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SOME PROPERTIES OF COUPLED-EXPANDING MAPS IN COMPACT SETS

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ABSTRACT. In this paper, some properties of a strictly A-coupled-expanding map in compact subsets of a metric space are studied, where A is a transition matrix. It is shown that this map has a compact invariant set on which it is topologically semi-conjugate to the subshift for A. If the subshift for A has positive topological entropy, then the map is chaotic in the sense of Li-Yorke. Moreover, in the one-dimensional case, the map is at most two-to-one conjugate to the subshift for A and chaotic in the sense of Devaney.

1. INTRODUCTION

Sharkovskiĭ's amazing discovery [17], as well as Li and Yorke's famous work which introduced the concept of chaos [12], have activated sustained interest and provoked the recent rapid advancement of the frontier research on discrete chaos theory. For different purposes of studies, new concepts of chaos were introduced, such as Devaney chaos [6], Wiggins chaos [24], etc. Lately, criteria of chaos and relationships among different concepts of chaos have become two main objectives of research on chaos theory. For continuous interval maps which map intervals into themselves, many good results have been obtained: transitivity implies Devaney chaos [23]; a map has positive topological entropy if and only if it exhibits Devaney chaos on a closed invariant subset of the interval [11], which also implies Li-Yorke chaos [7]; a map with zero topological entropy may imply Li-Yorke chaos under certain conditions [22], and so on. In general, Devaney chaos is stronger than Li-Yorke chaos [9]. Moreover, Blanchard et al. showed that a continuous and surjective map with positive entropy, defined in a compact metric space, is chaotic in the sense of Li-Yorke [2].

Coupled-expanding maps (or horseshoe maps; see Definition 2.5) have been discussed by many researchers using different methods. This concept in a metric space was first introduced in [19]. Then, it was further extended to coupled-expanding maps associated with a transition matrix A, called A-coupled-expanding maps [20]. Recently, by applying symbolic dynamical systems theory, several criteria of chaos

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have been established for strictly coupled-expanding maps in metric spaces, by Shi, Chen, Yu, etc. [18, 20, 21, 26].

In particular, many important results about coupled-expanding maps in compact subsets of metric spaces have been established. For a continuous interval map with positive entropy, Misiurewicz showed that there are infinitely many kinds of finite forward iterations of the map such that under these iterations the map is strictly coupled-expanding [14]. Block et al. studied the relationship between the topological entropy of a coupled-expanding interval map associated with a matrix A and the eigenvalues of A, showing that the topological entropy of the map is no less than the logarithm of the spectral radius of A [4]. Block and Coppel introduced the concept of a turbulent map (called coupled-expanding map lately) in the study of continuous interval maps, and discussed the relationships of a turbulent map with period points, topological entropy, symbolic dynamics, and chaos [3]. Ruette surveyed chaos of continuous interval maps with detailed discussions about coupled-expanding maps [16]. Blokh and Teoh, and Yang and Tang studied the topological semi-conjugacy relationship of a coupled-expanding map in compact sets of a metric space with a fullshift on a finite alphabet [5, 25].

In many of these important works, the establishment of a relationship between a given map and a fullshift or subshift on a finite alphabet provides us with a lot of information about the complex dynamical behaviors of the map, where the relationship is often said to be a topological semi-conjugacy or conjugacy. For example, Block and Coppel showed that if a continuous interval map is strictly coupled-expanding in two nondegenerate compact intervals, then there exists an uncountable invariant subset on which the map is topologically semi-conjugate to the shift on two symbols [3, Chapter II, Proposition 15]. Ruette further proved that this kind of map is topologically mixing, has sensitive dependence on initial conditions, and has dense periodic points on an invariant set with the help of the topological semi-conjugacy [16, Proposition 6.1.3]. A natural question is whether a similar relationship could be established between a general strictly A-coupled-expanding map in compact sets of a metric space and the subshift for A, where A is a transition matrix. In the present paper, we study this problem and obtain some similar results (see Theorems 3.1 and 3.4), which generalize the results of Block and Coppel [3], and Yang and Tang [25, Theorem 1], respectively (see Remarks 3.2 and 3.5). If the subshift for A has positive topological entropy, then the map is chaotic in the sense of Li-Yorke. In the one-dimensional case, it is further proved that the map is also chaotic in the sense of Devaney (see Theorem 3.3).

