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EFFECTIVE EQUIDISTRIBUTION OF THE REAL PART OF ORBITS ON HYPERBOLIC SURFACES

JIMI L. TRUELSEN

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Abstract. For non-cocompact Fuchsian groups with finite covolume we prove that the real part of the orbit of a point in the upper half-plane is equidistributed with an effective error term. This extends previous results by A. Good, M. Risager, and Z. Rudnick. We use the equidistribution result to generalize a theorem by F. Chamizo.

1. Introduction

It is well known that the group $\mathrm{SL}_2(\mathbf{R})$ acts on the upper half-plane $\mathbf{H} \subset \mathbf{C}$ as Möbius transformations. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{R})$ be a Fuchsian group with finite covolume and let $d(\cdot,\cdot)$ denote the hyperbolic distance on $\mathbf{H} \cong \mathbf{R} \times \mathbf{R}_+$ derived from the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. With this metric elements in $\mathrm{SL}_2(\mathbf{R})$ act as orientation-preserving isometries on $\mathbf{H}$. It has been proved by numerous authors ([2], [4], [5], [6], [7], [9], [10]) that there exists $a < 1$ such that for z_1, z_2 fixed and $R > 0$,

$$\#\{\gamma \in \Gamma : d(\gamma z_1, z_2) \leq R\} = \frac{\mathfrak{K}_\Gamma \pi}{\mathrm{Vol}(\Gamma \backslash \mathbf{H})} e^R + O(e^{aR}), \tag{1.1}$$

where

$$\mathfrak{K}_\Gamma = \begin{cases} 1 & \text{if } \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \notin \Gamma, \\ 2 & \text{if } \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in \Gamma. \end{cases}$$

This is known as the hyperbolic lattice point problem. In [8] a very general approach to lattice point counting in locally compact groups is presented.

In this paper we consider a variant of the hyperbolic lattice point problem. Throughout this paper we let Γ denote a non-cocompact Fuchsian group of the first kind and we will assume that ∞ is a cusp of Γ of width 1. The last condition is not very strict, as we can always bring ourselves into this situation by conjugation. We let Γ_∞ denote the stability group of the cusp ∞. If $\mathfrak{K}_\Gamma = 1$ the group Γ_∞ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$. If $\mathfrak{K}_\Gamma = 2$ the group Γ_∞ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right)$.

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Define

$$N_\Gamma(T;z) = \#\{\gamma \in \Gamma_\infty \backslash \Gamma : \mathrm{Im}(\gamma z) \geq T^{-1}\}. \tag{1.2}$$

Our approach to investigating (1.2) is via spectral theory of the automorphic Laplacian on $\Gamma\backslash\mathbf{H}$. We let Δ denote the automorphic Laplacian which is the selfadjoint closure of the operator

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

restricted to smooth functions on $\Gamma\backslash\mathbf{H}$ with compact support. It is well known that Δ has a continuous spectrum $[1/4, \infty)$ (with multiplicity equal to the number of inequivalent cusps) as well as eigenvalues (counted with multiplicity)

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots,$$

which are finite in number or tending to ∞. We write $\lambda_j = s_j(1 - s_j)$, where $\sigma_j = \mathrm{Re}(s_j) \geq 1/2$. We remark that it is important for our applications that there is a spectral gap between the zeroth eigenvalue $\lambda_0 = 0$ and the first eigenvalue λ_1. In other words we have the strict inequality $\sigma_1 < \sigma_0 = 1$.

A. Good [4] proved that for z fixed and $T \geq 1$,

$$N_\Gamma(T;z) = \frac{T}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} + O(T^{\max(2/3,\sigma_1)}). \tag{1.3}$$

Here $\mathrm{Vol}(A) = \int_A y^{-2}dxdy$ denotes the hyperbolic volume on $\mathbf{H}$. The asymptotics in (1.3) is indeed the result we would expect since the hyperbolic volume of the infinite box

$$\#\{x + iy \in \mathbf{H} : x \in [0,1],\ y \geq T^{-1}\}$$

is T. In (2.8) below we obtain (1.3) with the error term $O(T^{(1+\sigma_1)/2+\varepsilon})$.

