The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n \ (n > 2)$. Let $\mathcal{C}^n$ denote the set of nonempty convex figures (compact, convex subsets) and let $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. We reserve the letter $u$ for unit vectors, and the letter $B$ is reserved for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. The volume of the unit $n$-ball is denoted by $\omega_n$. For $u \in S^{n-1}$, let $E_u$ denote the hyperplane, through the origin, that is orthogonal to $u$. We will use $K_u$ to denote the image of $K$ under an orthogonal projection onto the hyperplane $E_u$.

We use $V(K)$ for the $n$-dimensional volume of a convex body $K$. Let $h(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$, i.e., for $u \in S^{n-1}$,

$$h(K, u) = \max \{ u \cdot x : x \in K \},$$

where $u \cdot x$ denotes the usual inner product of $u$ and $x$ in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K, u) = \max \{ \lambda \geq 0 : \lambda u \in K \}.$$

If $\rho(K, \cdot)$ is positive and continuous, $K$ will be called a star body. Let $S^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, i.e., if $K, L \in S^n$, then $\tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty$.

If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then there is a convex body $K \tilde{+} L$ such that

$$S(K \tilde{+} L, \cdot) = S(K, \cdot) + S(L, \cdot),$$

where $S(K, \cdot)$ denotes the surface area measure of $K$. The operation $\tilde{+}$ is called the Blaschke sum (see e.g. [13]).
If $K$ is a convex body that contains the origin in its interior, the polar body $K^*$ of $K$ is defined by
\[ K^* := \{ x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K \}. \]

1. DUAL MIXED VOLUMES

The radial Minkowski linear combination, $\lambda_1 K_1 + \cdots + \lambda_r K_r$, is defined by
\[ \lambda_1 K_1 + \cdots + \lambda_r K_r = \{ \lambda_1 x_1 + \cdots + \lambda_r x_r : x_i \in K_i, i = 1, \ldots, r \}, \]
for $K_1, \ldots, K_r \in S^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

It has the following important property (see [13]):
\[ \rho(\lambda K + \mu L, \cdot) = \rho(K, \cdot) + \mu \rho(L, \cdot), \]
for $K, L \in S^n$ and $\lambda, \mu \geq 0$.

For $K_1, \ldots, K_r \in S^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial of degree $n$ in the $\lambda_i$,
\[ V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \tilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \]
where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding $r$. If we require the coefficients of the polynomial in (1.3) to be symmetric in their argument, then they are uniquely determined. The coefficient $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is called the dual mixed volume $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$.

If $K_1, \ldots, K_n \in S^n$, the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ can be represented in the form (see [14])
\[ \tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \]

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume is written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume $\tilde{V}_i(K, L) = \tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$.

For $K, L \in S^n$, the $i$-th dual mixed volume of $K$ and $L$, $\tilde{V}_i(K, L)$, can be extended to all $i \in \mathbb{R}$ by
\[ \tilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i dS(u), \quad i \in \mathbb{R}. \]

Thus, if $K \in S^n$ and $i \in \mathbb{R}$, then (see [14])
\[ \tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u), \quad i \in \mathbb{R}. \]

If $K$ and $L$ are star bodies in $\mathbb{R}^n$, $s \neq 0$ and $\lambda, \mu \geq 0$, then $\lambda \cdot K +_s \mu \cdot L$ is the star body whose radial function is given by
\[ \rho(\lambda \cdot K +_s \mu \cdot L, \cdot)^s = \lambda \rho(K, \cdot)^s + \mu \rho(L, \cdot)^s. \]

The addition $+_s$ is called the $L_s$ radial sum.

The $L_s$ dual Brunn-Minkowski inequality states: If $K, L \in S^n$, then
\[ V(K +_s L)^{-s/n} \geq V(K)^{-s/n} + V(L)^{-s/n}, \]
with equality if and only if $K$ and $L$ are dilates.
2. Radial Blaschke-Minkowski homomorphisms

Definition 2.1 ([16]). A map \( \Psi : S^n \to S^n \) is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

1. \( \Psi \) is continuous.
2. For all \( K, L \in S^n \),
   \[
   \Psi(K \hat{+} L) = \Psi(K) \hat{+} \Psi(L),
   \]
   where \( \hat{+} \) denotes the \( L_{n-1} \) radial sum of \( K \) and \( L \).
3. For all \( K, L \in S^n \) and every \( \vartheta \in SO(n) \),
   \[
   \Psi(\vartheta K) = \vartheta \Psi(K),
   \]
where \( SO(n) \) is the group of rotations in \( n \) dimensions.

