CONVERGENCE OF REFINEMENT SCHEMES ON METRIC SPACES

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Abstract. We analyze the convergence of iterative refinement processes on metric spaces, imposing the principle of contractivity to obtain convergence criteria. As a major result, we show that on Hadamard spaces a wide natural class of contractible barycentric subdivision schemes converges.

1. Introduction

Linear subdivision schemes have a short, yet eventful, history in different areas of pure and applied mathematics. Topics in which these refinement methods appear include harmonic analysis (see e.g. [7]) and the theory of functional equations (cf. [4]). A scheme of this type refines a data grid \( x : \mathbb{Z}^s \rightarrow \mathbb{R}^n \) according to

\[
S x_i = \sum_{k \in \mathbb{Z}^s} a_{i-2k} x_k, \quad \text{where } \# \text{supp}(a) < \infty.
\]

This actually amounts to \( 2^s \) linear subdivision rules, one for each residue class modulo \( 2 \mathbb{Z}^s \). The scheme \( S \) is said to converge if for each \( x \in \ell^\infty \) the powers of \( S \) acting on \( x \) approach a continuous limit \( S^\infty x : \mathbb{R}^s \rightarrow \mathbb{R}^n \) uniformly in a sense that

\[
\lim_{m \to \infty} \sup_i \| S^m x (i/2^m) - S^\infty x_i \| = 0.
\]

The convergence analysis of linear schemes with nonnegative mask coefficients \( a_k \), its beginnings comprehensively described in [2], was further developed by Zhou in [11], who showed that these schemes converge as long as their mask’s support has an appropriate shape and the property of affine invariance is fulfilled, meaning \( \sum_k a_{i-2k} = 1 \) for \( i \in \mathbb{Z}^s \).

The hypothesis of affine invariance is natural in a sense that every convergent scheme is subject to it. Moreover, it allows us to at least formally generalize equation (1.1) to arbitrary metric spaces:

\[
S x_i = \arg\min_{k \in \mathbb{Z}^s} \sum a_{i-2k} d(x_k, \cdot)^2.
\]

This nonlinear refinement method, referred to as barycentric subdivision, has the advantage of being well-adapted to the metric structure of the underlying space. As a drawback, however, the existence and uniqueness of the minimizer in (1.2) is not always guaranteed, even if the weights are nonnegative. Nevertheless there is a nice class of metric spaces in which barycentric schemes with nonnegative masks...
are well-defined, namely, the complete geodesic spaces of nonpositive Alexandrov curvature, known as Hadamard spaces; cf. [1].

As far as barycentric schemes on finite-dimensional Hadamard manifolds are concerned, at least smoothness properties are well-understood; cf. [5]. However, convergence analysis even in this setting remains to be developed. Apart from a univariate result to be found in [10], sufficient conditions known to imply convergence require tight density bounds on the input data, and, in any case, rely heavily on differentiability. This paper aims at a first understanding of barycentric schemes on general Hadamard spaces.

The article is organized as follows: In Section 2, we prove a convergence criterion valid on complete metric spaces based on contractivity. The second section handles the task of identifying contractive schemes. Section 4 uses the results obtained in the previous two to establish convergence of contractive schemes on Hadamard spaces; see Theorem 4.6. Making use of this theorem and the contractivity criteria obtained in Section 3, we show that a barycentric scheme with nonnegative mask whose support is sufficiently well-behaved converges; cf. Corollary 4.7. We thus generalize corresponding linear results from [2].

2. Refinement schemes on metric spaces

This section establishes a theorem on complete metric spaces ensuring convergence of contractive schemes comparable to convergent ones. To begin with, we introduce some notions used throughout the paper:

**Definition 2.1.** Consider a metric space \((X, d)\) and set \(X_s = \{x \mid x : \mathbb{Z}^s \to X\}\). A scheme \(S : X_s \to X_s\) is called convergent on \(\Omega \subseteq X_s\) if for all \(x \in \Omega\) there exists a continuous function \(S^\infty x : \mathbb{R}^s \to X\) such that \(d_\infty(S^\infty x(-/2^n), S^n x) = \sup_j d(S^\infty x(j/2^n), S^n x_j)\) tends to 0 as \(n \to \infty\).