For a closed interval $I \subset \mathbf{R}/\mathbf{Z}$ we define

$$N_\Gamma^I(T;z) = \#\{\gamma \in \Gamma_\infty\backslash\Gamma : \mathrm{Im}(\gamma z) \geq T^{-1},\ \mathrm{Re}(\gamma z) \in I\}. \tag{1.4}$$

The main theorem of this paper is the following effective equidistribution result:

Theorem 1. *Let $I \subset \mathbf{R}/\mathbf{Z}$ be a closed interval, $K \subset \Gamma\backslash\mathbf{H}$ a compact set, and $T \geq 1$. Then*

$$N_\Gamma^I(T;z) = \frac{|I|T}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} + O(T^{(3+\sigma_1)/4+\varepsilon})$$

uniformly for $z \in K$. The constant implied depends at most on Γ, ε, and K.

The asymptotics in the theorem was proved in [4] and [11] but without the error term. Our strategy is to combine the spectral theory arguments in [11] and [12].

Using the approximation argument in [12, Lemma 3.1] we can count lattice points in more general domains. We do not strive to obtain maximal generality on the domains, but we give a couple of examples. For $\Omega \subset \Gamma_\infty\backslash\mathbf{H}$ we define

$$\mathcal{N}_\Gamma(\Omega;z) = \#\{\gamma \in \Gamma_\infty\backslash\Gamma : \gamma z \in \Omega\}.$$

When estimating $\mathcal{N}_\Gamma(\Omega;z)$ we will view Ω as a subset of $[0,1] \times \mathbf{R}_+$ rather than $\Gamma_\infty\backslash\mathbf{H}$. It will be clear as we go along that this does not make any difference.

Let $[a,b] \subset [0,1]$ and $\phi : [a,b] \to \mathbf{R}_+$ be bounded (here bounded means that $\phi([a,b])$ is compact in $\mathbf{R}_+$) and integrable (thus $1/\phi$ is integrable). We define

$$G_\phi = \{x + iy \in \Gamma_\infty\backslash\mathbf{H} : a \le x \le b,\ y \ge \phi(x)\}.$$

We easily check that

$$\text{(1.5)} \qquad \operatorname{Vol}(G_\phi) = \int_a^b \int_{\phi(x)}^\infty y^{-2} dy dx = \int_a^b \phi(x)^{-1} dx.$$

The following theorem is natural and allows us to estimate the number of lattice points in rather general domains:

Theorem 2. *Let $[a,b] \subset [0,1]$ and $\phi : [a,b] \to \mathbf{R}_+$ be monotonic and bounded. Then*

$$\mathcal{N}_\Gamma(G_\phi; z) = \frac{\operatorname{Vol}(G_\phi)}{\operatorname{Vol}(\Gamma\backslash\mathbf{H})} + O((1/\inf\phi)^{(7+\sigma_1)/8+\varepsilon}).$$

The constant implied depends at most on Γ, ε, and z.

The following is an immediate consequence of Theorem 2:

Corollary 3. *Let $[a,b] \subset [0,1]$ and $f : [a,b] \to \mathbf{R}_+$ be monotonic and bounded. Define $\phi_T(x) = f(x)/T$ for $x \in [a,b]$, $T \ge 1$. Then*

$$\mathcal{N}_\Gamma(G_{\phi_T}; z) = \frac{\operatorname{Vol}(G_{\phi_T})}{\operatorname{Vol}(\Gamma\backslash\mathbf{H})} + O(T^{(7+\sigma_1)/8+\varepsilon}).$$

The constant implied depends at most on Γ, ε, z, and $\inf f$. Also we have

$$\operatorname{Vol}(G_{\phi_T}) = T \int_a^b f(x)^{-1} dx.$$

Let $[\alpha,\beta] \subset \mathbf{R}_+$ and $F : [\alpha,\beta] \to [0,1]$ be monotonic. We define:

$$\text{(1.6)} \qquad \Omega_{F,T} = \{x + iy \in \Gamma_\infty\backslash\mathbf{H} : y \in [\alpha/T, \beta/T],\ \min(F) \le x \le F(Ty)\}.$$

In [1] the following theorem was proved:

Theorem 4. *Let $[\alpha,\beta] \subset \mathbf{R}_+$ and $F : [\alpha,\beta] \to [0,1]$ be monotonic and continuously differentiable. For $T > 2$,*

$$\mathcal{N}_{\mathrm{SL}_2(\mathbf{Z})}(\Omega_{F,T}; i) = \frac{3T}{\pi} \int_{\beta^{-1}}^{\alpha^{-1}} (F(1/t) - \min(F)) dt + O(T^{7/8} \log T).$$

The constant implied depends only on α.

