

## MAXIMAL EIGENVALUES OF A CASIMIR OPERATOR AND MULTIPLICITY-FREE MODULES

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ABSTRACT. Let  $\mathfrak{g}$  be a finite-dimensional complex semisimple Lie algebra and  $\mathfrak{b}$  a Borel subalgebra. Then  $\mathfrak{g}$  acts on its exterior algebra  $\wedge \mathfrak{g}$  naturally. We prove that the maximal eigenvalue of the Casimir operator on  $\wedge \mathfrak{g}$  is one third of the dimension of  $\mathfrak{g}$ , that the maximal eigenvalue  $m_i$  of the Casimir operator on  $\wedge^i \mathfrak{g}$  is increasing for  $0 \leq i \leq r$ , where  $r$  is the number of positive roots, and that the corresponding eigenspace  $M_i$  is a multiplicity-free  $\mathfrak{g}$ -module whose highest weight vectors correspond to certain ad-nilpotent ideals of  $\mathfrak{b}$ . We also obtain a result describing the set of weights of the irreducible representation of  $\mathfrak{g}$  with highest weight a multiple of  $\rho$ , where  $\rho$  is one half the sum of positive roots.

### 1. INTRODUCTION

Let  $\mathfrak{g}$  be a finite-dimensional complex semisimple Lie algebra and  $U(\mathfrak{g})$  its universal enveloping algebra. The study of the  $\mathfrak{g}$ -module structure of its exterior algebra  $\wedge \mathfrak{g}$  has a long history. Although this module structure is still not fully understood, Kostant has done a lot of important work on it; see for example [2] and [3].

Let  $Cas \in U(\mathfrak{g})$  be the Casimir element with respect to the Killing form. Let  $m_i$  be the maximal eigenvalue of  $Cas$  on  $\wedge^i \mathfrak{g}$  and  $M_i$  be the corresponding eigenspace. Let  $p$  be the maximal dimension of commutative subalgebras of  $\mathfrak{g}$ . In [2] it is proved that  $m_i \leq i$  for any  $i$  and  $m_i = i$  for  $0 \leq i \leq p$ , and if  $m_i = i$ , then  $M_i$  is a multiplicity-free  $\mathfrak{g}$ -module whose highest weight vectors correspond to  $i$ -dimensional abelian ideals of  $\mathfrak{b}$ . The integer  $p$  for all the simple Lie algebras was determined by Malcev, and Suter gave a uniform formula for  $p$  in [6].

Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and a set  $\Delta^+$  of positive roots. Let  $\rho \in \mathfrak{h}^*$  be one half the sum of all the positive roots. For any  $\lambda \in \mathfrak{h}^*$ , let  $V_\lambda$  denote the irreducible representation of  $\mathfrak{g}$  with highest weight  $\lambda$ .

In this paper we will prove the following result, which extends some theorems of Kostant. Let  $n = \dim \mathfrak{g}$ ,  $l = \text{rank } \mathfrak{g}$  and  $r$  be the number of positive roots.

**Theorem 1.1** (Theorem 3.2). *(1) One has  $m_i \leq n/3$  for  $i = 0, 1, \dots, n$ , and  $m_i = n/3$  if and only if  $i = r, r+1, \dots, r+l$ . For  $s = 0, 1, \dots, l$ ,  $M_{r+s} = \binom{l}{s} V_{2\rho}$ .*

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(2) For  $0 \leq i < r$  one has  $m_i < m_{i+1}$ . For  $1 \leq i \leq r$ ,  $M_i$  is a multiplicity-free  $\mathfrak{g}$ -module, whose highest weight vectors correspond to certain ad-nilpotent ideals of  $\mathfrak{b}$ . In fact  $\bigoplus_{i=0}^r M_i$  is also a multiplicity-free  $\mathfrak{g}$ -module.

Note that Kostant already proved in Remark 1.2 of [4] that  $m_i \leq n/3$  for all  $i$ . Thus (1) of this theorem provides further information about the corresponding eigenspace of the maximal eigenvalue  $n/3$ .

This result relates  $M_i$  to ad-nilpotent ideals of  $\mathfrak{b}$ , which are classified in [5]. But it will be complicated to determine those ad-nilpotent ideals of  $\mathfrak{b}$  corresponding to the highest weight vectors of  $M_i$ .

To prove this theorem, we need the following interesting result.