Moreover, the scheme is called contractive with respect to some nonnegative function \(D : X_s \to \mathbb{R}_+\) if and only if

\[
D(Sx) < \gamma D(x), \quad \text{where} \quad \gamma < 1.
\]

We refer to \(D\) as a contractivity function for \(S\). Unless specified otherwise, a contractive scheme is called convergent if and only if it has this property on the set \(\Omega_D = \{x \in X_s \mid D(x) < \infty\}\).

**Remark 2.2.** Sometimes notation becomes more accessible if one views \(S^n x\) not as an element of \(X_s\), but rather as a function on the refined grid \(S^n x : 2^{-n} \mathbb{Z}^s \to X\) through the natural identification \(S^n x_j = S^n x(\frac{j}{2^n})\).

**Theorem 2.3.** Let \((X, d)\) be a complete metric space. Suppose a convergent iterative scheme \(T : X_s \to X_s\) satisfies

\[
d_\infty(Tx, Ty) \leq d_\infty(x, y).
\]

Moreover, assume the scheme \(S\) is contractive with respect to \(D\) and

\[
d_\infty(Tx, Sx) \leq C \cdot D(x)
\]

holds for any \(x \in X_s\). Then \(S\) is convergent.
Proof. We set \( f_n(y) := T^\infty(S^n x)(2^n y) \) and claim that this defines a Cauchy sequence in \( (C(\mathbb{R}^s, X), d_\infty) \). Note first that given \( n \in \mathbb{N} \) and \( y \in \mathbb{R}^s \), by continuity of \( f_n \), respectively \( f_{n+1} \), we find \( j \in \mathbb{Z}^s \) and \( m \in \mathbb{N} \) such that

\[
d(f_r(y), f_r(2^{-m} j)) < C \cdot D(x) \gamma^n \quad \text{for} \quad r = n, n + 1.
\]

Moreover, due to convergence of \( T \), by multiplying both the numerator and the denominator of the number \( j/2^m \) with a power of two if necessary we may assume \( m \) to be sufficiently large for

\[
d(f_r(2^{-m} j), T^{m-r}(S^r x)_j) = d(T^\infty S^r x(2^{r-m} j), T^{m-r}(S^r x)_j) < C \cdot D(x) \gamma^n
\]

to hold for \( r = n, n + 1 \), in addition to \((2.3)\). This together with \((2.1)\) and \((2.2)\) implies that

\[
d(f_n(y), f_{n+1}(y)) \leq d(f_n(y), f_n(2^{-m} j))
\]
\[
+ d(f_n(2^{-m} j), T^{m-n} S^n x_j)
\]
\[
+ d(T^{m-n} S^n x_j, T^{m-n-1} S^{n+1} x_j)
\]
\[
+ d(T^{m-n-1} S^{n+1} x_j, f_{n+1}(2^{-m} j))
\]
\[
+ d(f_{n+1}(2^{-m} j), f_{n+1}(y))
\]
\[
< 5C \cdot D(x) \gamma^n,
\]

showing that \( f_n \) is a Cauchy sequence. Since \( X \) is complete, we find a continuous \( f : \mathbb{R}^s \to X \) with \( f_n \to f \) uniformly. We claim that \( S^n x \) converges to \( f \) in the sense of Definition 2.1. For \( m \geq n \) and \( j \in \mathbb{Z}^s \), we obtain the inequality

\[
d(T^{m-n} S^n x_j, S^n x_j) \leq \sum_{k=n}^{m-1} d(T^{m-k} S^k x_j, T^{m-k-1} S^{k+1} x_j)
\]
\[
\leq \sum_{k=n}^{m-1} \gamma^k \cdot D(x) C \leq \gamma^n \left( \frac{D(x) C}{1 - \gamma} \right),
\]

which together with

\[
d(f_n(2^{-m} j), S^n x_j) \leq d(f_n(2^{-m} j), T^{m-n} S^n x_j) + d(T^{m-n} S^n x_j, S^n x_j)
\]

establishes the claim. \( \square \)