In 2006, Schuster [16] established the following Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms of star bodies.

If \( K \) and \( L \) are star bodies in \( \mathbb{R}^n \), then

\[
(2.1) \quad V(\Psi(K \hat{+} L))^{1/n(n-1)} \leq V(\Psi K)^{1/n(n-1)} + V(\Psi L)^{1/n(n-1)},
\]
with equality if and only if \( K \) and \( L \) are dilates.

In fact a more general version of the Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms holds (see [16]): If \( K \) and \( L \) are star bodies in \( \mathbb{R}^n \) and \( 0 \leq i \leq n-1 \), \( 0 \leq j < n-2 \), then

\[
(2.2) \quad W_i(\Psi_j(K \hat{+} L))^{1/(n-i)(n-j-1)} \leq W_i(\Psi_j K)^{1/(n-i)(n-j-1)} + W_i(\Psi_j L)^{1/(n-i)(n-j-1)},
\]
with equality if and only if \( K \) and \( L \) are dilates. Here \( \Psi_j \) denotes the mixed radial Blaschke-Minkowski homomorphism defined by:

**Theorem 2.2** (see [16]). Let \( \Psi : S^n \to S^n \) be a radial Blaschke-Minkowski homomorphism. There is a continuous operator \( \Psi : S^n \times \cdots \times S^n \to S^n \), symmetric in its arguments such that, for \( K_1, \ldots, K_m \in S^n \) and \( \lambda_1, \ldots, \lambda_m \leq 0 \),

\[
\Psi(\lambda_1 K_1 \hat{+} \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{n-1} \Psi(K_{i_1}, \ldots, K_{i_{n-1}}).
\]

Clearly, Theorem 2.2 generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call \( \Psi : S^n \times \cdots \times S^n \to S^n \) the mixed radial Blaschke-Minkowski homomorphism induced by \( \Psi \). The mixed radial Blaschke-Minkowski homomorphisms were first studied in more detail in [17]–[18]. If \( K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B \), we write \( \Psi_i K \) for \( \Psi(K, \ldots, K, B, \ldots, B) \) and call \( \Psi_i K \) the mixed Blaschke-Minkowski homomorphism of order \( i \) of \( K \). We write \( \Psi_i(K, L) \) for \( \Psi(K, \ldots, K, L, \ldots, L) \).

The aim of this paper is to establish the following new Brunn-Minkowski inequality for mixed radial Blaschke-Minkowski homomorphisms.
Theorem 2.3. If \( K, L \in S^n \) and \( i, j \in \mathbb{R}, s \in \mathbb{N} \) satisfy \( i \leq n - 1 \leq j \leq n, 0 \leq s \leq n - 1 \), then
\[
(2.3) \quad \left( \frac{\tilde{W}_i(\Psi_1 s(K^+L))}{\tilde{W}_j(\Psi_1 s(K^+L))} \right)^{1/(j-i)} \leq \left( \frac{\tilde{W}_i(\Psi_1 s K)}{\tilde{W}_j(\Psi_1 s K)} \right)^{1/(j-i)} + \left( \frac{\tilde{W}_i(\Psi_1 s L)}{\tilde{W}_j(\Psi_1 s L)} \right)^{1/(j-i)},
\]
with equality if and only if \( K \) and \( L \) are dilates.

3. POLAR BLASCHKE-MINKOWSKI HOMOMORPHISMS

Definition 3.1 (see [16]). A map \( \Phi : \mathcal{K}_n \to \mathcal{K}_n \) is called a Blaschke-Minkowski homomorphism if it satisfies the following conditions:
(a) \( \Phi \) is continuous.
(b) For all \( K, L \in \mathcal{K}_n \),
\[ \Phi(K+L) = \Phi(K) + \Phi(L). \]
(c) For all \( K, L \in \mathcal{K}_n \) and every \( \vartheta \in SO(n) \),
\[ \Phi(\vartheta K) = \vartheta \Phi(K). \]

In [16] it was shown that the polar body \((\Phi K)^*\) is well defined for every Blaschke-Minkowski homomorphism \( \Phi \) and \( K \in \mathcal{K}_n \). In the following we simply write \( \Phi^* K \) rather than \((\Phi K)^*\).