**Proposition 1.2** (Proposition 2.1). *Let  $k \in \mathbb{Z}^+$ . The set of weights of  $V_{k\rho}$  (whose dimension is  $(k + 1)^r$ ) is*

$$\left\{ \sum_{i=1}^r c_i \alpha_i \mid \alpha_i \in \Delta^+, c_i = -k/2, -k/2 + 1, \dots, k/2 - 1, k/2 \right\}.$$

2. WEIGHTS OF A REPRESENTATION  
WITH HIGHEST WEIGHT A MULTIPLE OF  $\rho$

Let  $\mathfrak{g}$  be a finite-dimensional complex semisimple Lie algebra. Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and a Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}$  containing  $\mathfrak{h}$ . Let  $\Delta$  be the set of roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  and  $\Delta^+$  be the set of positive roots whose corresponding root spaces lie in  $\mathfrak{b}$ . Let  $W$  be the Weyl group. Let  $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ . Then  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ . An ideal of  $\mathfrak{b}$  contained in  $\mathfrak{n}$  is called an ad-nilpotent ideal of  $\mathfrak{b}$ , as it consists of ad-nilpotent elements. Let  $\Gamma \subset \mathfrak{h}^*$  be the lattice of  $\mathfrak{g}$ -integral linear forms on  $\mathfrak{h}$  and  $\Lambda \subset \Gamma$  the subset of dominant integral linear forms. Let  $(, )$  be the bilinear form on  $\mathfrak{h}^*$  induced by the Killing form. Let  $l = \dim \mathfrak{h}$ ,  $r = |\Delta^+|$  and  $n = l + 2r = \dim \mathfrak{g}$ . Assume  $\Delta^+ = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ .

For any  $\lambda \in \Lambda$ , let  $\pi_\lambda : \mathfrak{g} \rightarrow \text{End}(V_\lambda)$  be the irreducible representation of  $\mathfrak{g}$  with highest weight  $\lambda$ , and  $\Gamma(V_\lambda)$  be the set of weights, with multiplicities. Any  $\gamma \in \Gamma(V_\lambda)$  will appear  $k$  times if the dimension of the  $\gamma$ -weight space is  $k$ . For example  $\Gamma(\mathfrak{g}) = \Delta \cup \{0, \dots, 0\}$  ( $l$  times). If  $U \subset V_\lambda$  is an  $\mathfrak{h}$ -invariant subspace, then we will also use  $\Gamma(U)$  to denote the the set of weights of  $U$  with multiplicities and define

$$\langle U \rangle = \sum_{\gamma \in \Gamma(U)} \gamma.$$

For any  $S \subset \Gamma(V_\lambda)$ , we also define  $\langle S \rangle = \sum_{\gamma \in S} \gamma$ .

Let  $\rho \in \mathfrak{h}^*$  be one half the sum of all the positive roots. For any  $k \in \mathbb{Z}^+$ , the representation  $V_{k\rho}$  of  $\mathfrak{g}$  has dimension  $(k + 1)^r$  by Weyl's dimension formula. The following result describes the set of weights of  $V_{k\rho}$ , which is well-known if  $k = 1$  (see e.g. [1]).

**Proposition 2.1.** *The set of weights of  $V_{k\rho}$  is*

$$\Gamma(V_{k\rho}) = \left\{ \sum_{i=1}^r c_i \alpha_i \mid \alpha_i \in \Delta^+, c_i = -k/2, -k/2 + 1, \dots, k/2 - 1, k/2 \right\},$$

or equivalently,

$$\Gamma(V_{k\rho}) = \{k\rho - \sum_{i=1}^r c_i \alpha_i \mid \alpha_i \in \Delta^+, c_i = 0, 1, \dots, k.\}.$$

*Proof.* By Weyl's denominator formula,

$$\prod_{i=1}^r (e^{\frac{k+1}{2}\alpha_i} - e^{-\frac{k+1}{2}\alpha_i}) = \sum_{w \in W} \text{sgn}(w) e^{w((k+1)\rho)}.$$