3. Recognition of contractivity

This section is devoted to a simple criterion of contractivity, namely Proposition 3.5 below. Referring e.g. to [3] for details, we start by recalling some fundamental facts on Hadamard spaces. These complete geodesic spaces allow for a notion of nonpositive curvature in a sense that geodesic triangles are ‘slim’ compared to the Euclidean ones of the same edge lengths: The defining property of such a space \( X \) is the so-called Hadamard inequality

\[
d(z, x_\frac{1}{2})^2 \leq \frac{1}{2} d(z, x_0)^2 + \frac{1}{2} d(z, x_1)^2 - \frac{1}{4} d(x_0, x_1)^2,
\]

where \( x_0, x_1, z \in X \), and \( x_\frac{1}{2} \) denotes some midpoint of \( x_0 \) and \( x_1 \) satisfying \( d(x_0, x_\frac{1}{2}) = d(x_\frac{1}{2}, x_1) = \frac{1}{2} d(x_0, x_1) \).

Hadamard spaces for instance play an important role in the theory of cost-minimizing networks; see [3]. Topological examples are trees as well as Euclidean
Bruhat-Tits buildings. Notably, for a measure space $M$, and $N$ Hadamard, the space of strongly measurable square-integrable functions $L^2(M, N)$ inherits the Hadamard property. It is remarkable that these spaces also occur as families of certain geometric and topological structures, such as spaces of Riemannian and Kähler metrics or spaces of connections. The latter examples actually are generically infinite-dimensional Hadamard manifolds; see [6]. An instance of a finite-dimensional Hadamard manifold significant in applications is the space of symmetric positive definite matrices, which occurs in Diffusion Tensor Imaging.

We define the barycenter or center of mass of an $L^2$-probability measure $\mu$ on the Borel algebra of a Hadamard space by

$$b(\mu) = \arg\min \int_X d(\cdot, x)^2 \mu(dx).$$

It is well known that on Hadamard spaces, this barycenter exists and is unique; see [8]. Recall that a coupling of probability measures $\mu$ and $\nu$ is a measure on $X \times X$ satisfying $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for all Borel sets $A$. We define the $L^1$-Wasserstein metric (see [9]) on the space of probability measures by

$$d^W(\mu, \nu) = \inf \left\{ \int_{X \times X} d(x, y)\pi(dx\,dy) \mid \pi \text{ is a coupling of } \mu \text{ and } \nu \right\}.$$  

**Proposition 3.1** ([8]). *On a Hadamard space, the barycenter $b$ is Lipschitz continuous as a map from the space of $L^2$-probability measures to $X$. More precisely,*

$$d(b(\mu), b(\nu)) \leq d^W(\mu, \nu).$$

We continue by adapting the theory of barycenters on Hadamard spaces to fit into the setup of nonlinear refinement schemes. Throughout the rest of the paper, we encounter situations in which two sets of data are given:

(i) A finite measure $\alpha$ on the discrete sigma algebra of $\mathbb{Z}^s$, whose weights $\alpha(\{i\}) = \alpha_i \geq 0$ satisfy $\sum_i \alpha_i = \alpha(\mathbb{Z}^s) = \sigma > 0$.

(ii) A multivariate sequence of data points $x : \mathbb{Z}^s \to X$.

In this setting, we consider the induced probability measure $\mu_x = \frac{1}{\sigma} x_* \alpha = \frac{1}{\sigma} \alpha \circ x^{-1}$ on $X$ and define

$$\bar{x}_\alpha = b(\mu_x) = \arg\min \left( \sum_i \alpha_i d(x_i, \cdot)^2 \right).$$

This little instance of double-thinking allows us to apply convenient properties of the barycenter in the convergence analysis of refinement schemes based on formula (1.2).
Corollary 3.2. Let $(X,d)$ be a Hadamard space. Suppose we are given weights $\alpha_i, \beta_i \in \mathbb{R}_{\geq 0}$ and data points $x_i, y_i \in X$, $i \in \mathbb{Z}$ such that $0 < \sigma = \sum_i \alpha_i < \infty$ and $0 < \tau = \sum_i \beta_i < \infty$. Moreover assume $\frac{1}{\sigma}x_i \alpha$, $\frac{1}{\tau}x_i \beta$ and $\frac{1}{\sigma}y_i \alpha$ constitute $L^2$-probability measures w.r.t. the metric $d$. Then the following inequalities hold:

\begin{align}
(3.1) \quad d(\bar{x}_\alpha, \bar{y}_\alpha) & \leq \frac{1}{\sigma} \sum_i \alpha_i d(x_i, y_i), \\
(3.2) \quad d(\bar{x}_\alpha, \bar{x}_\beta) & \leq \frac{1}{\sigma \tau} \sum_{i,j} \alpha_i \beta_j d(x_i, x_j).
\end{align}