The rest of this paper is organized as follows. In Section 2, some concepts, notation and useful lemmas are introduced. Section 3 pays attention to properties of a strictly A-coupled-expanding map on compact subsets of a metric space. In particular, some better results are obtained in the one-dimensional case.

2. Preliminaries

Two definitions of chaos are introduced, which will be useful in the sequel.

Definition 2.1. Let (X, d) be a metric space, $f : X \to X$ a map, and S a subset of X with at least two points. Then S is called a scrambled set of f if for any two distinct points $x, y \in S$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,\ \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

The map f is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of f.

Let (X, d) be a metric space. A continuous map $g : X \to X$ is said to be topologically transitive if for any two open subsets U and V of X, there exists a positive integer m such that $g^m(U) \cap V \neq \emptyset$; g is said to be topologically mixing if for any two open subsets U and V of X, there exists a positive integer k such that $g^n(U) \cap V \neq \emptyset$ for any integer $n \geq k$; g is said to have sensitive dependence on initial conditions in X if there exists a positive constant δ such that for any point $x \in X$ and any neighborhood U of x, there exist $y \in U$ and a positive integer m such that $d(f^m(x), f^m(y)) > \delta$.

Definition 2.2 ([6])**.** Let (X, d) be a metric space. A map $f : V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if

(i) the set of the periodic points of f is dense in V;
(ii) f is topologically transitive in V;
(iii) f has sensitive dependence on initial conditions in V.

In the above definition, condition (iii) is redundant if f is continuous in V by the result of [1], where V contains infinitely many points.

Some notations are now given. Let $A = (a_{ij})_{m\times m}$ $(m \geq 2)$, where a_{ij} is the (i, j) entry of matrix A, $1 \leq i, j \leq m$. For any positive integer k, the (i, j) entry of matrix A^k is denoted by $a_{ij}^{(k)}$. Let (X, d) be a metric space and $f : X \to X$ a continuous map. By $h(f)$ denote the topological entropy of f.

Lemma 2.3 ([2, Corollary 2.4])**.** *Let (X, d) be a compact metric space and $f : X \to X$ a continuous surjective map. If $h(f) > 0$, then f is chaotic in the sense of Li-Yorke.*

The following basic notions about matrices are referred to [8, 15]. A matrix $A = (a_{ij})_{m\times m}$ $(m \geq 2)$ is said to be a transition matrix if $a_{ij} = 0$ or 1 for all i, j; $\sum_{j=1}^{m} a_{ij} \geq 1$ for all i; and $\sum_{i=1}^{m} a_{ij} \geq 1$ for all j, $1 \leq i, j \leq m$. A is said to be nonnegative, denoted by $A \geq 0$, if all its entries $a_{ij} \geq 0$. A is said to be positive, denoted by $A > 0$, if all its entries $a_{ij} > 0$. A is said to be eventually positive if there exists a positive integer k such that $A^n > 0$ for all the integers $n \geq k$. A is called a permutation matrix if exactly one entry is equal to 1 and all the other entries are 0 in each row and each column. A is said to be reducible if there exists a permutation matrix P such that

$$P^T A P = \begin{pmatrix} E & C \\ 0 & D \end{pmatrix},$$

where E and D are square submatrices; otherwise, A is called irreducible. If a nonnegative matrix is eventually positive, then it is irreducible, but its converse is not true in general [13]. A nonnegative matrix $A = (a_{ij})_{m\times m}$ is irreducible if and only if for every pair (i, j), $1 \leq i, j \leq m$, there exists a positive integer k such that $a_{ij}^{(k)} > 0$ [13].

Next, some basic concepts about one-sided symbolic dynamical systems are introduced [10].