Then for  $c_i = -k/2, -k/2 + 1, \dots, k/2 - 1, k/2$  with  $i = 1, \dots, r$ ,

$$\begin{aligned} \sum_{c_1, \dots, c_r} e^{\sum_{i=1}^r c_i \alpha_i} &= \prod_{i=1}^r (e^{(-\frac{k}{2})\alpha_i} + e^{(-\frac{k}{2}+1)\alpha_i} + \dots + e^{(\frac{k}{2}-1)\alpha_i} + e^{(\frac{k}{2})\alpha_i}) \\ &= \prod_{i=1}^r \frac{e^{\frac{k+1}{2}\alpha_i} - e^{-\frac{k+1}{2}\alpha_i}}{e^{\frac{1}{2}\alpha_i} - e^{-\frac{1}{2}\alpha_i}} \\ &= \frac{\sum_{w \in W} \text{sgn}(w) e^{w((k+1)\rho)}}{\prod_{i=1}^r (e^{\frac{1}{2}\alpha_i} - e^{-\frac{1}{2}\alpha_i})} \\ &= \text{char}(V_{k\rho}). \quad \square \end{aligned}$$

Let  $Cas \in U(\mathfrak{g})$  be the Casimir element corresponding to the Killing form. For any  $\lambda \in \Gamma$ , define

$$Cas(\lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho).$$

The following result is well-known.

**Lemma 2.2.** *If  $\lambda \in \Lambda$ , then  $Cas(\lambda)$  is the scalar value taken by  $Cas$  on  $V_\lambda$ . For any  $\mu \in \Gamma(V_\lambda)$  one has  $Cas(\mu) \leq Cas(\lambda)$  and  $Cas(\mu) < Cas(\lambda)$  if  $\mu \neq \lambda$ .*

### 3. MAXIMAL EIGENVALUES OF A CASIMIR OPERATOR AND THE CORRESPONDING EIGENSPACES

Let  $\wedge \mathfrak{g}$  be the exterior algebra of  $\mathfrak{g}$ . Then  $\mathfrak{g}$  acts on  $\wedge \mathfrak{g}$  naturally. Let  $m_i$  be the maximal eigenvalue of  $Cas$  on  $\wedge^i \mathfrak{g}$  and  $M_i$  be the corresponding eigenspace.

One knows that  $\wedge^i \mathfrak{g}$  is isomorphic to  $\wedge^{n-i} \mathfrak{g}$  as  $\mathfrak{g}$ -modules for each  $i$ , so one has

$$m_i = m_{n-i}$$

and

$$M_i \cong M_{n-i}.$$

Let  $p$  be the maximal dimension of commutative subalgebras of  $\mathfrak{g}$ . Kostant showed that  $m_i \leq i$  and  $m_i = i$  for  $0 \leq i \leq p$ , and if  $m_i = i$ , then  $M_i$  is spanned by  $\wedge^k \mathfrak{a}$ , where  $\mathfrak{a}$  runs through  $k$ -dimensional commutative subalgebras of  $\mathfrak{g}$ .

A nonzero vector  $w \in \wedge \mathfrak{g}$  is called decomposable if  $w = z_1 \wedge z_2 \wedge \dots \wedge z_k$  for some positive integer  $k$ , where  $z_i \in \mathfrak{g}$ . In this case let  $\mathfrak{a}(w)$  be the corresponding  $k$ -dimensional subspace spanned by  $z_1, z_2, \dots, z_k$ .

**Theorem 3.1** (Proposition 6 and Theorem 7 of [2]). (1) Let

$$w = z_1 \wedge z_2 \wedge \cdots \wedge z_k \in \wedge^k \mathfrak{g}$$

be a decomposable vector. Then  $w$  is a highest weight vector if and only if  $\mathfrak{a}(w)$  is  $\mathfrak{b}$ -normal, i.e.,  $[\mathfrak{b}, \mathfrak{a}(w)] \subset \mathfrak{a}(w)$ . In this case the highest weight of the simple  $\mathfrak{g}$ -module generated by  $w$  is  $\langle \mathfrak{a}(w) \rangle$ .

Thus there is a one-to-one correspondence between all the decomposably-generated simple  $\mathfrak{g}$ -submodules of  $\wedge^k \mathfrak{g}$  and all the  $k$ -dimensional  $\mathfrak{b}$ -normal subspaces of  $\mathfrak{g}$ .

(2) Let  $\mathfrak{a}_1, \mathfrak{a}_2$  be any two ideals of  $\mathfrak{b}$  lying in  $\mathfrak{n}$ . Then  $\langle \mathfrak{a}_1 \rangle = \langle \mathfrak{a}_2 \rangle$  if and only if  $\mathfrak{a}_1 = \mathfrak{a}_2$ . Thus, if  $V_1 \subset \wedge^k \mathfrak{g}, V_2 \subset \wedge^j \mathfrak{g}$  are two decomposably-generated simple  $\mathfrak{g}$ -submodules which correspond to ideals of  $\mathfrak{b}$  lying in  $\mathfrak{n}$ , then  $V_1$  is equivalent to  $V_2$  if and only if  $V_1 = V_2$ .

