

PROJECTIVE ISOMONODROMY AND GALOIS GROUPS

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ABSTRACT. In this article we introduce the notion of projective isomonodromy, which is a special type of monodromy-evolving deformation of linear differential equations, based on the example of the Darboux-Halphen equation. We give an algebraic condition for a parameterized linear differential equation to be projectively isomonodromic, in terms of the derived group of its parameterized Picard-Vessiot group.

1. INTRODUCTION

Classically, monodromy-preserving deformations of Fuchsian systems have been investigated by many authors who described them in terms of the Schlesinger equation and its links to Painlevé equations. In [15], Landesman developed a new Galois theory for parameterized differential equations. A special case was developed in [8], where the authors consider parameterized *linear* differential equations and discuss various properties of the parameterized Picard-Vessiot group, the PPV-groups for short. This is a linear differential algebraic group in the sense of Cassidy [6]. As is well known, the differential Galois group of a system with regular singularities is, as a linear algebraic group, Zariski topologically generated by the monodromy matrices with respect to a fundamental solution. Cassidy and Singer have shown that a parameterized family of such systems is isomonodromic if and only if its PPV-group is conjugate to a (constant) linear algebraic group.

Analogous to the Schlesinger and Painlevé equations' relation to isomonodromic deformations of Fuchsian systems, the Darboux-Halphen V equation accounts for a special type of monodromy-evolving deformation of Fuchsian systems, as was shown by Chakravarty and Ablowitz in [9]. After reviewing the notion of isomonodromy in Section 2, we follow and detail, in Section 3, the description by Ohyama [17] of the Darboux-Halphen system in a way that will illustrate the general notion of projective isomonodromy that is introduced in Section 4 for general parameterized (not necessarily Fuchsian) differential systems with analytic coefficients. For Fuchsian systems, our definition naturally extends the classical notion of isomonodromy (not necessarily of the Schlesinger type) given for instance in [3]. For these systems, we show in Section 5 that projective isomonodromy is indeed the type of monodromy-evolving deformation introduced by Ohyama in ([17], Section 4) and we characterize it by a condition on the residue matrices. In Section 6, we consider

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general parameterized systems with regular singularities that are not necessarily Fuchsian, and we characterize projective isomonodromy by the purely algebraic condition that the derived group (G, G) of the PPV-group G be conjugate to a constant linear algebraic group when the given system is absolutely irreducible.

2. CLASSICAL ISOMONODROMY

In many of the classical studies of isomonodromic deformations, only parameterized *Fuchsian* systems are considered. Furthermore, these systems are assumed to be parameterized in a very special way; that is, the systems are written as

$$(2.1) \quad \frac{dY}{dx} = \sum_{i=1}^m \frac{A_i(a)}{x - a_i}, \quad \sum_{i=1}^m A_i(a) = 0,$$

where the $n \times n$ matrices $A_i(a)$ depend holomorphically on the multi-parameter $a = (a_1, \dots, a_m)$ in some open polydisk $D(a^0)$ and the condition on the residue matrices guarantees, for simplicity, that ∞ is not singular. The polydisk $D(a^0) = D_1 \times \dots \times D_m$ has center at the initial location $a^0 = (a_1^0, \dots, a_m^0) \in \mathbb{C}^m$ of the poles, with $D_i \subset \mathbb{C}$ a disk with center a_i^0 and $D_i \cap D_j \neq \emptyset$ for all $i \neq j$. Let $x_0 \in \mathcal{D} = \mathbb{P}^1(\mathbb{C}) \setminus \bigcup_i D_i$.

For fixed $a \in D(a^0)$ and local fundamental solution Y_a of (2.1) at x_0 , analytic continuation along a loop γ from x_0 in $\mathcal{D}_a = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_m\}$ yields a solution Y_a^γ . The *monodromy representation* with respect to Y_a is

$$(2.2) \quad \chi_a : \pi_1(\mathcal{D}_a; x_0) \rightarrow \text{GL}_n(\mathbb{C}),$$

defined by

$$Y_a^\gamma = Y_a \cdot \chi_a(\gamma),$$

for all $[\gamma] \in \pi_1(\mathcal{D}_a; x_0)$.