Let $S_0 := \{1, 2, ..., m\}$, $m \geq 2$. The one-sided sequence space

$$\textstyle\sum_m^+ := \{\alpha = (a_0, a_1, a_2, ...) : \ a_i \in S_0, \ i \geq 0\}$$

is a metric space equipped with the distance

$$d(\alpha, \beta) = \sum_{i=0}^{\infty} \frac{d(a_i, b_i)}{2^i},$$

where $\alpha = (a_0, a_1, a_2, ...)$, $\beta = (b_0, b_1, b_2, ...) \in \sum_m^+$, $d(a_i, b_i) = 1$ if $a_i \neq b_i$, and $d(a_i, b_i) = 0$ if $a_i = b_i$, $i \geq 0$. Then, $(\sum_m^+, d)$ is a complete metric space and also a Cantor set. Define the shift map $\sigma : \sum_m^+ \to \sum_m^+$ by $\sigma(\alpha) = (a_1, a_2, ...)$, where $\alpha = (a_0, a_1, ...)$. Then, $(\sum_m^+, \sigma)$ is called the one-sided symbolic dynamical system on m symbols.

For a transition matrix $A = (a_{ij})_{m\times m}$, denote

$$\textstyle\sum_m^+(A) := \{\beta = (b_0, b_1, ...) \in \sum_m^+ : a_{b_i b_{i+1}} = 1,\ i \geq 0\}.$$

Then, $\sum_m^+(A)$ is a compact subset of $\sum_m^+$ and invariant under σ. The map

$$\sigma_A := \sigma|_{\sum_m^+(A)} : \textstyle\sum_m^+(A) \to \sum_m^+(A) \tag{2.1}$$

is said to be the subshift of finite type for A. It is briefly called the subshift for A, which is continuous and surjective.

For a transition matrix $A = (a_{ij})_{m\times m}$, a finite sequence $w = (s_1, s_2, ..., s_k)$ is called an allowable word of length k for A if $a_{s_i s_{i+1}} = 1$, $1 \leq i \leq k-1$, where $s_1, s_2, ..., s_k \in S_0$. The length of w is denoted by $|w|$ [15, page 74].

Lemma 2.4 ([15, page 74])**.** *Let $A = (a_{ij})_{m\times m}$ be a transition matrix. Then, for any $k \geq 1$ and any $1 \leq i, j \leq m$, there are exactly $a_{ij}^{(k)}$ allowable words of length $k+1$ for A, starting at i and ending at j, in the form of $(i, s_1, s_2, ..., s_{k-1}, j)$.*

Finally in this section, the main concept, A-coupled-expanding map, is introduced.

Definition 2.5 ([20, Definition 2.5])**.** Let (X, d) be a metric space, $f : D \subset X \to X$ a map, and $A = (a_{ij})_{m\times m}$ a transition matrix ($m \geq 2$). If there exist m subsets V_i ($1 \leq i \leq m$) of D with $V_i \cap V_j = \partial_D V_i \cap \partial_D V_j$ for every pair (i, j), $1 \leq i \neq j \leq m$, where $\partial_D V_i$ is the relative boundary of V_i with respect to D, such that

$$f(V_i) \supset \bigcup_{a_{ij}=1} V_j,\ 1 \leq i \leq m,$$

then f is said to be coupled-expanding for matrix A (or simply, A-coupled-expanding) in V_i, $1 \leq i \leq m$. Further, the map f is said to be strictly coupled-expanding for matrix A (or strictly A-coupled-expanding) in V_i, $1 \leq i \leq m$, if $d(V_i, V_j) > 0$ for all $1 \leq i \neq j \leq m$. Moreover, if each entry of matrix A is equal to 1, then f is briefly called coupled-expanding in V_i, $1 \leq i \leq m$.

3. Properties of strictly A-coupled-expanding maps

In this section, some properties of a strictly A-coupled-expanding map on compact subsets of a metric space are discussed. It is shown that the map has a compact invariant set Λ on which it is topologically semi-conjugate to the one-sided subshift for A, where A is a transition matrix. If the subshift for A has positive topological entropy, then it is chaotic in the sense of Li-Yorke. In the one-dimensional case, the map on Λ is at most two-to-one semi-conjugate to the subshift for A and chaotic in the sense of Devaney.

