second demonstrates the “shifting” property at its first occurrence, $f_{0,p}^{(p)}$. The third is the general case; it builds inductively upon the methods of the first two cases.

3.1. **Case 1:** $(m, p) = 1$.

**Proof.** This proof is almost identical to Lehner’s proof of Theorem 3 in [8]; however, it applies not only to functions which have poles of order bounded by $p$ but to all basis elements with $(m, p) = 1$. For ease of notation, let $f(\tau) = f_{0,m}(\tau)$.

We will demonstrate the method with $m = 1$, then generalize it to all $m$ relatively prime to $p$. First, we will write $U_p f(\tau)$ as a polynomial in $\phi(\tau)$ with integral coefficients, all of which are divisible by the desired power of $p$. Since $U_p$ isolates the coefficients whose index is divisible by $p$, we will have proven the theorem for $\beta = 1$. We will then apply $U_p$ repeatedly to the polynomial, showing that the result is always another polynomial in $\phi$ with integral coefficients, all of which are divisible by the desired higher power of $p$. 
Consider the level $p$ modular function $g(\tau) = pU_p f(\tau) + p^{\lambda/2} \phi(\tau)$. Notice that $g(\tau)$ is holomorphic at $\infty$ since both $U_p f(\tau)$ and $\phi(\tau)$ are holomorphic there. The $q$-expansion at 0 for $g(\tau)$ is given by

$$g(-1/(p\tau)) = p(U_p f)(-1/(p\tau)) + p^{\lambda/2} \phi(-1/(p\tau)),$$

which, by Lemmas 3 and 4 becomes

$$g(-1/(p\tau)) = p(U_p f)(p\tau) + f(-1/(p^2\tau)) - f(\tau) + \psi(\tau).$$

When we notice that $f(\tau) = \psi(\tau)$ in this $m = 1$ case, we obtain

$$g(-1/(p\tau)) = p(U_p f)(p\tau) + \psi(-1/(p^2\tau)) - \psi(\tau) + \psi(\tau) = p(U_p f)(p\tau) + p^{\lambda/2} \phi(p\tau),$$

which is holomorphic at $\infty$. Hence, $g(\tau)$ is a holomorphic modular function on $\Gamma_0(p)$, so it must be constant. Therefore,

$$U_p f(\tau) = c_0 - p^{\lambda/2 - 1} \phi(\tau)$$

for some constant $c_0$. The proof is complete for $\beta = 1$.

Note: The prime 13, having genus zero, would work in this construction; however, in that case $\lambda = \frac{24}{13-1} = 2$, so $13^{\lambda/2 - 1} = 1$, and we gain no new information.

We now iterate the above process to prove the theorem for $\beta > 1$. Notice that

$$U_p(U_p f(\tau)) = c^{(p)} - p^{\lambda/2 - 1} U_p \phi(\tau).$$

We know from Lemma 5 that $U_p \phi$ is a polynomial in $\phi$; in fact, by inspection of the $b_j^{(p)}$ values we see that we may write

$$U_2 \phi^{(2)}(\tau) = 2^3 (d_1^{(2)} \phi^{(2)}(\tau) + \sum_{n=2}^{2} d_n^{(2)} 2^{8(n-1)} \phi^{(2)}(\tau)^n),$$

$$U_3 \phi^{(3)}(\tau) = 3^2 (d_1^{(3)} \phi^{(3)}(\tau) + \sum_{n=2}^{3} d_n^{(3)} 3^{4(n-1)} \phi^{(3)}(\tau)^n),$$

$$U_5 \phi^{(5)}(\tau) = 5^3 (d_1^{(5)} \phi^{(5)}(\tau) + \sum_{n=2}^{5} d_n^{(5)} 5^n \phi^{(5)}(\tau)^n),$$

$$U_7 \phi^{(7)}(\tau) = 7^3 (d_1^{(7)} \phi^{(7)}(\tau) + \sum_{n=2}^{7} d_n^{(7)} 7^n \phi^{(7)}(\tau)^n)$$

for some integers $d_n^{(p)}$. This shows that the second $U_p$ iteration is divisible by the correct power of $p$. Further, it gives us a polynomial of a suitable form to iterate the process using Lemma 6. In each of the polynomials above, notice that $U_p \phi(\tau) = p^\delta r$ for some $r \in R^{(p)}$. Using Lemma 6, we find that

$$U_p(U_p \phi)(\tau) = p^{2\delta} r'$$

for some $r' \in R^{(p)}$, and further

$$U_p^\beta \phi(\tau) = p^{\beta \delta} r^{(\beta)}$$

for some $r_\beta \in R^{(p)}$. This completes the proof for $m = 1$.

Now, if $(m, p) = 1$, then $U_p f(\tau)$ is holomorphic at $\infty$, just as it was with $m = 1$. Moving to the cusp at 0 we find that $(U_p f)(-1/(p\tau))$ can be written as a polynomial.
in $\psi(\tau)$ which appears as a polynomial in $\phi(\tau)$ when we return to $\infty$. Similar to (3.1), we obtain the equality

$$U_p f(\tau) = c_0 + \sum_{i=1}^{M} p^{\lambda_i/2-1} c_i \phi(\tau)^i$$

for some $c_i \in \mathbb{Z}$ and $M \in \mathbb{Z}^+$. The only difference between this equation and (3.1) is that in this more general case, we find that $U_p f$ is a higher-degree polynomial in $\phi$. This formula can easily be iterated as before to obtain the desired result. \[\square\]

3.2. Case 2: $m = p$.

**Proof.** Again, for ease of notation, denote $f_{0,p}^{(p)}(\tau)$ by $f(\tau)$. For the $m = p$ case, we will proceed as before; however, we will find that $U_p f(\tau)$ has poles at both $\infty$ and $0$ and that $U_p f(\tau)$ does not possess any interesting divisibility properties, but $U_p^2 f(\tau)$ does. This property will manifest itself as the “shifting” previously mentioned.