In 2006, Schuster [16] also established the following Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms \( \Phi \) of convex bodies.

If \( K \) and \( L \) are convex bodies in \( \mathbb{R}^n \), then
\[
(3.1) \quad V(\Phi^*(K + L))^{-1/n(n-1)} \geq V(\Phi^* K)^{1/(n(n-1))} + V(\Phi^* L)^{1/n(n-1)},
\]
with equality if and only if \( K \) and \( L \) are homothetic.

In fact a more general version of the Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms holds (see [16]): If \( K \) and \( L \) are convex bodies in \( \mathbb{R}^n \) and \( 0 \leq j \leq n - 3 \), then
\[
(3.2) \quad V(\Phi_j^*(K + L))^{-1/n(n-j-1)} \geq V(\Phi_j^* K)^{1/(n(n-j-1))} + V(\Phi_j^* L)^{1/n(n-j-1)},
\]
with equality if and only if \( K \) and \( L \) are homothetic. Here, \( \Phi_j K \) denotes the mixed Blaschke-Minkowski homomorphism induced by \( \Phi \) defined by:

Theorem 3.2 (see [16]). There is a continuous operator \( \Phi : \mathcal{K}_n^* \times \cdots \times \mathcal{K}_n^* \), symmetric in its arguments such that, for \( K_1, \ldots, K_r \) and \( \lambda_1, \ldots, \lambda_r \geq 0 \),
\[
\Phi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \ldots, K_{i_{n-1}}).
\]

Clearly, Theorem 3.2 generalizes the notion of Blaschke-Minkowski homomorphisms. We call \( \Phi : \mathcal{K}_n^* \times \cdots \times \mathcal{K}_n^* \to \mathcal{K}_n^* \) the mixed Blaschke-Minkowski homomorphism induced by \( \Phi \). Mixed Blaschke-Minkowski homomorphisms were first studied in more detail in [13]. If \( K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B \), we write \( \Phi_i K \) for \( \Phi(K, \ldots, K, B, \ldots, B) \) and call \( \Phi_i K \) the mixed Blaschke-Minkowski homomorphism of order \( i \). We write \( \Phi_i(K, L) \) for \( \Phi(K, \ldots, K, L, \ldots, L) \) and write \( \Phi_0 K = \Phi K \).
Blaschke-Minkowski homomorphisms are an important notion in the theory of convex body-valued valuations (see, e.g., [5–6], [9–10], [15], [19], [22] and [1–2], [7–8], [11–12], [20–21]). They are natural duals to radial Blaschke-Minkowski homomorphisms which are important examples of star body-valued valuations.

Another aim of this paper is to establish the following Brunn-Minkowski inequality for polars of even Blaschke-Minkowski homomorphisms.

**Theorem 3.3.** Let $K, L$ be convex bodies in $\mathbb{R}^n$ and $i, j \in \mathbb{R}$ satisfy $i \geq n + 1 \geq j \geq n$. Then

$$
\left( \frac{W_i(\Phi^*(K+L))}{W_j(\Phi^*(K+L))} \right)^{1/(i-j)} \leq \left( \frac{W_i(\Phi^*K)}{W_j(\Phi^*K)} \right)^{1/(i-j)} + \left( \frac{W_i(\Phi^*L)}{W_j(\Phi^*L)} \right)^{1/(i-j)},
$$

with equality if and only if $K$ and $L$ are homothetic.