Using the ideas in Theorem 2 and Corollary 3 we can extend Theorem 4 to any non-cocompact Fuchsian group of the first kind with a cusp at ∞ of width 1:

Proposition 5. *Let $[\alpha,\beta] \subset \mathbf{R}_+$ and $F : [\alpha,\beta] \to [0,1]$ be monotonic and continuous. For $T \ge 1$ we have*

$$\mathcal{N}_\Gamma(\Omega_{F,T}; z) = \frac{T}{\operatorname{Vol}(\Gamma\backslash\mathbf{H})} \int_{\beta^{-1}}^{\alpha^{-1}} (F(1/t) - \min(F)) dt + O(T^{(7+\sigma_1)/8+\varepsilon}).$$

The constant implied depends at most on Γ, ε, z, and α.

The error term in Proposition 5 is worse than the error term in Theorem 4 in the case of the modular group. However, it is important to stress that the asymptotics found in Proposition 5 is not an "arithmetic" feature; rather it is a general property of non-cocompact finite volume hyperbolic surfaces. From Proposition 5 we easily deduce:

Corollary 6. *Let $\Omega \subset \mathbf{H}$ be a closed and connected set bounded by straight line segments and m_1 curves ξ_j of the form $\xi_j(t) = (t, f_j(t))$, where $f_j : [a_j, b_j] \to \mathbf{R}_+$ is monotonic. Assume $\Omega \subset [0, m_2] \times \mathbf{R}_+$ and set*

$$\{\alpha_0, \dots, \alpha_m\} = \{a_1, b_1, \dots, a_{m_1}, b_{m_1}\} \cup \{0, 1, \dots, m_2 - 1, m_2\},$$

where $\alpha_{j-1} < \alpha_j$ for $j = 1, \dots, m$. Define

$$L_j = \{z \in \mathbf{H} : \mathrm{Re}(z) \in [\alpha_{j-1}, \alpha_j]\}.$$

Then

$$\#\{\gamma \in \Gamma : \gamma z \in \Omega\} = \frac{\mathfrak{K}_\Gamma \mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} + O\bigg(\sum_{j=1}^{m} (1/\inf\{\mathrm{Im}(z) : z \in \Omega \cap L_j\})^{(7+\sigma_1)/8+\varepsilon}\bigg),$$

where the constant implied depends at most on Γ, ε, and z.

2. Equidistribution

In this section we prove Theorem 1. Following [11] we define

$$V_m(z, s) = \sum_{\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s e(m\mathrm{Re}(\gamma z)),$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$. The function $V_m(z, s)$ is not square integrable on $\Gamma\backslash\mathbf{H}$, but as remarked in [11] the function

$$W_m(z, s) = V_m(z, s) - h(y) y^s e(mx)$$

is. Here h is a smooth, increasing function such that $h(y) = 0$ for $y < T$ and $h(y) = 1$ for $y > T + 1$, $T \geq 1$. The series $V_m(z, s)$ satisfies the equation

$$(\Delta - s(1 - s)) V_m(z, s) = (2\pi m)^2 V_m(z, s + 2).$$

Moreover,

$$(\Delta - s(1 - s)) W_m(z, s) = (2\pi m)^2 W_m(z, s + 2) - H_m(z, s), \tag{2.1}$$