Theorem 3.1. *Let (X,d) be a metric space, $V_1,...,V_m$ $(m \geq 2)$ pairwise-disjoint compact subsets of X, and $A=(a_{ij})_{m\times m}$ a transition matrix. If a continuous map $f: D := \bigcup_{i=1}^m V_i \to X$ is strictly A-coupled-expanding in $V_1,...,V_m$, then there exists a compact subset $\Lambda \subset D$ such that $f(\Lambda)=\Lambda$ and $f:\Lambda\to\Lambda$ is topologically semi-conjugate to $\sigma_A : \sum_m^+(A) \to \sum_m^+(A)$. Furthermore, if $h(\sigma_A)>0$, then f is chaotic in the sense of Li-Yorke.*

Proof. For any $\alpha=(a_0,a_1,a_2,...)\in\sum_m^+(A)$, set

$$V_{a_0a_1\cdots a_n} = \bigcap_{i=0}^{n} f^{-i}(V_{a_i}).$$

Obviously, $V_{a_0a_1\cdots a_n}$ is compact and

$$V_{a_0a_1\cdots a_{n+1}} \subset V_{a_0\cdots a_n},\ n\geq 0. \tag{3.1}$$

It follows from the relation $f(U\cap f^{-1}(V)) = f(U)\cap V$ that

$$\begin{aligned} f\left(\bigcap_{i=0}^{n} f^{-i}(V_{a_i})\right) &= f(V_{a_0})\cap\left(\bigcap_{i=0}^{n-1} f^{-i}(V_{a_{i+1}})\right) \\ &= f(V_{a_0})\cap V_{a_1\cdots a_n} \\ &= V_{a_1\cdots a_n}; \end{aligned}$$

that is,

$$f(V_{a_0a_1\cdots a_n}) = V_{a_1\cdots a_n},\ n\geq 1, \tag{3.2}$$

which implies that

$$f^n(V_{a_0a_1\cdots a_n}) = V_{a_n},\ n\geq 0. \tag{3.3}$$

Hence,

$$V_{a_0a_1\cdots a_n} \neq \emptyset,\ n\geq 0. \tag{3.4}$$

From (3.1), (3.4) and the compactness of $V_{a_0a_1\cdots a_n}$, one has

$$V_\alpha := \bigcap_{i=0}^{\infty} V_{a_0a_1\cdots a_i} \neq \emptyset.$$

Next, we show that for any two different allowable words $w_1=(c_1,...,c_k)$ and $w_2=(d_1,...,d_k)$ for A,

$$V_{c_1\cdots c_k}\cap V_{d_1\cdots d_k} = \emptyset. \tag{3.5}$$

In fact, (3.5) holds obviously for $k=1$. Suppose that (3.5) holds for $k=l\geq 1$. It will be shown that (3.5) holds for $k=l+1$. Indeed, if $c_1=d_1$, then $(c_2,...,c_{l+1})\neq(d_2,...,d_{l+1})$. It follows from the assumption of induction that $V_{c_2\cdots c_{l+1}}\cap V_{d_2\cdots d_{l+1}}=\emptyset$. So, (3.5) holds in this case since $f(V_{c_1\cdots c_{l+1}}) = V_{c_2\cdots c_{l+1}}$ and $f(V_{d_1\cdots d_{l+1}}) = V_{d_2\cdots d_{l+1}}$. If $c_1\neq d_1$, then (3.5) holds since $V_{c_1\cdots c_{l+1}}\subset V_{c_1}$ and $V_{d_1\cdots d_{l+1}}\subset V_{d_1}$. Hence, (3.5) holds if $w_1\neq w_2$ by induction. Consequently, for any $\alpha,\beta\in\sum_m^+(A)$ with $\alpha\neq\beta$,