4. **BRUNN–MINKOWSKI TYPE INEQUALITIES FOR RADIAL AND POLAR BLASCHKE–MINKOWSKI HOMOMORPHISMS**

An extension of Beckenbach’s inequality (see [3], p. 27) was obtained by Dresher [4] by means of moment-space techniques:

**Lemma 4.1** (The Beckenbach-Dresher inequality). If $p \geq 1 \geq r \geq 0$, $f, g \geq 0$, and $\phi$ is a distribution function, then

$$
\left( \frac{\int (f+g)^p d\phi}{\int (f+g)^r d\phi} \right)^{1/(p-r)} \leq \left( \frac{\int f^p d\phi}{\int f^r d\phi} \right)^{1/(p-r)} + \left( \frac{\int g^p d\phi}{\int g^r d\phi} \right)^{1/(p-r)},
$$

with equality if and only if the functions $f$ and $g$ are proportional.

**Lemma 4.2** (see [16]). If $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a Blaschke-Minkowski homomorphism, then there is a function $g \in \mathcal{C}(S^n-1, \hat{e})$ such that

$$
h(\Phi K, \cdot) = S_n-1(K, \cdot)^{n-1} \ast g,
$$

where $\mathcal{C}(S^n-1, \hat{e})$ denotes the set of continuous zonal functions on $S^n-1$.

As a consequence of Lemma 4.2, we have for the mixed Blaschke-Minkowski homomorphism induced by $\Phi$,

$$
h(\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}; \cdot)^{n-1} \ast g,
$$

where $S(K_1, \ldots, K_{n-1}; \cdot)$ is the mixed surface area measure of $K_1, \ldots, K_{n-1}$.

Let $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \rightarrow \mathcal{K}^n$ be a mixed Blaschke-Minkowski homomorphism. If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then (see [16])

$$
\rho(\Phi^*(K_1, \ldots, K_{n-1}), \cdot)^{-1} = h(\Phi^*(K_1, \ldots, K_{n-1}), \cdot).
$$

**Lemma 4.3** (see [16]). A map $\Psi : S^n \rightarrow S^n$ is a radial Blaschke-Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^n-1, \hat{e})$ such that

$$
\rho(\Psi K, \cdot) = \rho(K, \cdot)^{n-1} \ast \mu,
$$

where $\mathcal{M}_+(S^n-1, \hat{e})$ denotes the set of nonnegative zonal measures on $S^n-1$. 
For the mixed radial Blaschke-Minkowski homomorphism induced by $\Psi$, we have
\[
\rho(\Psi(K_1, \ldots, K_{n-1}), \cdot) = \rho(K_1, \cdot) \cdots \rho(K_{n-1}, \cdot) * \mu.
\]

We are now in a position to prove Theorem 2.3. The following statement is just a slight reformulation of Theorem 2.3.

**Theorem 4.4.** If $K, L \in S^n$ and $p, r \in \mathbb{R}, j \in \mathbb{N}$ satisfy $0 \leq r \leq 1 \leq p$, $0 \leq j \leq n - 1$, then
\[
\left( \frac{\tilde{W}_{n-p}(\Psi_j(K_jL))}{\tilde{W}_{n-r}(\Psi_j(K_jL))} \right)^{1/(p-r)} \leq \left( \frac{\tilde{W}_{n-p}(\Psi_jK)}{\tilde{W}_{n-r}(\Psi_jK)} \right)^{1/(p-r)} + \left( \frac{\tilde{W}_{n-p}(\Psi_jL)}{\tilde{W}_{n-r}(\Psi_jL)} \right)^{1/(p-r)},
\]
with equality if and only if $K$ and $L$ are dilates.

