

A DICHOTOMY FOR EXPANSIONS OF THE REAL FIELD

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ABSTRACT. A dichotomy for expansions of the real field is established: Either \mathbb{Z} is definable or every nonempty bounded nowhere dense definable subset of \mathbb{R} has Minkowski dimension zero.

Given $E \subseteq \mathbb{R}^n$ bounded and $r > 0$, let $N(E, r)$ be the number of closed balls of radius r needed to cover E . Put $\overline{\dim}_M E = \overline{\lim}_{r \downarrow 0} \log N(E, r) / \log(1/r)$ (with $\log 0 := -\infty$), the **upper Minkowski dimension** of E (but there are many different names and equivalent formulations in the literature). We refer to Falconer [2] for basic facts. We say that E is **M-null** if $\overline{\dim}_M E \leq 0$.

Theorem. *Given an expansion of the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot)$, either \mathbb{Z} is definable or every bounded nowhere dense definable subset of \mathbb{R} is M-null.*

In other words: *An expansion of $\overline{\mathbb{R}}$ avoids defining \mathbb{Z} if and only if every bounded definable subset of \mathbb{R} is either somewhere dense or M-null.* (See van den Dries and Miller [1, §2] for definitions and basic facts about definability in expansions of $\overline{\mathbb{R}}$.) A standard exercise is that $\overline{\dim}_M \{1/(n+1) : n \in \mathbb{N}\} = 1/2$, so only the forward implication needs to be established. We begin with a result in geometric measure theory. For $E \subseteq \mathbb{R}$, put

$$Q(E) = \{ (a-b)/(c-d) : a, b, c, d \in E \text{ \& } c \neq d \}.$$

Lemma. *Let $E \subseteq \mathbb{R}$ be bounded and $\overline{\dim}_M E > 0$. Then there exist $n \in \mathbb{N}$ and linear $T: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $Q(T(E^n))$ is dense in \mathbb{R} .*

Proof. That $\overline{\dim}_M E^n = n \overline{\dim}_M E$ is an exercise, so $\lim_{n \rightarrow \infty} \overline{\dim}_M E^n = \infty$. By Falconer and Howroyd [3, Theorem 3], there exist $n \in \mathbb{N}$ and linear $T: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\overline{\dim}_M T(E^n) > 1/2$. Hence, it suffices to consider the case that $\overline{\dim}_M E > 1/2$ and show that $Q(E)$ is dense in \mathbb{R} . (We thank Kenneth Falconer for the following elegant geometric argument. Our original proof was based on additive combinatorics.) Suppose not. Observe that $Q(E)$ is the set of slopes of nonvertical lines connecting pairs of points in E^2 . Thus, the difference set $\{u-v : u, v \in E^2\}$ of E^2 is disjoint from some open double cone $C \subseteq \mathbb{R}^2$ centered at the origin. Let ℓ be the line through the origin perpendicular to the axis of C . Then the restriction to E^2 of the projection of \mathbb{R}^2 onto ℓ is injective, and the compositional inverse is Lipschitz. Hence, E^2 is contained in a rotation of the graph of a Lipschitz function from some bounded subinterval of \mathbb{R} into \mathbb{R} . It follows from [2, Corollary 11.2] that $\overline{\dim}_M E^2 \leq 1$. But then $\overline{\dim}_M E = (\overline{\dim}_M E^2)/2 \leq 1/2$, a contradiction. \square

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Proof of Theorem. It suffices to let $E \subseteq \mathbb{R}$ be bounded and nowhere dense with $\overline{\dim}_M E > 0$ and show that $(\overline{\mathbb{R}}, E)$ defines \mathbb{Z} . As $\overline{\dim}_M$ is preserved under topological closure, we reduce to the case that E is compact and has no interior. The set A of left endpoints of the complementary intervals of E is dense in E , so $\overline{\dim}_M A > 0$. By the lemma, there exist $n \in \mathbb{N}$ and linear $T: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $Q(T(A^n))$ is dense in \mathbb{R} . Let D be the set of midpoints of the bounded complementary intervals of E . Define $g: D \rightarrow \mathbb{R}$ by $g(x) = \sup(E \cap (-\infty, x])$. As g is definable in $(\overline{\mathbb{R}}, E)$ and $g(D) = A$, there exist $m \in \mathbb{N}$ and a function $f: D^m \rightarrow \mathbb{R}$ definable in $(\overline{\mathbb{R}}, E)$ such that $f(D^m)$ is somewhere dense. As D is discrete, $(\overline{\mathbb{R}}, E)$ defines \mathbb{Z} by [6, Theorem A]. \square

We believe the result to be optimal. There are Cantor sets $K \subseteq \mathbb{R}$ such that $(\overline{\mathbb{R}}, K)$ defines sets in every projective level, yet every subset of \mathbb{R} definable in $(\overline{\mathbb{R}}, K)$ either has interior or is nowhere dense (Friedman *et al.* [4]); we suspect that, for at least some such K , there exist dense $X \subseteq \mathbb{R}$ having empty interior such that $(\overline{\mathbb{R}}, K, X)$ still does not define \mathbb{Z} . In any case, \mathbb{Z} is not definable in the expansion of $\overline{\mathbb{R}}$ by $\{(2^j, 2^k 3^l) : j, k, l \in \mathbb{Z}\}$ (Günaydın [5]), yet it evidently defines both an infinite discrete set and a dense subset of $\mathbb{R}^{>0}$ that has empty interior.

Consequences, extensions and variations of the theorem are numerous, but will be detailed elsewhere by various subsets of the authors.

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