$$V_\alpha\cap V_\beta = \emptyset. \tag{3.6}$$

Set

$$\Lambda = \bigcup_{\alpha\in\sum_m^+(A)} V_\alpha.$$

Then, Λ is closed, so is compact. In fact, suppose that $\{x_n\}_{n=1}^{\infty} \subset \Lambda$ is a sequence with $x_n \to x$ as $n \to \infty$. Then, there exists $\alpha_n \in \sum_m^+(A)$ such that $x_n \in V_{\alpha_n}$, $n \geq 1$. Since $\sum_m^+(A)$ is compact, $\{\alpha_n\}_{n=1}^{\infty}$ has a convergent subsequence. Without loss of generality, suppose that $\{\alpha_n\}_{n=1}^{\infty}$ is convergent with limit $\alpha = (a_0, a_1, ...)$. Thus, for any given integer $k \geq 0$, there exists N_k such that $V_{\alpha_n} \subset V_{a_0 \cdots a_k}$ for all integers $n \geq N_k$. So, $x_n \in V_{a_0 \cdots a_k}$, which implies that $x \in V_{a_0 \cdots a_k}$ for all integers $k \geq 0$. Hence, $x \in V_\alpha$, and consequently, Λ is closed.

By using (3.2), it is possible to show that for any $\alpha = (a_0, a_1, a_2, ...) \in \sum_m^+(A)$, one has

$$f(V_\alpha) = \bigcap_{n=0}^{\infty} f(V_{a_0 \cdots a_n}) = \bigcap_{n=1}^{\infty} V_{a_1 \cdots a_n} = V_{\sigma_A(\alpha)}. \tag{3.7}$$

Obviously, it suffices to verify the first relation. For any $y \in \bigcap_{n=0}^{\infty} f(V_{a_0 \cdots a_n})$, there exists $x_n \in V_{a_0 \cdots a_n} \subset D$ such that $f(x_n) = y$, $n \geq 0$. By the compactness of D, $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence. Without loss of generality, assume that $\{x_n\}_{n=0}^{\infty}$ is convergent, and its limit is denoted by x. It is evident that $x \in V_\alpha$. By the continuity of f on V_{a_0}, one has $y = f(x) \in f(V_\alpha)$. Hence, $\bigcap_{n=0}^{\infty} f(V_{a_0 \cdots a_n}) \subset f(V_\alpha)$. The converse inclusion relation can be directly derived from the definition of V_α. Therefore, (3.7) holds.

Since σ_A is surjective, there exists $\alpha' = (a_{-1}, a_0, a_1, ...) \in \sum_m^+(A)$ such that $\sigma_A(\alpha') = \alpha$. This, together with (3.7), yields that $f(V_{\alpha'}) = V_{\sigma_A(\alpha')} = V_\alpha$. Hence, $f(\Lambda) = \Lambda$.

Now, define a map $g : \Lambda \to \sum_m^+(A)$ by $g(x) = \alpha$ for $x \in V_\alpha$. From (3.6), g is well defined. Obviously, g is surjective. It follows from (3.7) that $g \circ f(x) = \sigma_A \circ g(x)$ for all $x \in \Lambda$.

Next, it is possible to prove that g is continuous. For any given $\epsilon > 0$, there exists an integer $n \geq 1$ such that $2^{-n} < \epsilon$. By Lemma 2.4, the number of all the pairwise-distinct allowable words of length $n+1$ for matrix A is equal to $\sum_{1 \leq i,j \leq m} a_{ij}^{(n)}$. Each of these allowable words for A, $w = (a_0, a_1, ..., a_n)$, corresponds to a subset, $V_{a_0 \cdots a_n}$. By (3.5), any two of these subsets are disjoint. Let δ_n be the least distance between any two of these subsets. Then, $\delta_n > 0$, since these subsets are pairwise-disjoint and compact, and the number of these subsets is finite. For any $x, y \in \Lambda$, if $d(x, y) < \delta_n$, then x and y must belong to a common subset $V_{a_0 \cdots a_n}$, implying that $d(g(x), g(y)) \leq 2^{-n} < \epsilon$. Hence, g is continuous in Λ.

Hence, $f : \Lambda \to \Lambda$ is topologically semi-conjugate to $\sigma_A : \sum_m^+(A) \to \sum_m^+(A)$.

