CRITERIA FOR TOEPLITZ OPERATORS ON THE SPHERE

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Abstract. Let $H^2(S)$ be the Hardy space on the unit sphere $S$ in $\mathbb{C}^n$. We show that a set of inner functions $\Lambda$ is sufficient for the purpose of determining which $A \in B(H^2(S))$ is a Toeplitz operator if and only if the multiplication operators $\{M_u : u \in \Lambda\}$ on $L^2(S,d\sigma)$ generate the von Neumann algebra $\{M_f : f \in L^\infty(S,d\sigma)\}$.

1. Introduction

Throughout the paper, we denote the unit sphere $\{z \in \mathbb{C}^n : |z| = 1\}$ in $\mathbb{C}^n$ by $S$. Let $\sigma$ be the positive, regular Borel measure on $S$ which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ which fix 0. Furthermore we normalize $\sigma$ such that $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is the closure in $L^2(S,d\sigma)$ of polynomials in the coordinate variables $z_1, \ldots, z_n$. Let $P : L^2(S,d\sigma) \to H^2(S)$ be the orthogonal projection. For each $f \in L^\infty(S,d\sigma)$, we have the Toeplitz operator $T_f$ on $H^2(S)$ defined by the formula

$$T_f = PM_f|H^2(S).$$

That is, $T_f$ is the compression of the multiplication operator $M_f$ to the subspace $H^2(S)$.

Toeplitz operators on various reproducing-kernel Hilbert spaces have been extensively studied in the literature. This paper concerns one of the most elementary questions in the theory, namely, how does one characterize a Toeplitz operator on $H^2(S)$?

In the case where the complex dimension $n$ equals 1, i.e., in the unit circle case, there is a very simple answer due to Brown and Halmos. In [2], Brown and Halmos showed that if $A$ is a bounded operator on the Hardy space $H^2$ of the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$, then $A$ is a Toeplitz operator if and only if it satisfies the equation

$$(1.1) \quad T_z \bar{z} A T_z = A.$$

This criterion for Toeplitz operators was later generalized to an arbitrary complex dimension $n$. In [3], Davie and Jewell showed that, for whatever $n$, a bounded
operator $A$ on $H^2(S)$ is a Toeplitz operator if and only if it satisfies the equation

$$\sum_{j=1}^n T_{z_j} A T_{\bar{z}_j} = A.$$  

By any reasonable standard, (1.2) is a satisfactory generalization of (1.1) to the high-dimensional case. But the fact that (1.2) is a satisfactory generalization of (1.1) also causes one to neglect the other side of the story, namely, when $n \geq 2$, (1.2) is really a different kind of test for Toeplitz operators. The substantive difference between (1.1) and (1.2) is the simple fact that when $n = 1$, $z$ is an inner function on the unit disc; in contrast, if $n \geq 2$, then the functions $z_1, \ldots, z_n$ are far from being inner.

Let $B$ denote the open unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ in $\mathbb{C}^n$. Recall that an analytic function $u$ on $B$ is said to be inner if

$$\lim_{r \uparrow 1} |u(r\zeta)| = 1 \quad \text{for } \sigma\text{-a.e. } \zeta \in S.$$  

As usual, we identify the function $u$ on $B$ with its boundary value on $S$. In this paper, we consider the problem of characterizing Toeplitz operators in terms of inner functions, which is not addressed by (1.2) in the case $n \geq 2$. When [3] was published in 1977, it was not yet known that there are non-constant inner functions on $B$ in the case $n \geq 2$. Therefore (1.2) was the best that could be managed in terms of characterizing Toeplitz operators at the time. Later, Aleksandrov [1] and Løw [6] showed that there are non-constant inner functions on $B$ for every $n \geq 2$. This makes it possible to consider the problem of characterizing Toeplitz operators in terms of inner functions.