Proof. Set $u = x \times y, v = x \times x : \mathbb{Z}^2 \to X \times X$, i.e. $u(i,j) = (x_i, y_j)$ and $v(i,j) = (x_i, x_j)$ for $i, j \in \mathbb{Z}$. Moreover, define measures $\mu$ and $\nu$ on $\mathbb{Z}^2$ via $\mu(i,j) = \delta_{ij} \alpha_i$, where $\delta_{ij}$ denotes the Kronecker delta, and $\nu(i,j) = \alpha_i \beta_j$. Let $\pi_i^X : X \times X \to X$ and $\pi_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ denote the projections on the $i$-th components for $i = 1, 2$. Then, using the fact that for any pair of functions $f_1, f_2 : \mathbb{Z} \to X$ one has $\pi_i^X \circ (f_1 \times f_2) = f_i \circ \pi_i$, it is easy to verify that $\frac{1}{\sigma}u_* \mu$ is a coupling of $\frac{1}{\sigma}x_* \alpha$ and $\frac{1}{\sigma}y_* \alpha$. Similarly, one shows that $\frac{1}{\sigma \tau}v_* \nu$ is a coupling of $\frac{1}{\sigma}x_* \alpha$ and $\frac{1}{\sigma}x_* \beta$. Thus the statement directly follows from Proposition 3.1. 

\[ \blacksquare \]

Remark 3.3. Note that both inequalities (3.1) and (3.2) are found in the Euclidean case by simple computations exploiting linearity and the triangle inequality. In their general form proven above they turn out to be valuable tools for detecting contractive barycentric schemes.

Lemma 3.4. Suppose for $i = 1, \ldots, n$ we are given nonnegative weights $\alpha_i, \beta_i$ and control points $x_i \in X$ such that $\sum_i \alpha_i = \sum_i \beta_i = \sigma > 0$. Then there exist a positive integer $r$, nonnegative weights $\gamma_j$ and control points $y_j, z_j \in \{x_1, \ldots, x_n\}$, $j = 1, \ldots, r$, such that

\[ \sum_j \gamma_j d(y_j, \cdot)^2 = \sum_i \alpha_i d(x_i, \cdot)^2 \quad \text{and} \quad \sum_j \gamma_j d(z_j, \cdot)^2 = \sum_i \beta_i d(x_i, \cdot)^2. \]

Moreover, we may require $\gamma_1 = \min(\alpha_1, \beta_1)$.

Proof. We use induction over $n$ to prove the statement. Note first that w.l.o.g. we may assume $\alpha_1 = \min(\alpha_1, \beta_1) =: \gamma_1$. Set $k = \max\{\ell \mid \beta_1 - \sum_{i=1}^\ell \alpha_i \geq 0\}$ and $\delta = \beta_1 - \sum_{i=1}^k \alpha_i \geq 0$. In case $k = n$, we have $\sigma = \beta_1$ and $\beta_\ell = 0$ for $\ell > 1$. The $\gamma_i, y_i$ and $z_i$ are then given by

| $i$ | 1 \ldots n \\
|---|---|---|
| $\gamma$ | $\alpha_1$ \ldots $\alpha_n$ \\
| $y$ | $x_1$ \ldots $x_n$ \\
| $z$ | $x_1$ \ldots $x_1$. |

If $k < n$, we conclude that $\sigma = \beta_1 > 0$ and $\delta < \alpha_{k+1}$. Similar to the above, the first $k + 1$ members of $\gamma, y$ and $z$ are given by

| $i$ | 1 \ldots $k$ \ldots $k+1$ \\
|---|---|---|
| $\gamma$ | $\alpha_1$ \ldots $\alpha_k$ $\delta$ \\
| $y$ | $x_1$ \ldots $x_k$ $x_{k+1}$ \\
| $z$ | $x_1$ \ldots $x_1$. |
The remaining weights $\tilde{\alpha}$, $\tilde{\beta}$ and data points $\tilde{x}$ defined by