Definition 1. Equation (2.1) is *isomonodromic*, or an *isomonodromic deformation*, if for each $a \in D(a^0)$ there is a matrix $C(a) \in \text{GL}_n(\mathbb{C})$ such that

$$\chi_a = C(a) \chi_{a^0} C(a)^{-1}.$$

Bolibrukh ([3], [4]) has characterized isomonodromic deformations as follows.

Theorem 2 (Bolibruch). *Equation (2.1) is isomonodromic if and only if the following equivalent conditions hold.*

- (1) *There is a differential 1-form ω on $(\mathbb{P}^1(\mathbb{C}) \times D(a^0)) \setminus \bigcup_{i=1}^m \{(x, a) \mid x - a_i = 0\}$ such that*
 - *for each fixed $a \in D(a^0)$,*

$$\omega = \sum_{i=1}^m \frac{A_i(a)}{x - a_i} dx;$$

- $d\omega = \omega \wedge \omega$.
- (2) *For each $a \in D(a^0)$ there is a fundamental solution Y_a of (2.1) such that $Y_a(x)$ is analytic in x and a , and the corresponding monodromy representation χ_a does not depend on a , that is, $\chi_a = \chi_{a^0}$.*

A special type of isomonodromic deformation is given by the Schlesinger differential form

$$(2.3) \quad \omega_s = \sum_{i=1}^m \frac{A_i(a)}{x - a_i} d(x - a_i)$$

whose integrability condition is known as the *Schlesinger equation*

$$(2.4) \quad dA_i(a) = - \sum_{j=1, j \neq i}^m \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j), \quad i = 1, \dots, m.$$

Bolibrukh gave examples [3] of isomonodromic deformations that are not of the Schlesinger type and he described the general differential forms that occur in Theorem 2.

In the special case of order-two Fuchsian systems with four singularities one can, generically, reduce each system to an order-two linear scalar differential equation satisfied by the first component of the dependent variable Y , namely a Fuchsian scalar equation with an additional apparent singularity λ . It is well known that the Schlesinger isomonodromy condition then translates into a nonlinear equation of Painlevé VI type satisfied by λ . For basic results about Painlevé equations and isomonodromic deformations, we refer to [12] and [1].

3. AN EXAMPLE OF A MONODROMY EVOLVING DEFORMATION

In [9], Chakravarty and Ablowitz describe the Darboux-Halphen system

$$(3.1) \quad \begin{cases} \omega'_1 &= \omega_2\omega_3 & - \omega_1(\omega_2 + \omega_3) & + \phi^2 \\ \omega'_2 &= \omega_3\omega_1 & - \omega_2(\omega_3 + \omega_1) & + \theta^2 \\ \omega'_3 &= \omega_1\omega_2 & - \omega_3(\omega_1 + \omega_2) & - \theta\phi \\ \phi' &= \omega_1(\theta - \phi) & - \omega_3(\theta + \phi) & \\ \theta' &= -\omega_2(\theta - \phi) & - \omega_3(\theta + \phi) & \end{cases}$$

as a prototype of a class of nonlinear systems arising as the integrability conditions of an associated Lax pair in the same way as the Painlevé and Schlesinger equations do. This system occurs in the Bianchi IX cosmological models and arises from a special reduction of the self-dual Yang-Mills (SDYM) equation (*cf.* [1], [9], [10], [17]). It is also related to the Chazy and Painlevé VI equations (see [1] for a complete study of such equations and reductions of the SDYM equation). We will review and detail Ohyama's study [17] of this equation and refer to (3.1) as the Darboux-Halphen V Equation or DH-V for short.

Originally (*cf.* [9]) the DH-V system with the special condition $\theta = \phi = 0$ arose from a geometrical problem studied by Darboux, who in 1878 obtained it as the integrability condition for the existence in Euclidean space of a one-parameter family of surfaces of second degree orthogonal to two arbitrary given independent families of parallel surfaces. Halphen solved this system in 1881.

Ohyama ([17], [18]) shows how DH-V is, in the generic case, equivalent to Halphen's second equation

$$(3.2) \quad x'_i = Q(x_i), \quad i = 1, 2, 3,$$

where

$$(3.3) \quad Q(x) = x^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2$$

with constants a, b, c such that $a + b = c + b = -1/4$ (all derivatives are with respect to the complex variable t).

