COMPLETELY REGULAR PROPER REFLECTION OF LOCALES
OVER A GIVEN LOCALE

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Abstract. Let $X$ be a completely regular locale. We present a construction
which shows that every locale $f : Y \to X$ over $X$ has a completely regular
proper reflection in the slice category $\text{Loc}/X$ and the reflection map is a dense
embedding if and only if $Y$ is completely regular.

Let $X$ be a locale. We know that the category $\text{Loc}(\text{Sh}(X))$ of internal locales in
the sheaves topos $\text{Sh}(X)$ is equivalent to the slice category $\text{Loc}/X$ of locales over $X$
(see [1], [2]). Also a localic map $f : Y \to X$ is proper if and only if it corresponds to
a compact internal locale in the topos $\text{Sh}(X)$ (see [3]). By a classical result given
by B. Banaschewski and C. J. Mulvey in [3], we know that the category of compact
completely regular locales (compact regular locales) is reflective in the category of
locales. So the following question is interesting:

Dose there exist a compact completely regular reflection for every internal locale
in $\text{Sh}(X)$?

Suppose $f : Y \to X$ is a locale over $X$. Let $g : Z \to X$ be a proper map with
$Z$ completely regular. We call $g : Z \to X$ a completely regular proper reflection of
$f : Y \to X$ if there exists a localic map $h : Y \to Z$ such that $f = gh$,

\[
\begin{tikzcd}
Y \arrow{dr}{f} \arrow{rr}{h} & & Z \\
& X \arrow{ur}{g}
\end{tikzcd}
\]

and $h : Y \to Z$ is universal among all those localic maps from $f : Y \to X$ to a
completely regular proper map over $X$. Now we consider a similar question in the
slice category $\text{Loc}/X$ as follows.

Does there exist a completely regular proper reflection for every locale $f : Y \to X$
over $X$? If so, can we give an explicit description of the reflection?

Suppose $f : Y \to X$ is a proper surjection and $Y$ is regular. Then $X$ is regular by
proposition 2 in [9]. Hence if every locale over $X$ has a completely regular proper
reflection, then we take a surjection $g : Y \to X$ such that its completely regular
proper reflection $p : Z \to X$ will still be surjective. This requires $X$ to be regular,

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which shows that for a non-regular locale $X$ the answer to the above question is negative.

**Lemma 1.** If every locale over $X$ has a completely regular proper reflection (regular proper reflection), then $X$ is regular.

For $X$ being the terminal locale 2, $\text{Sh}(X) \cong \text{Set}$, then it is clear that the compact completely regular reflection for any locale $Y$ is just the Stone–Čech compactification of $Y$. In this paper we will show that if $X$ is completely regular, then every locale $f : Y \to X$ over $X$ has a completely regular proper reflection $P_f : P_Y \to X$ in the slice category $\text{Loc}/X$. Moreover, the reflection map $\varepsilon_f : Y \to P_Y$ is a dense embedding if and only if $Y$ is completely regular. In this case, $P_f : P_Y \to X$ becomes a maximal proper reflection of $f : Y \to X$ like that in the classical case.

Recall that a localic map $f : X \to Y$ is said to be proper whenever $f$ is closed and the right adjoin $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ of the frame morphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ preserves directed joins. This is a point-free version of the classical notion of what are called perfect maps in the category of topological spaces. Properness of $f : X \to Y$ has the following characterizations (see [6], [7], [8]):

(i) $f$ is proper;
(ii) $f$ is stably closed; i.e., its pullback along any morphism with codomain $Y$ is closed;
(iii) the internal locale in the topos $\text{Sh}(Y)$ corresponding to $f$ is compact;
(iv) $\text{id}_Z \times f : Z \times X \to Z \times Y$ is closed for every locales $Z$.

If $X$ is a compact locale. Then the projection $p_Y : X \times Y \to Y$ is closed for any $Y$ (see [5]). Thus we have $\text{id}_Z \times p_Y = p_{Z \times Y} : Z \times X \times Y \to Z \times Y$ is closed for all locale $Z$. Hence we have the following result.