Note that if $u$ is an inner function, then for every $f \in L^\infty(S, d\sigma)$ we have

$$T_u T_f T_u = T_{uf} = T_f.$$  

Thus, in order for an operator $A$ on $H^2(S)$ to be a Toeplitz operator, it is necessary that

$$T_u A T_u = A \quad \text{for every inner function } u \text{ on } B.$$  

Our question is the following. Suppose that $\Lambda$ is a non-empty set of inner functions on $B$. If $A$ is a bounded operator on $H^2(S)$ and if it satisfies the condition

$$T_u A T_u = A \quad \text{for every } u \in \Lambda,$$

can we conclude that $A$ is a Toeplitz operator on $H^2(S)$? Note that even in the case $n = 1$, this question goes beyond the Brown-Halmos criterion for Toeplitz operators.

We will characterize those sets $\Lambda$ which yield the answer “yes” to the above question. Interestingly, this characterization involves von Neumann algebras and a rare use of the double-commutant relation of Murray and von Neumann. We will then deal with specific $\Lambda$’s which yield the answer “yes” and specific $\Lambda$’s which yield the answer “no” to the above question.

To conclude the introduction, let us summarize these specific results.

(1) Let $\text{Aut}(B)$ denote the group of biholomorphic bijections on the ball $B$. Let $u$ be any non-constant inner function on $B$. Then the set $\{u \circ \psi : \psi \in \text{Aut}(B)\}$ is an example of $\Lambda$ that yields the answer “yes” to the above question. That is, if $A$ is a bounded operator on $H^2(S)$ such that $T_{u\circ\psi} A T_{u\circ\psi} = A$ for every $\psi \in \text{Aut}(B)$, then $A$ is a Toeplitz operator.
(2) Suppose that \( n \geq 2 \). Then for each singleton set \( \Lambda = \{ u \} \), the answer to the above question is always “no”. In other words, for every inner function \( u \) on \( B \), there is a bounded operator \( Y \) on \( H^2(S) \) such that \( T_u Y T_u = Y \) and yet \( Y \) is not a Toeplitz operator.

(3) Suppose that \( n = 1 \). Let \( u \) be an inner function on the unit disc. If \( u \) does not have the form 
\[
e^{i\theta} \frac{a - z}{1 - \bar{a}z},
\]
where \( \theta \in \mathbb{R} \) and \( |a| < 1 \), then there is a bounded operator \( Y \) on \( H^2 \) such that \( T_u Y T_u = Y \) and yet \( Y \) is not a Toeplitz operator. These comments put the Brown-Halmos criterion in the proper perspective.

2. Transformations on the unit ball

For each \( a \in B \setminus \{0\} \), we have the Möbius transformation
\[
\varphi_a(z) = \frac{1}{1 - \langle z, a \rangle} \left\{ a - \frac{\langle z, a \rangle}{|a|^2} a - (1 - |a|^2)^{1/2} \left( z - \frac{\langle z, a \rangle}{|a|^2} a \right) \right\}, \quad z \in B.
\]
Each \( \varphi_a \) is an involution, i.e., \( \varphi_a \circ \varphi_a = \text{id} \) [7, Theorem 2.2.2]. We also define \( \varphi_0(z) = -z \) on \( B \).

By Theorems 3.3.8 and 2.2.2 in [7], the formula
\[
(U_a f)(z) = \frac{(1 - |a|^2)^{n/2}}{(1 - \langle z, a \rangle)^n} f(\varphi_a(z)), \quad f \in L^2(S, d\sigma),
\]
defines a unitary operator with the property \([U_a, P] = 0\).

Let \( \mathcal{U} = \mathcal{U}(n) \) denote the collection of unitary transformations on \( \mathbb{C}^n \). For each \( V \in \mathcal{U} \), define the operator \( W_V : L^2(S, d\sigma) \rightarrow L^2(S, d\sigma) \) by the formula
\[
(W_V g)(z) = g(Vz), \quad g \in L^2(S, d\sigma).\]
By the invariance of \( \sigma \), \( W_V \) is a unitary operator on \( L^2(S, d\sigma) \).