where

$$H_m(z, s) = h''(y) y^{s+2} e(mx) + 2h'(y) y^{s+1} e(mx).$$

Note that $H_m(z, s)$ is compactly supported for $z \in \Gamma\backslash\mathbf{H}$. From [11, Proposition A.1] we know that $V_m(z, s)$ and $W_m(z, s)$ admit meromorphic continuations to $\mathrm{Re}(s) > 1/2$. The potential poles are at $s = \sigma_j$, where $\lambda_j = \sigma_j(1 - \sigma_j)$ are the small eigenvalues of Δ (i.e. the eigenvalues smaller than $1/4$). For $m \neq 0$, $V_m(z, s)$ and $W_m(z, s)$ are regular at $s = 1$. For $m = 0$ we have

$$\mathrm{res}_{s=1} W_0(z, s) = \mathrm{res}_{s=1} V_0(z, s) + O(1/T) = 1/\mathrm{Vol}(\Gamma\backslash\mathbf{H}) + O(1/T).$$

The constant implied is absolute.

We need the following estimates on $W_m(z,s)$:

Lemma 7. *Let $s=\sigma+it$ and $K\subset F$ be a fixed compact set. For $|t|>1$ and $\sigma>1/2+\kappa$, $\kappa>0$ we have*

$$W_m(z,s)=O((m^2+1)|t|).$$

For $|t|\le 1$ and $\sigma\in(\sigma_1+\kappa,1-\kappa)$, $\kappa>0$ we have

$$W_m(z,s)=O(m^2+1).$$

The constants implied may depend on K, Γ, and κ.

Proof. The proof follows the proof of Lemma 2.1 in [12] closely. Assume first that $|t|>1$. We have

$$\|R(s)\|\le\frac{1}{\operatorname{dist}(s(1-s),\operatorname{spec}(\Delta))}\le\frac{1}{|t|\,(2\sigma-1)},\tag{2.2}$$

where $\|\cdot\|$ denotes the operator norm and $R(s)=(\Delta-s(1-s))^{-1}$ is the resolvent. We see that

$$\|W_m(z,s)\|_2\le\|W_0(z,3/2)\|_2$$

for $\sigma\ge 3/2$. By (2.1) we obtain the L^2-bound

$$\begin{aligned}\|W_m(z,s)\|_2&\le\|R(s)\|\left((2\pi m)^2\|W_m(z,s+2)\|_2+\|H_m(z,s)\|_2\right)\\&=O((m^2+1)/|t|)\end{aligned}$$

for $\operatorname{Re}(s)>1/2+\kappa$. From (2.1) we also get

$$\begin{aligned}&\|\Delta W_m(z,s)\|_2\\&\quad\le(2\pi m)^2\|W_m(z,s+2)\|_2+\|H_m(z,s)\|_2+\|s(1-s)W_m(z,s)\|_2\\&\quad=O((m^2+1)\,|t|),\end{aligned}$$

for $\operatorname{Re}(s)>1/2+\kappa$.

To obtain the desired pointwise bound (in fact we get a uniform bound) we use the same strategy as in [12, Lemma 2.1], which uses the Sobolev imbedding theorem and elliptic regularity theory. We omit the details. This leads to the bound

$$\sup_{z\in K}|W_m(z,s)|=O(\|W_m(z,s)\|_2+\|\Delta W_m(z,s)\|_2)=O((m^2+1)\,|t|).$$

Still the constants implied depend at most on K, Γ, and κ. This proves the case where $|t|>1$. For $|t|\le 1$ and $\sigma\in(\sigma_1+\kappa,1-\kappa)$ we have the bound

$$\|R(s)\|\le\sup_{\substack{|t|\le 1\\ \sigma\in(\sigma_1+\kappa,1-\kappa)}}\max(|\lambda_1-(s(1-s))|^{-1},|1-(s(1-s))|^{-1}),$$

which comes from the first inequality in (2.2). Note that this is finite since $[\sigma_1+\kappa,1-\kappa]\times[-1,1]$ is compact. With this estimate we can proceed as before to obtain the desired bound on $W_m(z,s)$. □