**Lemma 2.** Suppose $X$ is a compact locale. Then the projection $X \times Y \to Y$ is proper for any locale $Y$.

Let $X$ be a completely regular locale and let $f : Y \to X$ be a localic map. Consider the diagonal $\langle \beta, f \rangle : Y \to \beta Y \times X$, where $\beta Y$ is the compact completely regular reflection of $Y$. Write $P_Y = \uparrow \{ (u, x) \mid \beta^*(u) \land f^*(x) = 0 \}$, the closure of the image of $Y$ under $\langle \beta, f \rangle$, and $P_f : P_Y \to X$ the composite of the closed inclusion $P_Y \hookrightarrow \beta Y \times X$ and the projection $\beta Y \times X \to X$. By Lemma 2, we know that $P_f : P_Y \to X$ is proper and $P_Y$ is completely regular since both $X$ and $\beta Y$ are completely regular. Write $\varepsilon_f : Y \to P_Y$ to be the co-restriction of $\langle \beta, f \rangle$. Then we have a commutative triangle of localic maps

$$
\begin{array}{ccc}
Y & \xrightarrow{\varepsilon_f} & P_Y \\
\downarrow{f} & & \downarrow{P_f} \\
X & & \\
\end{array}
$$

We shall show $P_f : P_Y \to X$ to be the completely regular proper reflection of $f : Y \to X$ in the slice category $\text{Loc}/X$. We note that the above construction can be applied to any locale $X$ so that it becomes a functor $P : \text{Loc}/X \to \text{Loc}/X$.

**Proposition 1.** $P : \text{Loc}/X \to \text{Loc}/X$ is a functor such that $\varepsilon$ becomes a natural transformation from the identity to $P$. 

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Proof. Let $h : Y \to Z$ be a localic map such that the following triangle commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & Z \\
\downarrow f & & \downarrow g \\
X & \xrightarrow{\beta} & \beta Z
\end{array}
\]

Write $\beta h : \beta Y \to \beta Z$ to be the extension of $h$; i.e., the following square commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & \beta Y \\
\downarrow \beta & & \downarrow \beta \\
Z & \xrightarrow{\beta h} & \beta Z
\end{array}
\]

Denote $0_{P_Y} = \{(u, x) \mid \beta^*_Y(u) \wedge f^*(x) = 0\}$ the least element of $P_Y$, and $0_{P_Z} = \{(v, x) \mid \beta^*_Z(v) \wedge g^*(x) = 0\}$ the least element of $P_Z$ respectively. Consider the product $\tilde{h} \times id_X : \beta Y \times X \to \beta Z \times X$. For $(v, x) \in 0_{P_Z}$, we have $\beta^*_Z(v) \wedge h^*(x) = h^*(\beta^*_Z(v)) \wedge g^*(x) = h^*(\beta^*_Z(v) \wedge g^*(x)) = 0$, i.e. $(\tilde{h} \times id_X)^*(0_{P_Z}) \leq 0_{P_Y}$. Now define $P(h) : P_Y \to P_Z$ for which the corresponding frame morphism is defined by $P(h)^*(I) = (\tilde{h} \times id_X)^*(I) \lor 0_{P_Y}$. Then the following triangle commutes:

\[
\begin{array}{ccc}
P_Y & \xrightarrow{P(h)} & P_Z \\
& \downarrow P_f & \downarrow P_g \\
X & \xrightarrow{\beta} & \beta Z
\end{array}
\]

$P$ clearly preserves composites. Hence $P : Loc/X \to Loc/X$ is a functor. To show that $\varepsilon : id \to P$ is a natural transformation, it suffices to check the commutativity of the following square. But it is straightforward.

\[
\begin{array}{ccc}
Y & \xrightarrow{\varepsilon_f} & P_Y \\
\downarrow h & & \downarrow P(h) \\
Z & \xrightarrow{\varepsilon_g} & P_Z
\end{array}
\]

Lemma 3. If $f : Y \to X$ is proper with $Y$ completely regular, then $\varepsilon_f : Y \to P_Y$ is an isomorphism.

Proof. By the fact already proved in [8] (or [9]), the diagonal $\langle \beta, f \rangle : Y \to \beta Y \times X$ is proper. Hence $\langle \beta, f \rangle$ is a closed inclusion since $\beta : Y \to \beta Y$ is an inclusion (see [3]). This shows that the co-restriction $\varepsilon_f : Y \to P_Y$ is an isomorphism.