**Theorem 3.2.** (1) One has

$$m_i = \max\{\|\rho + \gamma_1 + \cdots + \gamma_i\|^2 - \|\rho\|^2 \mid \{\gamma_t \mid t = 1, \dots, i\} \subset \Gamma(\mathfrak{g})\}$$

for any  $i$ .

(2) One has  $m_i \leq n/3$  for  $i = 0, 1, \dots, n$ , and  $m_i = n/3$  if and only if  $i = r, r + 1, \dots, r + l$ . For  $s = 0, 1, \dots, l$ ,  $M_{r+s} = \binom{l}{s} V_{2\rho}$ .

(3) For  $0 \leq k < r$  one has  $m_k < m_{k+1}$ . For  $1 \leq k \leq r$ ,  $M_k$  is a multiplicity-free  $\mathfrak{g}$ -module, whose highest weight vectors correspond to those  $k$ -dimensional ad-nilpotent ideals  $\mathfrak{a}$  of  $\mathfrak{b}$  such that  $Cas(\langle \mathfrak{a} \rangle) = m_k$ . In fact  $\bigoplus_{k=0}^r M_k$  is also a multiplicity-free  $\mathfrak{g}$ -module.

*Proof.* (1) For  $j = 1, \dots, r$ , let  $x_j$  (resp.  $y_j$ ) be a weight vector corresponding to  $\alpha_j$  (resp.  $-\alpha_j$ ). Let  $\{h_1, \dots, h_l\}$  be a basis of  $\mathfrak{h}$ . Then

$$A = \{x_1, \dots, x_r, y_1, \dots, y_r, h_1, \dots, h_l\}$$

is a basis of  $\mathfrak{g}$  consisting of weight vectors. Then

$$B_i = \{a_1 \wedge a_2 \wedge \cdots \wedge a_i \mid a_j \in A\}$$

is a basis of  $\wedge^i \mathfrak{g}$  consisting of weight vectors. Let

$$C_i = \{v \in B_i \mid Cas(\langle \mathfrak{a}(v) \rangle) = m_i\}.$$

Then by Corollary 2.1 of [2],  $M_i$  is the direct sum of simple  $\mathfrak{g}$ -modules with highest weight vectors  $v \in C_i$ . It is clear that

$$Cas(\langle \mathfrak{a}(v) \rangle) = \|\rho + \gamma_1 + \cdots + \gamma_i\|^2 - \|\rho\|^2$$

if the weight of  $a_j$  is  $\gamma_j$ ; thus (1) follows.

(2) For any  $S = \{\gamma_j \mid j = 1, \dots, i\} \subset \Gamma(\mathfrak{g})$ ,  $\langle S \rangle$  is a weight of  $\pi_{2\rho}$  by Proposition 2.1. Thus by Lemma 2.2,  $Cas(\langle S \rangle) \leq Cas(2\rho) = 8\|\rho\|^2 = n/3$ , as  $\|\rho\|^2 = n/24$  by Freudenthal's strange formula. So  $m_i = n/3$  if and only if there exists  $S \subset \Gamma(\mathfrak{g})$  such that  $|S| = i$  and  $\langle S \rangle = 2\rho$ . Then  $S$  must be of the form  $\{x_1, \dots, x_r, h_{j_1}, \dots, h_{j_s}\}$  and thus  $r \leq i \leq r + l$ . For  $0 \leq s \leq l$ , it is clear that

$$C_{r+s} = \{x_1 \wedge \cdots \wedge x_r \wedge h_{j_1} \wedge \cdots \wedge h_{j_s} \mid 1 \leq j_1 < j_2 < \cdots < j_s \leq l\};$$

thus  $M_{r+s} = \binom{l}{s} V_{2\rho}$ .