Let \( \text{Aut}(B) \) denote the group of biholomorphic bijections on \( B \). If \( \psi \in \text{Aut}(B) \) and if \( a \in B \) is such that \( \psi(a) = 0 \), then
\[
(2.3) \quad \psi = V \varphi_a
\]
for some \( V \in \mathcal{U} \) [7, Theorem 2.2.5]. For such a \( \psi \), set
\[
(2.4) \quad R_\psi = U_a W_V.
\]
Then \( R_\psi \) is a unitary operator on \( L^2(S, d\sigma) \) which has the properties that
\[
R_\psi M_f R_\psi^* = M_{f \circ \psi}
\]
for every \( f \in L^\infty(S, d\sigma) \) and that \([R_\psi, P] = 0\). Consequently, \( H^2(S) \) is a reducing subspace for \( R_\psi \), and if we regard \( R_\psi \) as a unitary operator on \( H^2(S) \), then
\[
R_\psi T_f R_\psi^* = T_{f \circ \psi}
\]
for every \( f \in L^\infty(S, d\sigma) \).

Let \( H^\infty(S) \) be the collection of bounded analytic functions on \( B \). As usual, each \( h \in H^\infty(S) \) is identified with its boundary value on \( S \). Our first proposition is based on ideas in Section VI of the paper [4] by Feldman and Rochberg, where the main interest was Hankel operators with conjugate analytic symbols.
Proposition 2.1. Let \( h \in H^\infty(S) \). If \( h \) is not a constant then, on the Hilbert space \( L^2(S, d\sigma) \), the von Neumann algebra generated by
\[
\{ M_{h\psi} : \psi \in \text{Aut}(B) \}
\]
equals \{ M_f : f \in L^\infty(S, d\sigma) \}.

Proof. Let \( \partial_1, \ldots, \partial_n \) denote the differentiations with respect to the complex variables \( z_1, \ldots, z_n \) respectively. We first show that if \( h \) is not a constant, then there exist \( \psi_1, \ldots, \psi_n \in \text{Aut}(B) \) such that
\[
(2.5) \quad \partial_j(h \circ \psi_j)(0) \neq 0, \quad j = 1, \ldots, n.
\]
For any analytic function \( g \) on \( B \), write \( \text{grad}(g) = (\partial_1 g, \ldots, \partial_n g) \). If \( h \) is not a constant, then there exists an \( a \in B \) such that \( \text{grad}(h)(a) \neq 0 \), that is, \( \text{grad}(h)(\varphi_a(0)) \neq 0 \). Since the derivative \( \varphi_a'(0) \) is an invertible \( n \times n \) matrix \([7] \) Theorem 2.2.2(ii)], by the chain rule, we have \( \text{grad}(h \circ \varphi_a)(0) \neq 0 \). That is, there is at least one \( \nu \in \{1, \ldots, n\} \) such that
\[
\partial_\nu(h \circ \varphi_a)(0) \neq 0.
\]

By simple transpositions of coordinates, we see that there are \( V_1, \ldots, V_n \in U \) such that \( \partial_j(h \circ \varphi_a \circ V_j)(0) \neq 0 \) for \( j = 1, \ldots, n \). Thus (2.5) holds for \( \psi_j = \varphi_a \circ V_j = V_j \varphi_{T_j a}, j = 1, \ldots, n \).

Let \( T^n \) denote the \( n \)-dimensional torus \( \{ (\tau_1, \ldots, \tau_n) : |\tau_1| = \cdots = |\tau_n| = 1 \} \). Let \( dm_n \) be the Lebesgue measure on \( T^n \) with the normalization \( m_n(T^n) = 1 \). Now, for each pair of \( j \in \{1, \ldots, n\} \) and \( \tau = (\tau_1, \ldots, \tau_n) \in T^n \), define the function
\[
\eta^{(j)}_\tau(\tau_1, \ldots, \tau_n) = (h \circ \psi_j)(\tau_1 z_1, \ldots, \tau_n z_n).
\]