As pointed out in [17], these equations do not satisfy the Painlevé property (on their movable singularities) and may therefore not be expected to be monodromy-preserving conditions. Nevertheless Chakravarty and Ablowitz in [9], and Ohyama in [17], showed how these nonlinear equations actually express a special type of monodromy-evolving deformation, in the same way as the Schlesinger and Painlevé VI equations rule the isomonodromic deformations of Schlesinger type.

Using the connection relating the self-dual Yang-Mills equation and the conformally self-dual Bianchi equations, these authors showed that DH-V, and hence Equation (3.2), actually is the compatibility condition of a Lax pair

$$(3.4) \quad \frac{\partial Y}{\partial x} = \left(\frac{\mu}{P} I + \sum_{i=1}^3 \frac{\lambda_i S}{x - x_i} \right) Y,$$

$$(3.5) \quad \frac{\partial Y}{\partial t} = \left(\nu I + \sum_{i=1}^3 \lambda_i x_i S \right) Y - Q(x) \frac{\partial Y}{\partial x}$$

of 2×2 matrix equations, where x_1, x_2, x_3 depend on t and $P(x) = (x - x_1)(x - x_2)(x - x_3)$, and S is a traceless constant matrix and where μ and the λ_i are constants with $\mu \neq 0, \lambda_1 + \lambda_2 + \lambda_3 = 0$, and $\nu(x, t)$ satisfies the auxiliary equation

$$(3.6) \quad \frac{\partial \nu}{\partial x} = - \frac{x + x_1 + x_2 + x_3}{P} \mu.$$

Assume that the Lax pair is integrable, *i.e.* that Equation (3.2) is satisfied by the x_i . Equation (3.4) is for fixed t a Fuchsian system with three singular points x_1, x_2, x_3 , and Equation (3.6) implies that ν is not a rational function of x . Therefore the Lax pair ((3.4), (3.5)) does not describe an isomonodromic deformation, since otherwise the coefficients of (3.5) would be rational (*cf.* [20], Remark A.5.2.5).

Let us fix $t_0 \in \mathbb{C}$, and open disjoint disks D_i with center at $x_i(t_0), i = 1, 2, 3$. Let $U(t_0)$ be a neighborhood of t_0 in \mathbb{C} such that $x_i(t) \in D_i$ for each i and all $t \in U(t_0)$, and let $x_0 \in \mathbb{C}$ be a fixed base-point, $x_0 \notin \bigcup_i D_i$.

Let $Y(t, x)$, for $t \in U(t_0)$, denote a fundamental solution, in a neighborhood of x_0 , of the Lax pair ((3.4), (3.5)). It is therefore analytic in both t and x . For fixed $t \in U(t_0)$, we can write an analytic continuation of the fundamental solution $Y(t, x)$ to a punctured neighborhood of x_i as

$$(3.7) \quad Y(t, x) = Y_i(t, x - x_i(t)) \cdot (x - x_i(t))^{L_i(t)},$$

where $Y_i(t, x - x_i(t))$ is single-valued, and the matrix $L_i(t)$ does not depend on x . Note that $Y_i(t, x - x_i(t))$ is analytic in t and x and $L_i(t)$ is analytic in t . Indeed, for fixed $t \in U(t_0)$, analytic continuation of Y along an elementary loop around $x_i(t)$ yields a fundamental solution $\tilde{Y}(t, x)$ of (3.4) which is again analytic in both t and x , by the theorem about analytic dependence on initial conditions (*cf.* [5]). The monodromy matrix $M_i(t)$ is therefore analytic in t , as well as $L_i = (1/2\pi i) \log M_i(t)$, and hence $Y_i(t, x - x_i(t)) = Y(t, x) (x - x_i(t))^{-L_i(t)}$ is analytic in t and x in $(U(t_0) \times D_i) \setminus \{(t, x) \mid x - x_i(t) = 0\}$.