(3) We first show that  $m_{k+1} > m_k$  for  $0 \leq k < r$ , which clearly holds in the case  $k = 0$ . Assume  $1 \leq k < r$ . Let  $v = a_1 \wedge \cdots \wedge a_k \in C_k$ . Then  $v$  is a highest weight vector of  $M_k$ , whose weight is  $\langle S \rangle$  with  $S = \Gamma(\mathfrak{a}(v))$ . Then  $Cas(\langle S \rangle) = m_k$ , and  $[\mathfrak{b}, \mathfrak{a}(v)] \subset \mathfrak{a}(v)$  by Theorem 3.1 (1). Recall that for  $\gamma = \sum_{i=1}^l k_i \gamma_i \in \Delta^+$ , where  $\{\gamma_i | i = 1, \dots, l\}$  is the set of simple roots, its height is defined as  $\sum_{i=1}^l k_i$ . Choose a positive root  $\alpha$  in  $\Delta^+ \setminus S$  (which is nonempty as  $k < r$ ) with largest height. Set  $T = S \cup \{\alpha\}$ . Let  $a \in A$  be the  $\alpha$ -weight vector and let  $u = v \wedge a \in B_{k+1}$ . By the choice of  $\alpha$  it is clear that  $[\mathfrak{b}, \mathfrak{a}(u)] \subset \mathfrak{a}(u)$ ; thus  $u$  is also a highest weight vector, whose weight is  $\langle T \rangle = \langle S \rangle + \alpha$ . As

$$\langle T \rangle, \alpha = \langle S \rangle, \alpha + \alpha, \alpha > 0,$$

$\langle S \rangle \in \Gamma(V_\lambda)$  with  $\lambda = \langle T \rangle$ . Then

$$m_{k+1} \geq Cas(\langle T \rangle) > Cas(\langle S \rangle) = m_k.$$

Now assume  $1 \leq k \leq r$ . Let  $v = a_1 \wedge \cdots \wedge a_k \in C_k$ , and let  $S = \Gamma(\mathfrak{a}(v))$ . We will show that  $S \subset \Delta^+$ . If not, let  $S' = S \setminus (S \cap (-S))$ . Then  $\langle S' \rangle = \langle S \rangle$  and  $|S'| = t < k$ . Thus  $m_k = Cas(S) = Cas(S') \leq m_t$ , which contradicts the previous result. Thus for  $1 \leq k \leq r$  one always has  $S \subset \Delta^+$ .

Any  $v \in C_k$  is a highest weight vector, so  $[\mathfrak{b}, \mathfrak{a}(v)] \subset \mathfrak{a}(v)$ . If  $1 \leq k \leq r$ , we have just showed  $\Gamma(\mathfrak{a}(v)) \subset \Delta^+$ . Thus  $\mathfrak{a}(v)$  is an ad-nilpotent ideal of  $\mathfrak{b}$ . Let  $\lambda(v) = \langle \mathfrak{a}(v) \rangle$ . Then

$$M_k = \bigoplus_{v \in C_k} V_{\lambda(v)}.$$

By Theorem 3.1 (2), if  $v_1, v_2 \in C_k$  with  $v_1 \neq v_2$ , then  $\mathfrak{a}(v_1) \neq \mathfrak{a}(v_2)$  and  $\lambda(v_1) \neq \lambda(v_2)$ . Thus  $M_k$  is a multiplicity-free  $\mathfrak{g}$ -module, whose highest weight vectors corresponding to the ad-nilpotent ideals  $\mathfrak{a}$  of  $\mathfrak{b}$  such that  $Cas(\langle \mathfrak{a} \rangle) = m_k$ . By Theorem 3.1 (2) one can further get that  $\bigoplus_{k=0}^r M_k$  is also a multiplicity-free  $\mathfrak{g}$ -module. □

Note that Kostant already showed in Remark 1.2 of [4] that  $m_i \leq n/3$  for all  $i$  using different arguments.

*Remark 3.3.* Considering the isomorphism of  $\mathfrak{g}$ -modules  $\wedge^k \mathfrak{g}$  and  $\wedge^{n-k} \mathfrak{g}$ ,  $\wedge^k \mathfrak{g}$  is multiplicity-free for  $0 \leq k \leq r$  and  $n-r \leq k \leq n$ . For  $r \leq k \leq r+l$  ( $r+l = n-r$ ), we have showed that  $M_k$  is primary of type  $\pi_{2\rho}$ . As  $\mathfrak{g}$ -modules one has  $\wedge \mathfrak{g} = 2^l V_\rho \otimes V_\rho$  (see [3]), so  $\wedge \mathfrak{g}$  contains exactly  $2^l$  copies of  $V_{2\rho}$ , which is just  $\bigoplus_{s=0}^l M_{r+s}$ .

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