Then, of course, we still have \( \eta^{(j)}_\tau \in \{ h \circ \psi : \psi \in \text{Aut}(B) \} \). Using a power-series expansion, it is straightforward to verify that for each \( j \in \{1, \ldots, n\} \),
\[
\int \tau_j M_{\eta^{(j)}_\tau} dm_n(\tau) = \partial_j(h \circ \psi_j)(0) M_{z_j}.
\]

By (2.5), this means that the von Neumann algebra generated by \( \{ M_{h\psi} : \psi \in \text{Aut}(B) \} \) contains \( M_{z_1}, \ldots, M_{z_n} \). Since \( M_{z_1}, \ldots, M_{z_n} \) generate the von Neumann algebra \( \{ M_f : f \in L^\infty(S, d\sigma) \} \), this completes the proof. \( \Box \)

3. Main results

To better state our results, let us introduce the following terminology:

Definition 3.1. Let \( \Lambda \) be a non-empty set of inner functions. We say that the set \( \Lambda \) is Toeplitz-determining if it has the property that for \( A \in B(H^2(S)) \), the condition that \( T_u A T_u = A \) for every \( u \in \Lambda \) implies that \( A \) is a Toeplitz operator on \( H^2(S) \).

The following is our main result:

Theorem 3.2. Let \( \Lambda \) be a non-empty set of inner functions. Let \( \mathcal{N}(\Lambda) \) denote the von Neumann algebra generated by
\[
\{ M_u : u \in \Lambda \}
\]
on the Hilbert space \( L^2(S, d\sigma) \). Then the set \( \Lambda \) is Toeplitz-determining if and only if \( \mathcal{N}(\Lambda) = \{ M_f : f \in L^\infty(S, d\sigma) \} \).
Proof. First, assuming that \(N(A) = \{M_f : f \in L^\infty(S, d\sigma)\}\), we will show that \(A\) is Toeplitz-determining. The set \(A\) is, of course, a subset of \(H^2(S)\). Since \(H^2(S)\) is separable and since separability is a hereditary property for metric spaces, there is a countable subset \(\Lambda_0\) of \(A\) which is dense in \(A\) with respect to the norm topology of \(H^2(S)\). We can always list \(\Lambda_0\) as

\[
\Lambda_0 = \{u_1, u_2, \ldots, u_k, \ldots\}
\]

if we allow the possibility \(u_j = u_k\) for distinct \(j\) and \(k\).

Now suppose that \(A\) is a bounded operator on \(H^2(S)\) such that

\[
(3.1) \quad T_u A T_u = A \quad \text{for every } u \in \Lambda.
\]

To show that \(A = T_\varphi\) for some \(\varphi \in L^\infty(S, d\sigma)\), we follow the ideas in the proof of [3 Lemma 2.5]. Define the operator

\[
\tilde{A} = A \oplus 0
\]

on \(L^2(S, d\sigma)\), where the direct sum corresponds to the space decomposition

\[
L^2(S, d\sigma) = H^2(S) \oplus \{H^2(S)\}^\perp.
\]

For each natural number \(k\), define the operator

\[
L_k = \frac{1}{k^k} \sum_{1 \leq i_1, \ldots, i_k \leq k} M_{u_{i_k}}^* \cdots M_{u_1}^* \tilde{A} M_{u_1} \cdots M_{u_k}^*.
\]

It is easy to see that if \(1 \leq j \leq k\), then

\[
(3.2) \quad \|M_{\tilde{u}_j} L_k M_{u_j} - L_k\| \leq 2 \frac{k^{k-1}}{k^k} \|\tilde{A}\| = \frac{2}{k} \|A\|.
\]