Proposition 3. *With notation as above, let $M_i(t)$ for any fixed $t \in U(t_0)$ denote the monodromy matrix of (3.4) with respect to Y , defined by analytic continuation*

along an elementary loop around $x_i(t)$. Then

$$(3.8) \quad M_i(t) = c_i(t)G_i,$$

where G_i is a constant matrix and where $c_i(t) = e^{-2\pi\mu\sqrt{-1} \int_{t_0}^t \alpha_i(t)dt}$ and the α_i are the residues of

$$(3.9) \quad \frac{x + x_1 + x_2 + x_3}{P} = \sum_{i=1}^3 \frac{\alpha_i}{x - x_i}.$$

Proof. Let us show that

$$(3.10) \quad \frac{dL_i}{dt} = -\alpha_i\mu I,$$

where α_i is the x_i -residue of $(x + x_1 + x_2 + x_3)/P$; that is,

$$\alpha_i = \frac{x_i + \sigma}{\prod_{j \neq i} (x_i - x_j)}$$

with $\sigma = x_1 + x_2 + x_3$. Note that Y_i is a function (the sum of a series) of the local coordinate $x - x_i$. We have

$$\begin{aligned} \frac{\partial Y}{\partial t} &= -Q(x_i) \frac{\partial Y_i}{\partial x} \cdot (x - x_i)^{L_i} + Y_i \cdot (x - x_i)^{L_i} \left(\log(x - x_i) \frac{dL_i}{dt} - Q(x_i) \frac{L_i}{x - x_i} \right) \\ &= -Q(x_i) \frac{\partial Y}{\partial x} + Y \cdot \log(x - x_i) \frac{dL_i}{dt}. \end{aligned}$$

If we compare with Equation (3.5) of the Lax pair we get

$$-Q(x_i) \frac{\partial Y}{\partial x} + Y \cdot \log(x - x_i) \frac{dL_i}{dt} = -Q(x) \frac{\partial Y}{\partial x} + \left(\nu I + \sum_{i=1}^3 c_i x_i S \right) Y.$$

From Equation (3.6) we have that

$$\nu = \mu \log \prod_{i=1}^3 (x - x_i)^{-\alpha_i} + \phi(t)$$

for some function $\phi(t)$, and hence as x tends to x_i for fixed t (simplifying and then comparing the leading terms on each side) we get that

$$\log(x - x_i) \frac{dL_i}{dt} \sim -\alpha_i \mu \log(x - x_i) I,$$

that is,

$$\frac{dL_i}{dt} = -\alpha_i \mu I.$$

The monodromy matrix of (3.4) with respect to x_0 and Y around x_i is $M_i = e^{2\pi i L_i}$, which in view of (3.10) is of the form

$$M_i(t) = c_i(t)G_i,$$

where G_i is the initial monodromy matrix around $x_i(t_0)$, and

$$c_i(t) = e^{-2\pi\mu\sqrt{-1} \int_{t_0}^t \alpha_i(t)dt}.$$

□

This is an example of what we will call projectively isomonodromic deformations, and we study them from an algebraic point of view.

4. PROJECTIVE ISOMONODROMY

Let \mathcal{D} be an open connected subset of $\mathbb{P}^1(\mathbb{C})$, \mathcal{P} be an open connected subset of \mathbb{C}^r , and $x_0 \in \mathcal{D}$. Assume that $\pi_1(\mathcal{D}, x_0)$ is finitely generated by $\gamma_1, \dots, \gamma_m$. Let $A(x, t) \in \text{gl}_n(\mathcal{O})$, where \mathcal{O} denotes the ring of $n \times n$ matrices whose entries are functions analytic on $\mathcal{D} \times \mathcal{P}$. We will consider the behavior of solutions of the differential equation

$$(4.1) \quad \frac{dY}{dx} = A(x, t)Y.$$

In the following we let Scal_n be the group of nonzero $n \times n$ scalar matrices.

Definition 4. Equation (4.1) is *projectively isomonodromic* if there exist m analytic functions $c_i : \mathcal{P} \rightarrow \text{Scal}_n(\mathbb{C})$ and fixed matrices $G_1, \dots, G_m \in \text{GL}_n(\mathbb{C})$ such that for each $t \in \mathcal{P}$ there is a local solution $Y_t(x)$ of (4.1) at x_0 such that analytic continuation of $Y_t(x)$ along γ_i yields $Y_t(x) \cdot G_i c_i(t)$, for each i .