For $U>U_0\ge 1$ let $\psi_U:\mathbf{R}_+\to\mathbf{R}$ be a family of smooth decreasing functions such that

$$\psi_U(t)=\begin{cases}1 & \text{if } t\le 1-1/U,\\ 0 & \text{if } t\ge 1+1/U,\end{cases}\tag{2.3}$$

and $\psi_U^{(j)}(t) = O(U^j)$ as $U \to \infty$. For $\mathrm{Re}(s) > 0$ we let

$$\Psi_U(s) = \int_0^\infty \psi_U(y) y^{s-1} dy$$

denote the Mellin transform of ψ_U. Mean value estimates give us

$$\Psi_U(s) = s^{-1} + O(U^{-1}) \qquad \text{as } U \to \infty \tag{2.4}$$

and for any $c > 0$,

$$\Psi_U(s) = O\left(\frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^c\right) \qquad \text{as } |s| \to \infty. \tag{2.5}$$

The estimates are uniform for $\mathrm{Re}(s)$ bounded. By the Mellin inversion formula we get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(m\mathrm{Re}(\gamma z))(1-h(\mathrm{Im}(\gamma z)))\psi_U(\mathrm{Im}(\gamma z)^{-1}T^{-1}) = \frac{1}{2\pi i}\int_{(2)} W_m(z,s)\Psi_U(s)T^s ds.$$

Note that Lemma 7 ensures that the integral on the right-hand side is convergent. Moving the line of integration to the line $\mathrm{Re}(s) = \sigma_1 + \varepsilon$ we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)} W_m(z,s)\Psi_U(s)T^s ds &= \mathrm{res}_{s=1}(W_m(z,s)\Psi_U(s)T^s) + \frac{1}{2\pi i}\int_{(\sigma_1+\varepsilon)} W_m(z,s)\Psi_U(s)T^s ds \\ &= \delta_{m,0}(T/\mathrm{Vol}(\Gamma\backslash\mathbf{H}) + O(T/U+1)) + \frac{1}{2\pi i}\int_{(\sigma_1+\varepsilon)} W_m(z,s)\Psi_U(s)T^s ds.\end{aligned}$$

We need to estimate the last integral. Applying Lemma 7 and (2.5) with $c = 1+\varepsilon$ the part of the line integral with $|t| > 1$ is

$$O((m^2+1)T^{\sigma_1+\varepsilon}U^{1+\varepsilon}).$$

The part of the line integral with $|t| \leq 1$ is estimated using Lemma 7 and (2.4), and we obtain

$$O((m^2+1)T^{\sigma_1+\varepsilon}).$$

Consequently,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(m\mathrm{Re}(\gamma z))(1-h(\mathrm{Im}(\gamma z)))\psi_U(\mathrm{Im}(\gamma z)^{-1}T^{-1}) = \delta_{m,0}(T/\mathrm{Vol}(\Gamma\backslash\mathbf{H}) + O(T/U)) + O((m^2+1)T^{\sigma_1+\varepsilon}U^{1+\varepsilon}). \tag{2.6}$$

Clearly,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(m\mathrm{Re}(\gamma z))h(\mathrm{Im}(\gamma z))\psi_U(\mathrm{Im}(\gamma z)^{-1}T^{-1}) = O(1). \tag{2.7}$$

Let $\psi_U^-, \psi_U^+ : \mathbf{R}_+ \to \mathbf{R}$ satisfy (2.3) and

$$\psi_U^- \leq \mathbf{1}_{(0,1]} \leq \psi_U^+,$$

where $\mathbf{1}_{(0,1]}$ denotes the characteristic function on $(0,1]$. We see that

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi_U^-(\mathrm{Im}(\gamma z)^{-1}T^{-1})\leq N_\Gamma(T;z)\leq\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi_U^+(\mathrm{Im}(\gamma z)^{-1}T^{-1}).$$

Thus (2.6) and (2.7) with $m=0$ and $U=T^{(1-\sigma_1)/2}$ imply

$$N_\Gamma(T;z)=\frac{T}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}+O(T^{(1+\sigma_1)/2+\varepsilon}).\tag{2.8}$$