Since \(\|L_k\| \leq \|\tilde{A}\| = \|A\|\) for every \(k \geq 1\), there is a strictly increasing sequence of natural numbers \(k(1) < k(2) < \cdots < k(\ell) < \cdots\) such that the limit

\[
L = \lim_{\ell \to \infty} L_{k(\ell)}
\]

exists in the weak operator topology. Clearly, it follows from (3.2) that \(M_{\tilde{u}_j} L M_{u_j} - L = 0\) for every \(j \geq 1\). Since \(|u_j| = 1\) a.e. on \(S\), this means that

\[
LM_{u_j} = M_{u_j} L \quad \text{for every } j \geq 1.
\]

Since \(\Lambda_0\) is dense in \(A\) with respect to the \(L^2\)-norm, the above implies that \(L\) commutes with \(\{M_u : u \in \Lambda\}\). Since each \(M_u, u \in \Lambda\), is a unitary operator, it follows that \(L\) also commutes with \(\{M_u : u \in \Lambda\}\). The assumption \(N(A) = \{M_f : f \in L^\infty(S, d\sigma)\}\) then leads to the conclusion that \(L\) commutes with \(\{M_f : f \in L^\infty(S, d\sigma)\}\). Hence there is a \(\varphi \in L^\infty(S, d\sigma)\) such that \(L = M_\varphi\).

Thus to complete the proof that \(A\) is a Toeplitz operator, it suffices to show that the compression of \(L\) to the subspace \(H^2(S)\) equals \(A\). Note that for \(h, g \in H^2(S)\) and natural numbers \(1 \leq i_1, \ldots, i_k \leq k\), we have

\[
\langle M_{u_{i_k}}^* \cdots M_{u_1}^* \tilde{A} M_{u_1} \cdots M_{u_k}^* h, g \rangle = \langle \tilde{A} M_{u_1} \cdots M_{u_k}^* h, M_{u_1}^* \cdots M_{u_k}^* g \rangle
\]

\[
= \langle AT_{u_1}^* \cdots T_{u_k}^* h, T_{u_1} \cdots T_{u_k} g \rangle
\]

\[
= \langle T_{u_k}^* \cdots T_{u_1}^* A T_{u_1} \cdots T_{u_k} h, g \rangle = \langle Ah, g \rangle,
\]

where the last = follows from repeated applications of (3.1). Thus the compression of each \(L_k\) to \(H^2(S)\) equals \(A\). Hence \(A = PLH^2(S)\) as promised. This proves the “if” part of the theorem.
To prove the “only if” part, let us now assume that \( \mathcal{N}(\Lambda) \neq \{M_f : f \in L^\infty(S, \sigma)\} \). We will find a bounded operator \( Y \) on \( H^2(S) \) such that \( T_uYT_u = Y \) for every \( u \in \Lambda \) and such that \( Y \notin \{T_f : f \in L^\infty(S, \sigma)\} \).

By the double-commutant relation, the assumption that \( \mathcal{N}(\Lambda) \neq \{M_f : f \in L^\infty(S, \sigma)\} \) implies that the commutant of \( \mathcal{N}(\Lambda) \) is strictly larger than the commutant of \( \{M_f : f \in L^\infty(S, \sigma)\} \). That is, there is a bounded operator \( Z \) which commutes with \( \mathcal{N}(\Lambda) \) but which does not commute with \( \{M_f : f \in L^\infty(S, \sigma)\} \).

Now take any non-constant inner function \( v \) constructed by Aleksandrov [11] or Löw [4]. By Proposition 2.1, the unitary operators \( \{M_{v\psi} : \psi \in \text{Aut}(B)\} \) generate the von Neumann algebra \( \{M_f : f \in L^\infty(S, \sigma)\} \). Since \( Z \) does not commute with \( \{M_f : f \in L^\infty(S, \sigma)\} \), it follows that there is an inner function \( w \in \{v \circ \psi : \psi \in \text{Aut}(B)\} \) such that
\[
M_w Z M_w \neq Z.
\]