Further, if $h(\sigma_A) > 0$, then $h(f) \geq h(\sigma_A) > 0$, and consequently, f is chaotic in the sense of Li-Yorke by Lemma 2.3. This completes the proof. □

Remark 3.2. The approach used in the proof of Theorem 3.1 is inspired by that of Proposition 15 in Chapter II of [3], Theorem 1 of [25], and Theorem 4.1 of [20]. Theorem 1 in [25] discussed a special strictly A-coupled-expanding map in compact sets of a metric space, where every entry of A is equal to 1. By Lemma 2.3, it can be concluded that the map in [25, Theorem 1] is chaotic in the sense of Li-Yorke. So, Theorem 3.1 here extends [25, Theorem 1] to more general A-coupled-expanding maps.

In the one-dimensional case, we can get the following better results than those in Theorem 3.1:

Theorem 3.3. *Let $I_1, ..., I_m$ ($m \geq 2$) be pairwise-disjoint nondegenerate compact intervals and $A = (a_{ij})_{m\times m}$ a transition matrix with topological entropy $h(\sigma_A) > 0$. If a continuous map $f : I := \bigcup_{i=1}^m I_i \to \mathbb{R}$ is strictly A-coupled-expanding in $I_1, ..., I_m$, then there exists an invariant set on which f is chaotic in the sense of Devaney.*

Proof. For the transition matrix A, there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & \cdots & A_{s-1} & * \\ 0 & 0 & & \cdots & 0 & A_s \end{pmatrix},$$

where all A_j are irreducible and the entries $*$ are some block matrices [15, page 77]. Furthermore, the nonwandering set $\Omega(\sigma_A) = \sum_m^+(A_1)\cup\cdots\cup\sum_m^+(A_s)$, and $h(\sigma_A) = \max_{1\leq i\leq s}\{h(\sigma_{A_i})\}$ [27, page 128]. From the assumption $h(\sigma_A) > 0$, there exists some i_0, $1 \leq i_0 \leq s$, such that $h(\sigma_{A_{i_0}}) > 0$. Without loss of generality, assume that $i_0 = 1$, and A_1 is an $l \times l$ matrix. Obviously, $l > 1$. If B is a $k \times k$ irreducible transition matrix with each row-sum equal to 1, then B is a permutation matrix [15, page 76]. So the maximal eigenvalue of B equals 1, and consequently, $h(\sigma_B) = 0$ [15, page 341]. Hence, A_1 has at least one row-sum no less than 2. This implies that there exist $W, W_1, W_2 \in \{I_1, ..., I_m\}$ and positive integers k_1, k_2 such that

$$f(W) \supset W_1 \cup W_2, \quad f^{k_1}(W_1) \supset W, \quad f^{k_2}(W_2) \supset W,$$

which implies that

$$f^k(W_1) \cap f^k(W_2) \supset W_1 \cup W_2,$$

where $k = \max\{k_1 + 1, k_2 + 1\}$. That is, f^k is a strictly coupled-expanding map in W_1 and W_2. By the method used in the proof of [11, Proposition 4.9], it can be shown that there exists a compact invariant subset of $W_1 \cup W_2$ on which f^k is chaotic in the sense of Devaney. This, together with [11, Proposition 4.10], yields that there exists a compact invariant set of f on which f is chaotic in the sense of Devaney. The proof is complete. □

Theorem 3.4. *Let $I_1, ..., I_m$ ($m \geq 2$) be pairwise-disjoint nondegenerate compact intervals and $A = (a_{ij})_{m\times m}$ a transition matrix. If a continuous map $f : I := \bigcup_{i=1}^m I_i \to \mathbb{R}$ is strictly A-coupled-expanding in $I_1, ..., I_m$, then*

(i) *there exists a compact subset $D \subset I$ such that $f(D) = D$;*
(ii) *there exists a continuous and surjective map $g : D \to \sum_m^+(A)$ such that $g \circ f(x) = \sigma_A \circ g(x)$ for $x \in D$, namely, $f : D \to D$ is topologically semi-conjugate to $\sigma_A : \sum_m^+(A) \to \sum_m^+(A)$;*
(iii) *there exists a countable subset $E \subset \sum_m^+(A)$ such that for any $\alpha \in E$, α has exactly two preimages of g, and for any $\beta \in \sum_m^+(A)\backslash E$, β has only one preimage of g.*

Proof. The whole proof is divided into four steps.

Step 1. The compact invariant set D is constructed.