**Proof.** From (1.7), we have
\[
\rho(K^+sL, \cdot)^s * \mu = \rho(K, \cdot)^s * \mu + \rho(L, \cdot)^s * \mu, \ s \neq 0,
\]
where $\mu$ is defined in Lemma 4.3. Hence, from (1.2) and (4.6), we obtain
\[
\rho(\Psi_s(K^+sL), \cdot) = \rho(\Psi_sK, \cdot) + \rho(\Psi_sL, \cdot) = \rho(\Psi_sK + \Psi_sL, \cdot).
\]
Namely,
\[
\Psi_s(K^+sL) = \Psi_s(K) + \Psi_s(L).
\]
Therefore, from (1.2) and (1.6), we have
\[
\tilde{W}_{n-p}(\Psi_j(K^+jL)) = \frac{1}{n} \int_{S^{n-1}} \rho(\Psi_j(K^+jL), u)^p dS(u)
\]
\[
= \frac{1}{n} \int_{S^{n-1}} \rho(\Psi_jK^+jL, u)^p dS(u) = \frac{1}{n} \int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^p dS(u)
\]
and
\[
\tilde{W}_{n-r}(\Psi_j(K^+jL)) = \frac{1}{n} \int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^r dS(u).
\]
From (4.8) and (4.9) and in view of Lemma 4.1, we obtain
\[
\left( \frac{\tilde{W}_{n-p}(\Psi_j(K^+jL))}{\tilde{W}_{n-r}(\Psi_j(K^+jL))} \right)^{\frac{1}{p-r}} = \left( \frac{\int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^p dS(u)}{\int_{S^{n-1}} (\rho(\Psi_jK, u) + \rho(\Psi_jL, u))^r dS(u)} \right)^{\frac{1}{p-r}}
\]
\[
\leq \left( \frac{\int_{S^{n-1}} \rho(\Psi_jK, u)^p dS(u)}{\int_{S^{n-1}} \rho(\Psi_jK, u)^r dS(u)} \right)^{\frac{1}{p-r}} + \left( \frac{\int_{S^{n-1}} \rho(\Psi_jL, u)^p dS(u)}{\int_{S^{n-1}} \rho(\Psi_jL, u)^r dS(u)} \right)^{\frac{1}{p-r}}
\]
\[
= \left( \frac{\tilde{W}_{n-p}(\Psi_jK)}{\tilde{W}_{n-r}(\Psi_jK)} \right)^{\frac{1}{p-r}} + \left( \frac{\tilde{W}_{n-p}(\Psi_jL)}{\tilde{W}_{n-r}(\Psi_jL)} \right)^{\frac{1}{p-r}}.
\]
Equality holds if and only if the functions $\rho(\Psi_jK, u)$ and $\rho(\Psi_jL, u)$ are proportional. Namely, $\rho(\Psi_jK, u) = \lambda \rho(\Psi_jL, u)$. From (4.5), we obtain $\rho(\Psi_jK, u) = \rho(\Psi_j(\lambda^{1/(n-j-1)}L), u)$. Hence, equality holds if and only if $K$ and $L$ are dilates.

Let $p = n - i$ and $r = n - j$. Since $0 \leq r \leq 1 \leq p$, we have
\[
r \leq 1 \leq p \Rightarrow i \leq n - 1 \leq j, \ 0 \leq r \Rightarrow j \leq n.
\]
Therefore,

\begin{equation}
(4.10) \quad i \leq n - 1 \leq j \leq n.
\end{equation}

Taking for \( p = n - i \) and \( r = n - j \) in (4.7) and using (4.10), we see that (4.7) changes to the inequality in Theorem 2.3.

Taking for \( p = n - i, j = 0 \) and \( r = 1 \) in (4.7), (4.7) changes to the following inequality:

\begin{equation}
(4.11) \quad \left( \frac{\tilde{W}_i(\Psi(K+L))}{\tilde{W}_{n-1}(\Psi(K+L))} \right)^{1/(n-i-1)} \leq \left( \frac{\tilde{W}_i(\Psi K)}{\tilde{W}_{n-1}(\Psi K)} \right)^{1/(n-i-1)} + \left( \frac{\tilde{W}_i(\Psi L)}{\tilde{W}_{n-1}(\Psi L)} \right)^{1/(n-i-1)},
\end{equation}

with equality if and only if \( K \) and \( L \) are dilates.

For \( K \in S^n \), there is a unique star body \( IK \) whose radial function satisfies for \( u \in S^{n-1} \),

\[ \rho(IK, u) = v(K \cap E_u). \]

It is called the intersection bodies of \( K \). The volume of intersection bodies is given by

\[ V(IK) = \frac{1}{n} \int_{S^{n-1}} v(K \cap E_u)^n dS(u). \]

The mixed intersection body of \( K_1, \ldots, K_{n-1} \in S^n, I(K_1, \ldots, K_{n-1}) \), is defined by

\[ \rho(I(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u), \]

where \( \tilde{v} \) is \((n - 1)\)-dimensional dual mixed volume.