We follow the usual multi-index notation [2] page 3). For each \( \alpha \in \mathbb{Z}_+^n \), define the function \( \epsilon_{\alpha}(z) = z^\alpha \) on \( S \). Then, of course, the linear span of \( \{\epsilon_{\alpha} \epsilon_{\beta} : \alpha, \beta \in \mathbb{Z}_+^n\} \) is dense in \( L^2(S, \sigma) \). Hence there are \( a, b, c, d \in \mathbb{Z}_+^n \) such that
\[
\langle M_{a} Z M_{a}, \epsilon_{b}, \epsilon_{c}, \epsilon_{d} \rangle \neq \langle Z \epsilon_{a}, \epsilon_{b}, \epsilon_{c}, \epsilon_{d} \rangle.
\]

Now define the operator \( X = M_a Z M_{\epsilon_a} \). Then the above gives us
\[
(3.3) \quad \langle M_w X M_w \epsilon_{a}, \epsilon_{c} \rangle \neq \langle X \epsilon_{a}, \epsilon_{c} \rangle.
\]

Let \( Y \) be the compression of \( X \) to the subspace \( H^2(S) \). That is, \( Y = PX|H^2(S) \). Since \( \epsilon_{a}, \epsilon_{c} \in L^2(S) \) and since \( (1 - P)M_w P = 0 \), (3.3) tells us that \( T_w Y T_w \neq Y \).

Since \( w \) is an inner function, this means that \( Y \) is not a Toeplitz operator.

On the other hand, since \( Z \) is in the commutant of \( \mathcal{N}(\Lambda) \), we have \( M_u Z M_u = Z \) for every \( u \in \Lambda \). Since \( X = M_u Z M_{\epsilon_u} \), it follows that \( M_u X M_u = X \) for every \( u \in \Lambda \). Compressing to the subspace \( H^2(S) \), we see that \( T_u Y T_u = Y \) for every \( u \in \Lambda \). Thus we have produced the promised \( Y \). This completes the proof. \( \square \)

As an immediate consequence of Theorem 3.2 and Proposition 2.1, we have

**Corollary 3.3.** Let \( u \) be a non-constant inner function. Then a bounded operator \( A \) on \( H^2(S) \) is a Toeplitz operator if and only if
\[
\overline{T_{u\psi} A T_{u\psi}} = A
\]
for every \( \psi \in \text{Aut}(B) \).

Recall that the unitary operator \( R_{\psi} \) defined by (2.4) has the property that \( R_{\psi}^* T_f R_{\psi} = T_{f \circ \psi^{-1}}, f \in L^\infty(S, \sigma) \). In other words, an operator \( A \) is a Toeplitz operator if and only if \( R_{\psi}^* AR_{\psi} \) is a Toeplitz operator. In light of this, let us restate Corollary 3.3 as

**Corollary 3.4.** Let \( u \) be a non-constant inner function. Then a bounded operator \( A \) on \( H^2(S) \) is a Toeplitz operator if and only if
\[
T_u R_{\psi}^* A R_{\psi} T_u = R_{\psi}^* A R_{\psi}
\]
for every \( \psi \in \text{Aut}(B) \).
4. Set of a singleton

Let us now consider the case where $\Lambda$ is a set of a single inner function. In this case, the story is actually simpler in complex dimensions $n \geq 2$.

**Proposition 4.1.** Suppose that $n \geq 2$. Then for each inner function $u$, there is a bounded operator $Y$ on $H^2(S)$ such that $T_{\bar{u}}YT_u = Y$ and such that $Y$ is not a Toeplitz operator.

**Proof.** Let an inner function $u$ be given and let $\mathcal{N}(u)$ denote the von Neumann algebra generated by the single unitary operator $M_u$. To prove the proposition, according to Theorem 3.2, it suffices to show that $\mathcal{N}(u) \neq \{M_f : f \in L^\infty(S,d\sigma)\}$.