Let $\bar{Y}(x, t)$ be any solution of (4.1) analytic in $\mathcal{D}_0 \times \mathcal{P}$, where \mathcal{D}_0 is a neighborhood of x_0 in \mathcal{D} , and let $M_i(t)$ denote the monodromy matrix corresponding to analytic continuation of this solution around γ_i . Note that $M_i(t)$ depends analytically on t . If (4.1) is projectively isomonodromic, then there exists a function $C : \mathcal{P} \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$M_i(t) = C(t)^{-1} G_i c_i(t) C(t)$$

for all $t \in \mathcal{P}$. Since there may be many ways of selecting $C(t)$, this function need not depend analytically on t . However, we will show that one can find a function $C(t)$ satisfying the above *and analytic in t* . This fact can be deduced easily from the following result of Andrey Bolibruch, whose proof is contained in the proof of Proposition 1 of [3].

Proposition 5. *For each $i = 1, \dots, m$, let $H_i : \mathcal{P} \rightarrow \text{GL}_n(\mathbb{C})$ be analytic on \mathcal{P} and let $G_i \in \text{GL}_n(\mathbb{C})$. Assume that there is a function $C : \mathcal{P} \rightarrow \text{GL}_n(\mathbb{C})$ such that*

$$H_i(t) = C(t)^{-1} G_i C(t)$$

for all $t \in \mathcal{P}$ and $i = 1, \dots, m$. Then there exists an analytic function C with the same property.

We can now prove the following.

Proposition 6. *If (4.1) is projectively isomonodromic, then there exists a solution $Y(x, t)$ of (4.1) analytic in $\mathcal{D}_0 \times \mathcal{P}$, where \mathcal{D}_0 is a neighborhood of x_0 in \mathcal{D} such that for all $t \in \mathcal{P}$ the monodromy matrix of $Y(x, t)$ along γ_i is $G_i c_i(t)$.*

Proof. Let $\bar{Y}(x, t)$ be any solution of (4.1) analytic in $\mathcal{D}_0 \times \mathcal{P}$, where \mathcal{D}_0 is a neighborhood of x_0 in \mathcal{D} , and let $M_i(t)$ denote the monodromy matrix corresponding to analytic continuation of this solution around γ_i . Since (4.1) is projectively isomonodromic, there is a function $C : \mathcal{P} \rightarrow \text{GL}_n(\mathbb{C})$ such that $M_i(t) = C(t)^{-1} G_i c_i(t) C(t)$ for all $t \in \mathcal{P}$. Applying Proposition 5 to $H_i(t) = M_i(t) c_i(t)^{-1}$ and G_i , we may assume that $C(t)$ is analytic and thus $Y(x, t) = \bar{Y}(x, t) C(t)$ satisfies the conclusion of this proposition. □

5. ISOMONODROMY VERSUS PROJECTIVE ISOMONODROMY

We now turn to the relation between Fuchsian isomonodromic equations and Fuchsian projectively isomonodromic equations. Consider the equation

$$(5.1) \quad \frac{dY}{dx} = \sum_{i=1}^m \frac{A_i(t)}{x - x_i(t)} Y$$

together with

- (1) \mathcal{P} , a simply connected open subset of \mathbb{C}^r and
- (2) \mathcal{D} , an open subset in $\mathbb{P}^1(\mathbb{C})$ and $x_0 \in \mathcal{D}$ such that
- (3) the functions $A_i : \mathcal{P} \rightarrow \mathfrak{gl}_n(\mathbb{C})$ and the $x_i : \mathcal{P} \rightarrow \mathbb{C}$ are analytic functions,
- (4) $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$ is the union of m disjoint closed disks D_i and
- (5) for $t \in \mathcal{P}$ we have $x_i(t) \in D_i$.

Let $x_0 \in \mathcal{D}$ and $\gamma_i, i = 1, \dots, m$, be the obvious loops generating $\pi_1(\mathcal{D}, x_0)$. We then have that Equation (5.1) is analytic in $\mathcal{D} \times \mathcal{P}$ and we can speak of monodromy matrices $M_i(t)$ corresponding to analytic continuation of a fundamental solution matrix along γ_i . We can now state

Proposition 7. *Let \mathcal{D} and \mathcal{P} be as above. Equation (5.1) is projectively isomonodromic if and only if for each $i = 1, \dots, m$, there exist functions $b_i : \mathcal{P} \rightarrow \text{Scal}_n(\mathbb{C})$ and $B_i : \mathcal{P} \rightarrow \mathfrak{gl}_n(\mathbb{C})$, analytic on \mathcal{P} such that*

- (1) $A_i = B_i + b_i$ for $i = 1, \dots, m$ and
- (2)

$$(5.2) \quad \frac{dY}{dx} = \left(\sum_{i=1}^m \frac{B_i(t)}{x - x_i(t)} \right) Y$$

is isomonodromic.