If \( K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L \), then \( I(K_1, \ldots, K_{n-1}) \) is written as \( I(I, K) \). If \( L = B \), then \( I_0(K, L) \) is written as \( I_0 K \) and called the ith intersection body of \( K \). For \( I_0 K \) we simply write \( IK \).

If \( \Psi : S^n \times \cdots \times S^n \rightarrow S^n \) is the mixed intersection operator \( I : S^n \times \cdots \times S^n \rightarrow S^n \) in (4.7), we obtain

**Corollary 4.5.** If \( K, L \in S^n \) and \( p, r \in \mathbb{R}, j \in \mathbb{N} \) satisfy \( 0 \leq r \leq 1 \leq p \), \( 0 \leq j \leq n - 1 \), then

\begin{equation}
(4.12) \quad \left( \frac{\tilde{W}_{n-p}(I_j(K+L))}{\tilde{W}_{n-r}(I_j(K+L))} \right)^{1/(p-r)} \leq \left( \frac{\tilde{W}_{n-p}(I_j K)}{\tilde{W}_{n-r}(I_j K)} \right)^{1/(p-r)} + \left( \frac{\tilde{W}_{n-p}(I_j L)}{\tilde{W}_{n-r}(I_j L)} \right)^{1/(p-r)},
\end{equation}

with equality if and only if \( K \) and \( L \) are dilates.

The following statement is just a slight reformulation of Theorem 3.3.

**Theorem 4.6.** If \( K, L \in \mathcal{K}^n \) and \( p, r \in \mathbb{R} \) satisfy \( p \leq -1 \leq r \leq 0 \), then

\begin{equation}
(4.13) \quad \left( \frac{\tilde{W}_{n-p}(\Phi^*(K+L))}{\tilde{W}_{n-r}(\Phi^*(K+L))} \right)^{1/(r-p)} \leq \left( \frac{\tilde{W}_{n-p}(\Phi^* K)}{\tilde{W}_{n-r}(\Phi^* K)} \right)^{1/(r-p)} + \left( \frac{\tilde{W}_{n-p}(\Phi^* L)}{\tilde{W}_{n-r}(\Phi^* L)} \right)^{1/(r-p)},
\end{equation}

with equality if and only if \( K \) and \( L \) are homothetic.
Proof. From (1.6), (4.4) and in view of Definition 3.1, we have
\[
\tilde{W}_{n-p}(\Phi^*(K+L)) = \frac{1}{n} \int_{S^{n-1}} \rho(\Phi^*(K+L), u)^p dS(u)
\]
\[
= \frac{1}{n} \int_{S^{n-1}} h(\Phi(K+L), u)^{-p} dS(u) = \frac{1}{n} \int_{S^{n-1}} h(\Phi K + \Phi L, u)^{-p} dS(u)
\]
(4.14) \quad \quad \quad \quad = \frac{1}{n} \int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-p} dS(u)
\]
and
\[
\tilde{W}_{n-r}(\Phi^*(K+L)) = \frac{1}{n} \int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-r} dS(u).
\]
(4.15)
From (4.14), (4.15), and Lemma 4.1, we obtain
\[
\left( \frac{\tilde{W}_{n-p}(\Phi^*(K+L))}{\tilde{W}_{n-r}(\Phi^*(K+L))} \right)^{1/(r-p)} = \left( \frac{\int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-p} dS(u)}{\int_{S^{n-1}} (h(\Phi K, u) + h(\Phi L, u))^{-r} dS(u)} \right)^{1/(r-p)}
\]
\[
\leq \left( \frac{\int_{S^{n-1}} h(\Phi K, u)^{-p} dS(u)}{\int_{S^{n-1}} h(\Phi K, u)^{-r} dS(u)} \right)^{1/(r-p)} + \left( \frac{\int_{S^{n-1}} h(\Phi L, u)^{-p} dS(u)}{\int_{S^{n-1}} h(\Phi L, u)^{-r} dS(u)} \right)^{1/(r-p)}
\]
\[
= \left( \frac{\tilde{W}_{n-p}(\Phi^* K)}{\tilde{W}_{n-r}(\Phi^* K)} \right)^{1/(r-p)} + \left( \frac{\tilde{W}_{n-p}(\Phi^* L)}{\tilde{W}_{n-r}(\Phi^* L)} \right)^{1/(r-p)}.
\]
Equality holds if and only if the functions \( h(\Phi^* K, u) \) and \( h(\Phi^* L, u) \) are proportional, namely, \( h(\Phi^* K, u) = \lambda h(\Phi^* L, u) \), and from (4.3), we obtain \( h(\Phi^* K, u) = \lambda h(\Phi^*(\lambda^{-1/(n-1)} L), u) \). Hence, equality holds if and only if \( K \) and \( L \) are homothetic.