Let $\mathcal{P}$ denote the linear span of all $w^j$ and $\bar{u}^k$, $j,k = 0,1,2,\ldots$. Note that since $u\bar{u} = 1$, $\mathcal{N}(u)$ is just the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. Because of the assumption $n \geq 2$, we have the function

$$q(z_1, \ldots, z_n) = z_1 \bar{z}_2$$

on $S$. To complete the proof, it suffices to show that $M_q$ is not in the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. Assuming the contrary, there would be a sequence $\{\xi_i\} \subset \mathcal{P}$ such that

$$\lim_{i \to \infty} \langle \xi_i, q \rangle = \langle q, q \rangle = \|q\|^2 > 0$$

for some sequence $\{\xi_i\} \subset \mathcal{P}$. But for each integer $j \geq 0$, the analyticity of $w^j$ gives us

$$\langle w^j, q \rangle = \langle z_2 w^j, z_1 \rangle = 0.$$ 

Similarly, for each integer $k \geq 0$, we have

$$\langle \bar{u}^k, q \rangle = \langle z_2, z_1 u^k \rangle = 0.$$ 

Thus, in the Hilbert space $L^2(S,d\sigma)$, $q$ is orthogonal to the set $\mathcal{P}$. This clearly contradicts (4.1). This contradiction shows that $M_q$ is not in the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. $\square$

**Proposition 4.2.** Suppose that $n = 1$. If $u$ is an inner function on the unit disc and if $u$ does not have the form

$$e^{i\theta} \frac{a - z}{1 - \bar{a}z}, \quad \text{where } \theta \in \mathbb{R} \text{ and } |a| < 1,$$

then there is a bounded operator $Y$ on $H^2$ such that $T_{\bar{u}}YT_u = Y$ and such that $Y$ is not a Toeplitz operator.

**Proof.** For the given $u$, again let $\mathcal{N}(u)$ denote the von Neumann algebra generated by the single unitary operator $M_u$. To prove the proposition, according to Theorem 3.2, it suffices to show that if $u$ does not have the form (4.2), then $\mathcal{N}(u) \neq \{M_f : f \in L^\infty\}$.

We only need, of course, to consider the case where $u$ is not a constant. Then

$$u = bs,$$

where $b$ is a Blaschke product or a constant, and $s$ is a so-called singular inner function or a constant. Suppose that $u$ does not have the form (4.2). Then either $b$ has at least two zeros (counting multiplicity) or $s$ is a non-trivial singular inner function. For such a $u$ it is well known (and easy to verify) that the dimension of
the subspace $H^2 \ominus uH^2$ is at least 2. Hence there is a $q \in H^2 \ominus uH^2$ with $\|q\| \neq 0$ which is orthogonal to the one-dimensional subspace $C$, that is,

$$q \in H^2, \quad q \perp uH^2, \quad \text{and} \quad q \perp C. \quad (4.3)$$

Since $\|q\| \neq 0$, there exists a $p \in L^\infty$ such that $\langle p, q \rangle \neq 0$.

As in the previous proof, let $P$ denote the linear span of all $u^j$ and $\overline{u}^k$, $j, k = 0, 1, 2, \ldots$. Again, $\mathcal{N}(u)$ is just the weak closure of $\{M_\xi : \xi \in P\}$. To complete the proof, it suffices to show that $M_p$ is not in the weak closure of $\{M_\xi : \xi \in P\}$.

Assuming the contrary, there would be a sequence $\{\xi_i\} \subset P$ such that $\langle M_{\xi_i} 1, q \rangle \to \langle M_p 1, q \rangle$ as $i \to \infty$. In other words, we would have

$$\lim_{i \to \infty} \langle \xi_i, q \rangle = \langle p, q \rangle \neq 0 \quad (4.4)$$

for some sequence $\{\xi_i\} \subset P$. But clearly, (4.3) implies $q \perp P$, which contradicts (4.4). This contradiction shows that $M_p$ is not in the weak closure of $\{M_\xi : \xi \in P\}$. \hfill \Box

References

1. A. Aleksandrov, The existence of inner functions in a ball (Russian), Mat. Sb. (N.S.) 118(160) (1982), 147-163, 287; (English) Math. USSR Sbornik 46 (1983), 143-159. MR658785 (83i:32002)

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