\begin{equation}
\begin{array}{c|cccccccc}
  i & 1 & \ldots & k & k+1 & \ldots & n-1 \\
  \hline
  \tilde{\alpha} & 0 & \ldots & \alpha_{k+1} - \delta & \alpha_{k+2} & \ldots & \alpha_n \\
  \tilde{\beta} & \beta_2 & \ldots & \beta_{k+1} & \beta_{k+2} & \ldots & \beta_n \\
  \tilde{x} & x_2 & \ldots & x_{k+1} & x_{k+2} & \ldots & x_n \\
\end{array}
\end{equation}

fulfill $\tilde{\sigma} = \sigma - \beta_{1} = \sum_i \tilde{\alpha}_i = \sum_i \tilde{\beta}_i$, so we obtain the number $r \geq k + 1$ and the weights $\gamma_i$ and data points $y_i$, $z_i$, $i = k+2, \ldots, r$ by applying the induction hypothesis to (3.3).

We are now in a position to prove our central contractivity criterion:

**Proposition 3.5.** Suppose $\alpha_i$, $\beta_i$ and $x_i$ are given as in Lemma 3.4, and set $D(x) = \max_{k,\ell} d(x_k, x_\ell)$. Then the following contractivity property holds:

$$d(\tilde{x}_\alpha, \tilde{x}_\beta) \leq (1 - \max_i (\min(\alpha_i, \beta_i))/\sigma) D(x).$$

**Proof.** W.l.o.g. $\alpha_1 = \max_i (\min(\alpha_i, \beta_i))$. Using Lemma 3.4 we find $\gamma$, $y$ and $z$ with $\tilde{x}_\alpha = \tilde{y}_\gamma$, $\tilde{x}_\beta = \tilde{z}_\gamma$, $\gamma_1 = \alpha_1$ and $z_1 = y_1$. Now by inequality (3.1) of Corollary 3.2 we may estimate

$$d(\tilde{x}_\alpha, \tilde{x}_\beta) = d(\tilde{y}_\gamma, \tilde{z}_\gamma) \leq \sum_j (\gamma_j/\sigma) d(y_j, z_j) \leq (1 - \alpha_1/\sigma) \max_{k,\ell} d(x_k, x_\ell),$$

which proves the statement. \(\square\)

**Remark 3.6.** As a direct consequence of (3.3) we obtain

$$d(\tilde{x}_\alpha, \tilde{x}_\beta) \leq (1 - \frac{1}{\sigma^2} \sum_i \alpha_i \beta_i) \max_{k,\ell} d(x_k, x_\ell),$$

which gives a different (sometimes larger, sometimes smaller) contractivity constant. Thus Proposition 3.5 should also be regarded as a means to quantify the speed of convergence; see Proposition 4.9 below.

4. **Contractivity as a convergence criterion**

This section is devoted to the convergence analysis of barycentric schemes with nonnegative masks on Hadamard spaces, on which they are well-defined. As a central result (see Theorem 4.6) we show how contractivity leads to convergence. Throughout the section, nonnegativity of the mask will be assumed for any occurring barycentric scheme.

**Definition 4.1.** Let $S : X_s \to X_s$ and $T : X_t \to X_t$ be barycentric subdivision schemes with masks $a$ and $b$, respectively. Then we define the tensor product of $S$ and $T$ to be the scheme $S \otimes T : X_{s+t} \to X_{s+t}$ whose mask is given by $c_{(i,j)} = a_{i} b_{j}$.

The next proposition establishes a class of contractive schemes on Hadamard spaces, including the ones generating splines of arbitrary degree. This result and its linear counterpart (see Proposition 3.1 in [2]), are equally powerful in identifying contractivity.
Proposition 4.2. Suppose the mask of the scheme $S$ acting on data from a Hadamard space is supported on a convex set $\Omega$ for which there exists $x_0 \in \Omega$ such that $\Omega = \Omega - x_0$ is balanced. With the definitions

$$\rho_\Omega(\xi) = \inf\{\lambda \geq 0 \mid \xi \in \lambda \Omega\}, \quad \xi \in \mathbb{R}^s$$

