ON THE NORM CLOSURE PROBLEM
FOR COMPLEX SYMMETRIC OPERATORS

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Abstract. We prove that the set of all complex symmetric operators on a separable, infinite-dimensional Hilbert space is not norm closed.

In [2, Sect. 3], it is asked whether the set of all complex symmetric operators on a separable, infinite-dimensional Hilbert space is norm closed. We answer this question in the negative. Let $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, \ldots)$ denote the unilateral shift on $\ell^2(\mathbb{N})$ and let $\cong$ denote unitary equivalence. Note that

$$
T_n = \frac{n}{n+1} S \oplus (\bigoplus_{j=1}^{\infty} \frac{1}{j+1} S) \oplus (\bigoplus_{j=1}^{\infty} \frac{1}{j+1} S^*) \cong \bigoplus_{j=1}^{\infty} \frac{1}{j+1} (S \oplus S^*)
$$

is complex symmetric by [1, Ex. 5]. On the other hand, $T_n$ converges in norm to

$$
T = S \oplus (\bigoplus_{j=1}^{\infty} \frac{1}{j+1} S) \oplus (\bigoplus_{j=1}^{\infty} \frac{1}{j+1} S^*) \cong S \oplus \bigoplus_{j=1}^{\infty} \frac{1}{j+1} (S \oplus S^*).
$$

Since $\|S^k(1,0,0,\ldots)\| = 1$, there is an $x$ so that $\|T^k x\| = 1$ for $k \geq 0$. However,

$$
T^* = S^* \oplus \bigoplus_{j=1}^{\infty} \frac{1}{j+1} (S^* \oplus S) = S^* \oplus (a \text{ strict contraction})
$$

possesses no such vector since $(S^*)^k$ tends strongly to zero. This precludes the existence of a conjugation $C$ (i.e., an isometric, conjugate-linear involution) such that $T = CT^*C$. Thus $T$ is not complex symmetric. □

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References


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