We remark that (1.3) could be used in the estimate below, but it does not give rise to a better error term in the final result. Using (2.8) we see that

$$\begin{aligned}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}e(m\mathrm{Re}(\gamma z))\psi_U^+(\mathrm{Im}(\gamma z)^{-1}T^{-1})-\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \mathrm{Im}(\gamma z)\geq T^{-1}}}e(m\mathrm{Re}(\gamma z))&\\ =O(N_\Gamma(T(1+U^{-1});z)-N_\Gamma(T;z))&\\ =O(T/U+T^{(1+\sigma_1)/2+\varepsilon}).&\end{aligned}$$

This together with (2.6) and (2.7) gives

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \mathrm{Im}(\gamma z)\geq T^{-1}}}e(m\mathrm{Re}(\gamma z))=O(T/U+T^{(1+\sigma_1)/2+\varepsilon}+m^2T^{\sigma_1+\varepsilon}U^{1+\varepsilon})$$

for $m\neq 0$. Let $I\subset\mathbf{R}/\mathbf{Z}$ be an interval. The Erdös-Turán inequality [3, Theorem III] implies that

$$\frac{N_\Gamma^I(T;z)}{N_\Gamma(T;z)}=|I|+O(M^{-1}+(U^{-1}+T^{(\sigma_1-1)/2+\varepsilon})\log M+M^2T^{\sigma_1-1+\varepsilon}U^{1+\varepsilon}).$$

Theorem 1 follows by setting $M=U=T^{(1-\sigma_1)/4}$.

3. Lattice points in general domains

First we prove a technical lemma.

Lemma 8. *Let $[a,b]\subset[0,1]$ and $\phi:[a,b]\to\mathbf{R}_+$ be bounded. For $j=1,\dots,M$ set*

$$I_j=[a+(j-1)(b-a)/M,a+j(b-a)/M].$$

Then there exists a constant C depending at most on Γ, z and ε such that

$$\begin{aligned}\frac{b-a}{M\mathrm{Vol}(\Gamma\backslash\mathbf{H})}\sum_{j=1}^M 1/(\sup_{\omega\in I_j}\phi(\omega))-CM(1/\inf\phi)^{(3+\sigma_1)/4+\varepsilon}\leq\mathcal{N}_\Gamma(G_\phi;z)&\\ \leq\frac{b-a}{M\mathrm{Vol}(\Gamma\backslash\mathbf{H})}\sum_{j=1}^M 1/(\inf_{\omega\in I_j}\phi(\omega))+CM(1/\inf\phi)^{(3+\sigma_1)/4+\varepsilon}&\end{aligned}$$

for any $M\geq 1$.

Proof. Define

$$G_\phi^j=\{z\in G_\phi:\mathrm{Re}(z)\in I_j\}.$$

By Theorem 1 we see that

$$\mathcal{N}_\Gamma(G_\phi;z)=\sum_{j=1}^M\mathcal{N}_\Gamma(G_\phi^j;z)+O(M(1/\inf\phi)^{(3+\sigma_1)/4+\varepsilon}).$$

The error term comes from counting the end points twice. Applying Theorem 1 we get the estimate

$$\frac{b-a}{M(\sup_{\omega\in I_j}\phi(\omega))\mathrm{Vol}(\Gamma\backslash\mathbf{H})}-C'(1/\inf\phi)^{(3+\sigma_1)/4+\varepsilon}\leq\mathcal{N}_\Gamma(G_\phi^j;z)$$
$$\leq\frac{b-a}{M(\inf_{\omega\in I_j}\phi(\omega))\mathrm{Vol}(\Gamma\backslash\mathbf{H})}+C'(1/\inf\phi)^{(3+\sigma_1)/4+\varepsilon}.$$

The result now follows by summing over $j=1,\dots,M$. □

Our first application of the lemma is to prove Theorem 2.