Notice first that $U_p f(\tau) = q^{-1} + O(1)$ has a simple pole at $\infty$. Therefore, we shall deal primarily with the function $U_p f(\tau) - \psi(\tau)$, which is holomorphic at $\infty$. We can use Lemmas 3 and 4 to view this function at $0$:

$$p(U_p f) \left( \frac{-1}{p \tau} \right) - p \psi \left( \frac{-1}{p \tau} \right) = p(U_p f)(p \tau) + f \left( \frac{-1}{p^2 \tau} \right) - f(\tau) - p^{\lambda/2+1} \phi(\tau)$$

$$= pq^{-p} + O(1) + O(1) - q^{-p} + O(1) + O(q)$$

$$= c_0 + \sum_{i=1}^{p} c_i \psi(\tau)^i$$

for some integers $c_i$. Replacing $\tau$ by $-1/(p \tau)$, we obtain

$$\tag{3.2} (U_p f)(\tau) = \frac{c_0}{p} + \psi(\tau) + \sum_{i=1}^{p} c_i p^{\lambda_i/2-1} \phi(\tau)^i.$$ 

The $\psi(\tau)$ term in the equation makes any attempt at $p$-divisibility fail; for example, computation shows that the $7^{th}$ coefficient of $\psi^{(2)}(\tau)$ is odd. However, $\psi(\tau)$ satisfies Lehner’s divisibility properties, so $U_p f$ inherits its $p$-divisibility from $\psi(\tau)$. So the function

$$U_p^2 f(\tau) = c_0 + U_p \psi(\tau) + \sum_{i=1}^{p} c_i p^{\lambda_i/2-1} U_p \phi(\tau)^i$$

has the same $p$-divisibility as $f_{0,1}^{(p)}$, hence, the shift. \[\square\]

3.3. Case 3: $m = p^\alpha m'$.

**Proof.** We prove this case using induction on $\alpha$. Case 1 showed that the theorem is true for all $m'$ relatively prime to $p$, so the $\alpha = 0$ base case is complete. Assume Theorem 2 holds for all $m$ of the form $m = p^\ell m'$ with $\ell < \alpha$. We will show that it holds for $m = p^\alpha m'$. To simplify notation, let $f_{\alpha}(\tau) = f_{0,p^\alpha m'}^{(p)}(\tau)$.

Since $f_{\alpha}(\tau) = q^{-p^{\alpha-1} m'} + O(1)$, we find that $U_p f_{\alpha}(\tau) = q^{-p^{\alpha-1} m'} + O(1)$ has a pole of order $p^{\alpha-1} m'$ at $\infty$. So we focus our attention on $U_p f_{\alpha}(\tau) - f_{\alpha-1}(\tau)$, which
is holomorphic at \( \infty \). Using (2.3) we examine this function at 0:

\[
p(U_\alpha f_\alpha) \left( \frac{-1}{p\tau} \right) - pf_\alpha - 1 \left( \frac{-1}{p\tau} \right) = p(U_\alpha f_\alpha)(p\tau) + f_\alpha \left( \frac{-1}{p^2\tau} \right) - f_\alpha(\tau) - pf_\alpha - 1 \left( \frac{-1}{p\tau} \right) = pq^{-p^m'} + O(1) + O(1) - q^{-p^m'} - O(1) - O(1) = (p - 1)q^{-p^m'} + O(1).
\]

As before, we write this function as a polynomial in \( \psi(\tau) \) with integral coefficients \( c_i \):

\[
p(U_\alpha f_\alpha) \left( \frac{-1}{p\tau} \right) - pf_\alpha - 1 \left( \frac{-1}{p\tau} \right) = c_0 + \sum_{i=1}^{p^m'} c_i \psi(\tau)^i,
\]

which, after switching back to the \( q \)-expansion at \( \infty \), becomes

\[
U_\alpha f_\alpha(\tau) = \frac{c_0}{p} + f_\alpha - 1(\tau) + \frac{1}{p^2} \sum_{i=1}^{p^m'} c_i p^{\lambda_i/2} \phi(\tau)^i.
\]

Notice that (3.3) looks very similar to (3.2), where \( \psi(\tau) \) is replaced by \( f_\alpha - 1(\tau) \), so \( U_\alpha f_\alpha(\tau) \) inherits whatever divisibility properties \( f_\alpha - 1(\tau) \) has. Our inductive hypothesis states that \( f_\alpha - 1(\tau) \) exhibits Lehner’s divisibility properties only after \( U_\alpha \) is applied \( \alpha - 1 \) times. Therefore, applying \( U_\alpha \) to (3.3) \( \alpha - 1 \) times, we obtain

\[
U_\alpha^\alpha f_\alpha(\tau) = \frac{c_0}{p} + U_\alpha^-1 f_\alpha - 1(\tau) + \frac{1}{p} \sum_{i=1}^{p^m'} c_i p^{\lambda_i/2} U_\alpha^-1 \phi(\tau)^i,
\]

showing that \( U_\alpha^\alpha f_\alpha(\tau) \) exhibits Lehner’s divisibility properties. \( \square \)

References


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