(the Minkowski functional of $\Omega$) and

$$D_\Omega(x) = \sup_{\rho_\Omega(i-j) < 2} d(x_i, x_j), \quad x \in X,$$

we have

$$D_\Omega(Sx) \leq \gamma D_\Omega(x),$$

where $\gamma = \min(\gamma_1, \gamma_2)$, with

$$\gamma_1 = 1 - \min_{\rho_\Omega(i-j) < 2} \max_{k \in \mathbb{Z}^s} \min(a_{i-2k}, a_{j-2k}),$$

$$\gamma_2 = 1 - \min_{\rho_\Omega(i-j) < 2} \sum_{k \in \mathbb{Z}^s} a_{i-2k}a_{j-2k}.$$ 

In particular, if for each $i, j \in \mathbb{Z}^s$ with $\rho_\Omega(i-j) < 2$ one finds $k \in \mathbb{Z}^s$ such that $i-2k \in \text{supp}(a)$ and $j-2k \in \text{supp}(a)$, then $\gamma < 1$ and hence $S$ is contractive w.r.t. $D_\Omega$.

Proof. Note first that $y \in \tilde{\Omega} \implies \rho_\Omega(y - x_0) \leq 1$. Introducing sequences $\alpha^i$ by letting $\alpha^i_k = a_{i-2k}$, we have $Sx_i = \tilde{x}_{\alpha^i}$ for $i \in \mathbb{Z}^s$. Thus, with the notation

$$\eta_{ij} = \max \left( \max_{k \in \mathbb{Z}^s} \min(a_{i-2k}, a_{j-2k}), \sum_{k \in \mathbb{Z}^s} a_{i-2k}a_{j-2k} \right),$$

Proposition 3.5 and the subsequent remark imply that

$$D_\Omega(Sx) = \sup_{\rho_\Omega(i-j) < 2} d(\tilde{x}_{\alpha^i}, \tilde{x}_{\alpha^j}) \leq \sup_{\rho_\Omega(i-j) < 2} \left[ (1 - \eta_{ij}) \sup_{\rho_\Omega(i-2k-x_0), \rho_\Omega(i-2\ell-x_0) \leq 1} \max_{\rho_\Omega(i-j) < 2} d(x_k, x_\ell) \right].$$

By convexity of $\Omega$, its Minkowski functional is subadditive. Therefore $\rho_\Omega(i-j) < 2$ together with $\rho_\Omega(i-2\ell-x_0) \leq 1$ and $\rho_\Omega(j-2k-x_0) \leq 1$ implies that

$$\rho_\Omega(k-\ell) \leq \rho_\Omega(k - \frac{1}{2}(j + x_0)) + \rho_\Omega(\frac{1}{2}(\ell + x_0) - i) + \rho_\Omega(\frac{1}{2}(i-j)) < 2.$$

Combining this with (4.1), we obtain $D_\Omega(Sx) \leq \gamma D_\Omega(x)$, as required. \qed

Proposition 4.2 provides us with a contractivity criterion that solely depends on the structure of the mask’s support. Thus every linear scheme seen to be contractive using the linear version of the above proposition possesses a contractive barycentric analogue. In particular, this applies to the class of schemes identified in chapter 3 of [2]; see Corollary 4.3 below. Recall that a centered zonotope is defined as $Z(X) = \{ Xu \mid u \in \mathbb{R}^n, \|u\|_\infty \leq 1\}$ with $X \in \mathbb{Z}^s \times \mathbb{Z}^n$, $n > s$. $Z(X)$ is called unimodular if and only if each $s \times s$ minor of $X$ has determinant $-1, 0,$ or $1$, and $\text{rank}(X) = s$.

Corollary 4.3. Suppose the barycentric scheme $S$ possesses a mask whose support is an integer quadrangle with edges of length at least 2, or $\text{supp}(a) = Z(X) \cap \mathbb{Z}^s$ with $Z(X)$ unimodular. Then $S$ is contractive w.r.t. some contractivity function $D_\Omega$. 

Proof. This is a direct consequence of the proofs of Theorems 3.3 and 3.4 of \cite{2}, and Proposition 4.2.

Up to now it is not clear how Theorem \cite{2} is of value in detecting convergent schemes. It would be desirable to have some kind of convergent model scheme to compare a given contractive scheme with, leading to an implication

\[
\text{Contraction} \Longrightarrow \text{Convergence}.
\]

For contractivity functions of the form \(D_{\Omega} \) with \(\Omega\) balanced and convex, cf. Proposition 4.2, we are able to provide such a model scheme (see Lemma 4.3 below).