Proof of Theorem 2. Set

$$\phi_j^-=\sup_{\omega\in I_j}\phi(\omega),$$
$$\phi_j^+=\inf_{\omega\in I_j}\phi(\omega).$$

Using the fact that ϕ (and hence $1/\phi$) is monotonic we see that

$$\left|\int_a^b\phi(x)^{-1}dx-\frac{b-a}{M}\sum_{j=1}^M\phi_j^{\pm-1}\right|=\left|\sum_{j=1}^M\int_{I_j}(\phi(x)^{-1}-\phi_j^{\pm-1})dx\right|$$
$$\leq((1/\inf\phi)-(1/\sup\phi))/M.$$

Thus, Lemma 8 implies that

$$\left|\mathcal{N}_\Gamma(G_\phi;z)-\frac{\mathrm{Vol}(G_\phi)}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}\right|\leq\frac{(1/\inf\phi)}{M\mathrm{Vol}(\Gamma\backslash\mathbf{H})}+CM((1/\inf\phi))^{(3+\sigma_1)/4+\varepsilon}.$$

The result now follows by choosing $M=[(1/\inf\phi)^{(1-\sigma_1)/8}]$. □

We can now prove Proposition 5.

Proof of Proposition 5. We will assume that F is increasing (the case where F is decreasing can be treated in a similar fashion). Let $a=\min(F)$ and $b=\max(F)$ and define $f_+,f_-:[a,b]\to\mathbf{R}_+$ by

$$f_+(t)=\inf F^{-1}(t),$$
$$f_-(t)=\sup F^{-1}(t)$$

for $t\in[a,b]$. The functions f_+ and f_- are both well defined (by the intermediate value theorem) and they are both monotonic (increasing). Define

$$\phi_{+,T}(t)=f_+(t)/T,$$
$$\phi_{-,T}(t)=f_-(t)/T$$

and

$$\Omega_{-,T}=\{x+iy\in\mathbf{H}:y\geq\beta T^{-1},\ x\in I\},$$
$$\Omega_{+,T}=\{x+iy\in\mathbf{H}:y>\beta T^{-1},\ x\in I\}.$$

We see that

$$G_{\phi_{-,T}}-\Omega_{-,T}\subset\Omega_{F,T}\subset G_{\phi_{+,T}}-\Omega_{+,T}$$

and

$$(G_{\phi_{+,T}}-\Omega_{+,T})-(G_{\phi_{-,T}}-\Omega_{-,T})=(G_{\phi_{+,T}}-G_{\phi_{-,T}})\cup(\Omega_{-,T}-\Omega_{+,T}).$$

By Theorem 1 we get

$$\mathcal{N}_\Gamma((\Omega_{-,T} - \Omega_{+,T}); z) = O(T^{(3+\sigma_1)/4+\varepsilon}).$$

Corollary 3 implies that

$$\mathcal{N}_\Gamma(G_{\phi_{+,T}} - G_{\phi_{-,T}}; z) = \frac{\mathrm{Vol}(G_{\phi_{+,T}} - G_{\phi_{-,T}})}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} + O(T^{(7+\sigma_1)/8+\varepsilon})$$

and

$$\begin{aligned}
&\mathrm{Vol}(G_{\phi_{+,T}} - G_{\phi_{-,T}})/T \\
&\quad = \int_a^b (f_+(x)^{-1} - f_-(x)^{-1})dx \\
&\quad \leq \frac{b-a}{n}\sum_{j=1}^{n}(f_+((j-1)(b-a)/n+a)^{-1} - f_-(j(b-a)/n+a)^{-1}) \\
&\quad \leq \frac{2(b-a)(\alpha^{-1}-\beta^{-1})}{n}.
\end{aligned}$$

Since this holds for all $n \geq 1$ we conclude that

$$\mathrm{Vol}(G_{\phi_{+,T}} - G_{\phi_{-,T}}) = 0$$

for all T. Thus it follows that

$$\mathcal{N}_\Gamma(\Omega_{F,T}; z) = \frac{\mathrm{Vol}(\Omega_{F,T})}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} + O(T^{(7+\sigma_1)/8+\varepsilon}),$$

and since

$$\mathrm{Vol}(\Omega_{F,T}) = \int_{\alpha/T}^{\beta/T}\int_{\min(F)}^{F(Ty)}\frac{dxdy}{y^2} = T\int_{\beta^{-1}}^{\alpha^{-1}}(F(1/t) - \min(F))dt,$$

the proposition follows. □

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Department of Mathematical Sciences, Aarhus University, Ny Munkegade 118, DK-8000 Aarhus C, Denmark

E-mail address: lee@imf.au.dk