Theorem 1. Let $X$ be a completely regular locale, and let $f : Y \to X$ be a localic map. Then the localic map $\varepsilon_f : Y \to P_Y$ is universal among all localic maps from $f : Y \to X$ to a completely regular proper map over $X$; i.e., $P_f : P_Y \to X$ is the completely regular proper reflection of $f : Y \to X$ in the slice category $Loc/X$. Moreover, the reflection map $\varepsilon_f : Y \to P_Y$ is a dense embedding if and only if $Y$ is completely regular.
Proof. Suppose \( g: Z \rightarrow X \) is a proper map with \( Z \) a completely regular locale. Let \( h: Y \rightarrow Z \) be a localic map such that the following triangle commutes:

\[
\begin{array}{c}
Y \\
\downarrow f \\
\downarrow g \\
X
\end{array}
\begin{array}{c}
\downarrow h \\
Z \\
\downarrow \epsilon_g
\end{array}
\Rightarrow
\begin{array}{c}
\downarrow g \\
Y
\end{array}
\]

Then we have a commutative square

\[
\begin{array}{ccc}
Y & \xrightarrow{\epsilon_f} & P_Y \\
\downarrow h & & \downarrow \epsilon(h) \\
Z & \xrightarrow{\epsilon_g} & P_Z
\end{array}
\]

By Lemma 3, \( \epsilon_g: Z \rightarrow P_Z \) is an isomorphism. Hence \( h \) has a factorization \( h = \epsilon_g^{-1}P(h)\epsilon_f \). If \( h = r\epsilon_f \) is another factorization of \( h \), then the equalizer of \( \epsilon_g^{-1}P(h) \) and \( r \) is a closed sublocale of \( P_Y \) since \( Z \) is completely regular, hence must be Hausdorff. But the equalizer contains \( Y \) and so must be \( P_Y \) itself. Hence \( r = \epsilon_g^{-1}P(h) \). If \( Y \) is completely regular, then \( \epsilon_f: Y \rightarrow P_Y \) is an embedding since \( \beta: Y \rightarrow \beta Y \) is an embedding. Conversely if \( \epsilon_f: Y \rightarrow P_Y \) is an embedding, then \( Y \) is completely regular since \( P_Y \) is completely regular.

It is interesting that we can apply the above reflection construction to the category of topological spaces and thus get a completely regular perfect reflection for every topological space over a given completely regular space.

Let \( X \) be a completely regular space, and let \( f: Y \rightarrow X \) be a continuous map. Write \( P_Y \) for the closure of the image of \( Y \) under the diagonal \( \langle \beta, f \rangle: Y \rightarrow \beta Y \times X \), where \( \beta Y \) is the Stone–Čech compactification of \( Y \). Then the restriction \( P_f: P_Y \rightarrow X \) of the projection is a perfect map and \( P_Y \) is completely regular. Denote by \( \epsilon_Y: Y \rightarrow P_Y \) the co-restriction of the diagonal \( \langle \beta, f \rangle \). Assuming the axiom of choice, we have the following result.

**Corollary 1.** \( \epsilon_Y: Y \rightarrow P_Y \) is universal among all continuous maps from the topological space \( f: Y \rightarrow X \) over \( X \) to a completely regular perfect map over \( X \); i.e., \( P_f: P_Y \rightarrow X \) is the completely regular perfect reflection of \( f: Y \rightarrow X \) in the slice category \( \text{Top}/X \).

Now we consider a more general case when the given locale \( X \) is not completely regular. In this case we will show that for a special class of locales over \( X \), i.e. those locales \( f: Y \rightarrow X \) over \( X \) such that the closure \( f(Y) \) of the image \( f(Y) \) of \( Y \) under \( f \) is completely regular, it indeed has a completely regular proper reflection in \( \text{Loc}/X \).

**Lemma 4.** Let \( p: Y \rightarrow X \) be a proper map. If the image \( p(Y) \) of \( Y \) under \( p \) is contained in a sublocale \( Y_j \) of \( Y \), then the co-restriction map \( \bar{p}: Y \rightarrow Y_j \) is proper.