**Lemma 4.4.** Suppose \(\Omega\) and \(\Omega'\) are bounded, balanced and convex subsets of \(\mathbb{R}^s\) having nonempty interior. Then there exists a constant \(C > 0\) such that

\[
D_{\Omega'} \leq C \cdot D_{\Omega}.
\]

Proof. Choose \(x_1, \ldots, x_n \in \mathbb{R}^s\) such that \(2(\Omega')^o \subseteq \bigcup_{i=1}^n (x_i + 2\Omega^o)\). Thus, given \(i, j \in \mathbb{Z}^s\) with \(\rho_{\Omega^o}(i-j) < 2 \iff i-j \in 2(\Omega')^o\), one finds \(i = y_0, y_1, \ldots, y_{m-1}, y_m = j\) with \(y_k - y_{k-1} \in 2\Omega^o \iff \rho_{\Omega^o}(y_k - y_{k-1}) < 2\), such that \(m \leq n\). It is plain to show that, as a consequence, \(D_{\Omega'} \leq n \cdot D_{\Omega}\).

The next lemma identifies linear B-spline subdivision as a model scheme suitable for our convergence analysis.

**Lemma 4.5.** Let \(\bar{T}\) denote the univariate linear B-spline subdivision scheme defined by \(b_0 = 1\) and \(b_{-1} = b_1 = \frac{1}{2}\). Then the barycentric analogue of \(T = \bar{T} \otimes \cdots \otimes \bar{T}\) converges on any Hadamard space. Moreover, \(d_\infty(Tx, Ty) \leq d_\infty(x, y)\) for all \(x, y \in X_s\).

Proof. We begin by proving convergence. Define \(D_1 = D_{\mathbb{D}_\infty}\), where \(\mathbb{D}_\infty\) denotes the closed unit disk with respect to the maximum norm. By Corollary 4.3, \(D_1(Tx) \leq \gamma D_1(x)\), with \(\gamma < 1\). For \(n \in \mathbb{N}_0\), define \(f_n : \mathbb{R}^s \to X\) as follows:

1. For \(\xi \in \mathbb{R}\), set \(\varphi_0(\xi) = \max\{1 - |\xi|, 0\}\) and define \(\varphi(t_1, \ldots, t_s) = \prod_1 \varphi_0(t_i)\).
2. Set \(f_n(\xi) = \text{argmin}(\sum_k \varphi(2^{n-1}\xi - k)d(\cdot, T^{n-1}x_k)^2)\).

This function is continuous since the center of mass depends continuously on the weights. Moreover, \(f_r(j/2^r) = T^r x_j\) for each \(j \in \mathbb{Z}^s\) by construction of \(\varphi\). Suppose \(\xi \in \mathbb{R}^s\) is contained in some dyadic cube \(Q = \prod [k_i 2^{-r+1}, (k_i + 1)2^{-r+1}]\), where \(k_i \in \mathbb{Z}\). Clearly \(\varphi \equiv 0\) outside \(\{\xi \in \mathbb{R}^s \mid \|\xi\|_\infty < 1\}\), from which we conclude that

\[
f_r(\xi) = \text{argmin} \sum_{v \in V(Q)} \varphi(2^{r-1}(\xi - v))d(\cdot, T^{r-1}x(v))^2,
\]

where \(V(Q)\) denotes the vertex set of \(Q\). Applying the inequality (3.2), we obtain

\[
\max_{v \in V(Q)} d(T^{r-1}x(v), f_r(\xi)) \leq \max_{v, w \in V(Q)} d(T^{r-1}x(v), T^{r-1}x(w)) \leq \max_{v \in V(Q)} d(T^{r-1}x(v), f_r(\xi)) \leq D_1(x).
\]