For any allowable word (b_0, b_1) for the matrix A of length 2, $I_{b_1} \subset f(I_{b_0})$ since $a_{b_0 b_1} = 1$. It follows from the intermediate value theorem that there exists $I_{b_0 b_1}$, which is a compact subinterval of I_{b_0} of minimal length, such that $f(I_{b_0 b_1}) = I_{b_1}$. By repeatedly applying the intermediate value theorem and by using induction, it can be shown that, for any integer $n \geq 2$ and any allowable word $(b_0, b_1, ..., b_n)$ for A of length $n+1$, there exists $I_{b_0 b_1 \cdots b_n}$, which is a compact subinterval of $I_{b_0 b_1 \cdots b_{n-1}}$ of minimal length, such that

$$f(I_{b_0 b_1 \cdots b_n}) = I_{b_1 \cdots b_n} \subset I_{b_1 \cdots b_{n-1}}. \tag{3.8}$$

For any $\alpha = (a_0, a_1, a_2, ...) \in \sum_m^+(A)$, denote

$$I_\alpha := \bigcap_{n=0}^{\infty} I_{a_0 \cdots a_n}.$$

Obviously, I_α is either a nondegenerate compact interval or a singleton set.

On the other hand, for any two different allowable words $w_1 = (c_1, ..., c_k)$ and $w_2 = (d_1, ..., d_k)$ for A, by applying the similar method used in the proof of Theorem 3.1, one has

$$I_{c_1 \cdots c_k} \cap I_{d_1 \cdots d_k} = \emptyset. \tag{3.9}$$

Consequently, for any $\alpha, \beta \in \sum_m^+(A)$ with $\alpha \neq \beta$, one has

$$I_\alpha \cap I_\beta = \emptyset. \tag{3.10}$$

Set

$$D := \bigcup_{\alpha \in \sum_m^+(A)} \partial I_\alpha,$$

where $\partial I_\alpha = I_\alpha$ if I_α is a singleton set, and ∂I_α consists of the two endpoints of I_α if I_α is a nondegenerate interval.

Now, it remains to prove that D is closed. Let $\{x_n\}_{n=1}^{\infty} \subset D$ be any given sequence satisfying $x_n \to x$ as $n \to \infty$. Suppose $x_n \in \partial I_{\alpha_n}$, $n \geq 1$. Since $\sum_m^+(A)$ is compact, $\{\alpha_n\}_{n=1}^{\infty}$ has a convergent subsequence. Without loss of generality, assume that $\{\alpha_n\}_{n=1}^{\infty}$ is convergent, and its limit is denoted by $\alpha = (a_0, a_1, ...)$. So, for any given integer $k \geq 0$, there exists N_k such that $I_{\alpha_n} \subset I_{a_0 \cdots a_k}$ for all integers $n \geq N_k$, which implies that $x \in I_{a_0 \cdots a_k}$. Hence, $x \in I_\alpha$. If $\alpha_n \neq \alpha$ for all sufficiently large n, then x, the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, must be in ∂I_α by (3.10) and $x_n \in \partial I_{\alpha_n}$. Hence, $x \in D$ in this case. If there exists a sequence $\{n_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} n_k = \infty$ such that $\alpha_{n_k} = \alpha$, $k \geq 1$, then $x_{n_k} \in \partial I_{\alpha_{n_k}} = \partial I_\alpha$, $k \geq 1$. Hence, $x = \lim_{k\to\infty} x_{n_k} \in \partial I_\alpha$, implying that $x \in D$. Therefore, D is closed and consequently D is a compact subset of I.

Step 2. Show that $f(D) = D$.