Proof. \( \bar{p} \) is clearly closed. Suppose \( \{y_s \mid s \in S\} \subset Y_j \) is directed. Then \( \bar{p}^*(\vee^I y_s) = \bar{p}^*(j(\vee y_s)) = p^*(\vee y_s) = \vee p^*(y_s) = \vee \bar{p}^*(y_s) \), where \( \vee^I \) represents the join in \( Y_j \). Hence \( \bar{p}^* \) preserves directed joins. \( \square \)
Let $X$ be a locale. Suppose $f : Y \to X$ is a locale over $X$ such that the closure $\overline{f(Y)}$ of the image $f(Y)$ of $Y$ under $f$ is completely regular. We have a factorization $Y \xrightarrow{f} X = Y \xrightarrow{\overline{f(Y)}} \overline{f(Y)} \xrightarrow{\overline{f}} Y$. Similar to the case for $X$ being completely regular, we have a diagonal map $\langle \beta, f \rangle : Y \to \beta Y \times \overline{f(Y)}$. Write $\hat{P}_Y = \{(u, x) \mid \beta^*(u) \land \hat{f}^*(x) = 0\}$, the closure of the image of $Y$ under $\langle \beta, \hat{f} \rangle$, and $\hat{P}_f : \hat{P}_Y \to X$ the composite of the closed inclusion $\hat{P}_Y \to \beta Y \times \overline{f(Y)}$, the projection $\beta Y \times \overline{f(Y)} \to \overline{f(Y)}$ and the inclusion $\overline{f(Y)} \to X$. Then $\hat{P}_f : \hat{P}_Y \to X$ is proper and $\hat{P}_Y$ is completely regular since both $\overline{f(Y)}$ and $\beta Y$ are completely regular. Denote by $\varepsilon : Y \to \hat{P}_Y$ the co-restriction of $\langle \beta, f \rangle$. We have a commutative triangle

\[
\begin{array}{ccc}
Y & \xrightarrow{\varepsilon} & \hat{P}_Y \\
\downarrow{f} & & \downarrow{\hat{P}_f} \\
X & & \\
\end{array}
\]

**Theorem 2.** $\hat{P}_f : \hat{P}_Y \to X$ is a completely regular proper reflection of $f : Y \to X$ in the slice category $\text{Loc}/X$ and $\varepsilon : Y \to \hat{P}_Y$ is the reflection map. Moreover, the reflection map $\varepsilon : Y \to \hat{P}_Y$ is a dense embedding if and only if $Y$ is completely regular.

**Proof.** Suppose $g : Z \to X$ is a proper map with $Z$ completely regular. Let $h : Y \to Z$ be a localic map such that $f = gh$. Write for $\overline{h(Y)}$ the closure of the image $h(Y)$ of $Y$ under $h$. Then $h : Y \to Z$ has a factorization $Y \xrightarrow{\bar{h}} \overline{h(Y)} \xrightarrow{b} Z$. We have $g(\overline{h(Y)}) = g(\overline{h(Y)}) = \overline{f(Y)}$ since $g$ is closed. Note that the composite $\overline{h(Y)} \to Z \xrightarrow{\beta} X$ is proper, so its co-restriction $s : \overline{h(Y)} \to g(\overline{h(Y)}) = \overline{f(Y)}$ is proper by Lemma 4. Moreover we have $f' s h = gh \bar{h} = gh = f' f$; hence $s \bar{h} = f$ since $f'$ is monic. By Theorem 1, the co-restriction $\bar{h}$ can be uniquely factored through $\varepsilon : Y \to \hat{P}_Y$ by a map $r : \hat{P}_Y \to \overline{h(Y)}$. Thus $h : Y \to Z$ can be factored through $\varepsilon : Y \to \hat{P}_Y$ as $Y \xrightarrow{h} Z = Y \xrightarrow{\bar{h}} \overline{h(Y)} \to Z = Y \xrightarrow{\varepsilon} \hat{P}(Y) \xrightarrow{\hat{P}_f} \overline{h(Y)} \to Z$. The factorization through $\varepsilon$ is essentially unique since $\varepsilon : Y \to \hat{P}(Y)$ is dense and $Z$ is completely regular. 

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