Certainly, every dyadic cube of edge length \(2^{-n}\) shares a vertex with a dyadic cube of edge length \(2^{-n+1}\). Together with (4.2), applied to \(r = n, n + 1\), this implies \(d(f_n(\xi), f_{n+1}(\xi)) < 2\gamma^{n-1}D_1(x)\). It is straightforward to show that \(f := \lim_n f_n\) is a uniform limit of \(T^n x\).
The second statement is an easy consequence of inequality \((4.1)\). Indeed, denoting the mask of \(T\) by \(b\), for \(i \in \mathbb{Z}^s\) we have
\[
d(Tx_i, Ty_i) \leq \sum_{k \in \mathbb{Z}^s} b_{i-2k}d(x_k, y_k) \leq d_\infty(x, y),
\]
since \(T\) is affine invariant. \(\square\)

Now all the ingredients of our main theorem on the convergence of subdivision schemes on Hadamard spaces are at hand, and we proceed to

**Theorem 4.6.** Suppose the barycentric scheme \(S\) acting on data from a Hadamard space is contractive with respect to some \(D_\Omega\), with \(\Omega\) bounded, balanced, convex and having nonempty interior. Then \(S\) converges.

**Proof.** Throughout the proof let \(T\) denote the \(s\)-fold tensor product of the barycentric analogue of the linear B-spline scheme, whose mask \(b\) was defined in Lemma \(4.5\).

Choose \(\Omega' \subset \mathbb{R}^s\) bounded, balanced, and convex such that \(\text{supp}(a) \subset \Omega'\) and \(D_\infty(0,1) \subset \Omega'\). Applying \((3.2)\), for \(i \in \mathbb{Z}^s\) one obtains
\[
\text{(4.3)} 
\]
\[
d(Sx_i, Tx_i) \leq \sum_{k,\ell} a_{i-2k}b_{i-2\ell}d(x_k, x_\ell).
\]

Note that the summands in \((4.3)\) are nonzero only if \(\rho_{\Omega'}(i-2k) \leq 1\) and \(\|i-2\ell\|_\infty \leq 1 \implies \rho_{\Omega'}(i-2\ell) \leq 1\), which implies \(\rho_{\Omega'}(k-\ell) \leq \frac{1}{2}(\rho_{\Omega'}(i-2k) + \rho_{\Omega'}(i-2\ell)) \leq 1\). This together with \((4.3)\) gives \(d_\infty(Sx, Tx) \leq D_{\Omega'}(x)\). By Lemma \(4.4\) we find \(C > 0\) such that \(D_{\Omega'}(x) \leq C \cdot D_{\Omega}(x)\) for \(x \in X_s\). Hence \(d_\infty(Sx, Tx) \leq C \cdot D_{\Omega}(x)\), so the statement follows from Theorem \(2.3\). \(\square\)

Thus, Corollary \(4.3\) leads to identification of the following convergent schemes:

**Corollary 4.7.** Suppose the barycentric scheme \(S\) possesses a mask whose support is an integer quadrangle with edges of length at least 2, or \(\text{supp}(a) = Z(X) \cap \mathbb{Z}^s\) with \(Z(X)\) a unimodular centered zonotope. Then \(S\) converges.

**Corollary 4.8.** On Hadamard spaces, barycentric analogues of B-spline schemes of arbitrary degree converge.

We conclude our reasoning with a statement highlighting the effect of the contractivity constant on the quality of convergence.

**Proposition 4.9.** Suppose \(\Omega\) is a balanced convex set with nonempty interior such that \(\text{supp}(a) \subseteq 4\Omega^c\) and \(D_\Omega(Sx) \leq \gamma D_\Omega(x)\), \(\gamma < 1\). Then \(S\) converges, and
\[
d(S^\infty x(j/2^n), S^n x_j) \leq \gamma^n \left(\frac{D_\Omega(x)}{1 - \gamma}\right).
\]

**Proof.** By inequality \((3.2)\), we have
\[
d(Sx_{2j}, x_j) \leq \sum_{\rho_{\Omega}(2(j-k)) < 4} a_{2(j-k)}d(x_k, x_j) \leq D_\Omega(x),
\]
and consequently
\[
\text{(4.4)} 
\]
\[
d((S^m x)_{2m-n_j}, S^n x_j) \leq \gamma^n \left(\frac{D_\Omega(x)}{1 - \gamma}\right).
\]

According to Theorem \(4.6\) \(S\) converges; hence the first statement follows by taking the limit in \(m\) on the left-hand side of equation \((4.4)\). \(\square\)
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