First, for any $\alpha = (a_0, a_1, a_2, ...) \in \sum_m^+(A)$, from a similar discussion in the proof of (3.7), it follows that

$$f(I_\alpha) = \bigcap_{n=0}^{\infty} f(I_{a_0 \cdots a_n}) = \bigcap_{n=1}^{\infty} I_{a_1 \cdots a_n} = I_{\sigma_A(\alpha)}. \tag{3.11}$$

Next, we are ready to show that for any $\alpha = (a_0, a_1, ...) \in \sum_m^+(A)$, f maps the endpoints of I_α to the endpoints of $I_{\sigma_A(\alpha)}$. There are four cases:

(a) I_α and $I_{\sigma_A(\alpha)}$ are both singleton sets;
(b) I_α is a singleton set, $I_{\sigma_A(\alpha)}$ is a nondegenerate interval;
(c) I_α and $I_{\sigma_A(\alpha)}$ are both nondegenerate intervals;
(d) I_α is a nondegenerate interval, $I_{\sigma_A(\alpha)}$ is a singleton set.

It follows from (3.11) that the assertion is correct in Case (a). By (3.11), Case (b) cannot happen. In Case (c), $I_{a_0\cdots a_n}$ is a compact subinterval of $I_{a_0\cdots a_{n-1}}$ of minimal length such that $f(I_{a_0\cdots a_n}) = I_{a_1\cdots a_n}$ for all integers $n \geq 1$. This implies that f maps the endpoints of $I_{a_0\cdots a_n}$ to the endpoints of $I_{a_1\cdots a_n}$. Because the endpoints of nondegenerate intervals I_α and $I_{\sigma_A(\alpha)}$ are the limits of the endpoints of the compact intervals $I_{a_0\cdots a_n}$ and $I_{a_1\cdots a_n}$, $n \geq 1$, respectively, f maps the endpoints of I_α to the endpoints of $I_{\sigma_A(\alpha)}$. So, the assertion is correct in Case (c). It is evident that the assertion is correct in Case (d). Hence, f maps the endpoints of I_α to the endpoints of $I_{\sigma_A(\alpha)}$. Consequently, $f(D) \subset D$.

Finally, we need to show that $f(D) \supset D$. For any $y \in D$, there exists $\alpha = (a_0, a_1, a_2, ...) \in \sum_m^+(A)$ such that $y \in \partial I_\alpha$. Since $\sigma_A(\sum_m^+(A)) = \sum_m^+(A)$, there exists $\beta = (b_0, a_0, a_1, ...) \in \sum_m^+(A)$ such that $\sigma_A(\beta) = \alpha$. It follows from (3.11) that $f(I_\beta) = I_\alpha$. If I_α is a singleton set, then there exists $x \in \partial I_\beta \subset D$ such that $f(x) = y$. Hence, $y \in f(D)$ in this case. If I_α is a nondegenerate interval, then I_β is also a nondegenerate interval. Since f maps the endpoints of I_β to the endpoints of I_α, there exists $x \in \partial I_\beta \subset D$ such that $y = f(x) \in f(D)$. Hence, $f(D) \supset D$. Therefore, $f(D) = D$.

Consequently, assertion (i) holds.

Step 3. Show assertion (ii).

Define a map $g : D \to \sum_m^+(A)$ by $g(x) = \alpha$ if $x \in \partial I_\alpha$. By (3.10), g is well defined. Obviously, g is surjective. With a similar argument to that used in the proof of Theorem 3.1, one can easily show that g is continuous in D. In addition, it follows from (3.11) that $g \circ f(x) = \sigma_A \circ g(x)$ for $x \in D$. Therefore, (ii) holds.

Step 4. Show assertion (iii).

Denote

$$E := \{\alpha : \ I_\alpha \text{ is a nondegenerate closed interval}\}.$$

It is evident that E is countable by (3.10). For any $\alpha \in E$, it follows from the definition of g that $g^{-1}(\alpha) = \partial I_\alpha$, which consists of the two endpoints of I_α. On the other hand, for any $\beta \in \sum_m^+(A) \backslash E$, I_β is a singleton set, and consequently, $g^{-1}(\beta) = \partial I_\beta$ contains only one point. Hence, (iii) holds.

The entire proof of the theorem is complete. ☐

Remark 3.5. The method used in the proof of Theorem 3.4 is inspired by that of Proposition 15 in Chapter II of [3], in which the map is a special one-dimensional strictly A-coupled-expanding map where every entry of A is equal to 1. Yet, Theorem 3.4 here extends Proposition 15 in Chapter II of [3] to more general one-dimensional A-coupled-expanding maps.

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