Let \( p = n - i \) and \( r = n - j \). Since \( p \leq -1 \leq r \leq 0 \), we have
\[
(4.16) \quad \quad \quad \quad p \leq -1 \leq r \leq 0 \Rightarrow i \geq n + 1 \geq j \geq n.
\]
Taking for \( p = n - i \) and \( r = n - j \) in (4.13) and using (4.16), we see that (4.13) changes to the inequality in Theorem 3.3.

Taking \( p = -n, r = -1 \) and \( s = 0 \) in (4.13), we have
\[
(4.17) \quad \quad \quad \quad \left( \frac{\tilde{W}_{2n}(\Phi^*(K+L))}{\tilde{W}_{n+1}(\Phi^*(K+L))} \right)^{1/(n-1)} \leq \left( \frac{\tilde{W}_{2n}(\Phi^* K)}{\tilde{W}_{n+1}(\Phi^* K)} \right)^{1/(n-1)} + \left( \frac{\tilde{W}_{2n}(\Phi^* L)}{\tilde{W}_{n+1}(\Phi^* L)} \right)^{1/(n-1)},
\]
with equality if and only if \( K \) and \( L \) are homothetic.

If \( K_1, \ldots, K_r \in \mathbb{K}^n \) and \( \lambda_1, \ldots, \lambda_r \geq 0 \), then the projection body of the Minkowski linear combination \( \lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathbb{K}^n \) can be written as a symmetric homogeneous polynomial of degree \( (n-1) \) in the \( \lambda_i \) (see [15]):
\[
(4.18) \quad \quad \quad \quad \Pi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Pi_{i_1 \cdots i_{n-1}},
\]
where the sum is a Minkowski sum taken over all \((n-1)\)-tuples \((i_1, \ldots, i_{n-1})\) of positive integers not exceeding \( r \). The body \( \Pi_{i_1 \cdots i_{n-1}} \) depends only on the bodies \( K_{i_1}, \ldots, K_{i_{n-1}} \) and is uniquely determined by (4.18). It is called the mixed projection bodies of \( K_{i_1}, \ldots, K_{i_{n-1}} \), and is written as \( \Pi(K_{i_1}, \ldots, K_{i_{n-1}}) \).
If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_{i_1}, \ldots, K_{i_{n-1}})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, L)$ is denoted by $\Pi_i K$ and when $i = 0$, $\Pi_i K$ is denoted by $\Pi K$, where $\Pi K$ is the projection body of $K$.

If $\Phi : K^n \times \cdots \times K^n \rightarrow K^n$ is the mixed projection operator $\Pi : K^n \times \cdots \times K^n \rightarrow K^n$ in (4.13), we obtain

**Corollary 4.7.** If $K, L \in K^n$ and $p, r \in \mathbb{R}$ satisfy $p \leq -1 \leq r \leq 0$, then

\[
\left( \frac{\widehat{W}_{n-p}(\Pi^*(K^+L))}{\widehat{W}_{n-r}(\Pi^*(K+L))} \right)^{1/(r-p)} \leq \left( \frac{\widehat{W}_{n-p}(\Pi^* K)}{\widehat{W}_{n-r}(\Pi^* K)} \right)^{1/(r-p)} + \left( \frac{\widehat{W}_{n-p}(\Pi^* L)}{\widehat{W}_{n-r}(\Pi^* L)} \right)^{1/(r-p)},
\]

with equality if and only if $K$ and $L$ are homothetic.

We finally remark that inequalities for the intersection operator $I$ were also established in [23, 27], for the $L_p$-intersection operator $I_p$ in [21] and for the polar projection body operator $\Pi^*$ in [25–26].

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