Proof. Assume that Equation (5.1) is projectively isomonodromic and let $Y(x, t), G_i$ and c_i be as in the conclusion of Proposition 6. Since \mathcal{P} is simply connected and the $c_i(t)$ are nonzero for all t , there exist analytic $b_i : \mathcal{P} \rightarrow \text{Scal}_n(\mathbb{C})$ such that $e^{2\pi\sqrt{-1}b_i} = c_i$. Let

$$Z(x, t) = Y(x, t) \prod_{i=1}^m (x - x_i(t))^{-b_i(t)}.$$

One sees that the monodromy of Z along γ_i is given by G_i and so is independent of t . Therefore, letting $B_i = A_i - b_i$, we have that

$$\frac{dY}{dx} = \left(\sum_{i=1}^m \frac{B_i(t)}{x - x_i(t)} \right) Y$$

is isomonodromic.

Now assume that A_i, B_i, b_i are as in items (1) and (2) of the proposition and that Equation (5.2) is isomonodromic. If $Y(x, t)$ is a local solution of (5.2) with constant monodromy matrices G_i along γ_i , then $Z(x, t) = Y(x, t) \prod_{i=1}^m (x - x_i(t))^{b_i(t)}$ will have monodromy $G_i c_i(t)$ along γ_i , with $c_i(t) = e^{2\pi\sqrt{-1}b_i(t)}$. Thus Equation (5.1) is projectively isomonodromic. □

Proposition 7 applies to the DH-V example since we can rewrite Equation (3.4) of the Lax pair as

$$\frac{\partial Y}{\partial x} = \left(\sum_{i=1}^3 \frac{A_i(t)}{(x - x_i)} \right) Y,$$

where $A_i = B_i + b_i$, with

$$\begin{aligned} B_i &= \lambda_i S, \\ b_i &= \frac{\mu I_n}{\prod_{j \neq i} (x_i - x_j)}. \end{aligned}$$

An easy computation shows that since $x'_i - x'_j = Q(x_i) - Q(x_j) = x_i^2 - x_j^2$ for all i, j , we have

$$b'_i = \frac{db_i}{dt} = - \frac{x_i + \sigma}{\prod_{j \neq i} (x_i - x_j)} \mu = -\alpha_i \mu,$$

and we recover the result of Proposition 3 that the monodromy of this equation is evolving ‘projectively’ and is equal to

$$M_i(t) = e^{2\pi\sqrt{-1}b_i(t)} G_i = e^{-2\pi\mu\sqrt{-1} \int_{t_0}^t \alpha_i(t) dt} G_i.$$

6. PARAMETERIZED DIFFERENTIAL GALOIS GROUPS

In this section we examine the parameterized differential Galois groups of projectively isomonodromic equations. Parameterized differential Galois groups (cf. [8], [15]) generalize the concept of differential Galois groups of the classical Picard-Vessiot theory, and we begin this section by briefly describing the underlying theory.

Let

$$(6.1) \quad \frac{dY}{dx} = A(x)Y$$

be a differential equation where $A(x)$ is an $n \times n$ matrix with entries in $\mathbb{C}(x)$. The usual existence theorems for differential equations imply that if $x = x_0$ is a point in \mathbb{C} such that the entries of $A(x)$ are analytic at x_0 , then there exists a nonsingular matrix $Z = (z_{i,j})$ of functions analytic in a neighborhood of x_0 such that $\frac{dZ}{dx} = A(x)Z$. Note that the field $K = \mathbb{C}(z_{1,1}, \dots, z_{n,n})$ is closed with respect to taking the derivation $\frac{d}{dx}$ and this is an example of a *Picard-Vessiot extension*.¹ The set of field-theoretic isomorphisms of K that leave $\mathbb{C}(x)$ elementwise fixed and commute with $\frac{d}{dx}$ forms a group G called the *Picard-Vessiot group* or *differential Galois group* of (6.1). One can show that for any $\sigma \in G$, there exists a matrix $M_\sigma \in \text{GL}_n(\mathbb{C})$ such that $\sigma(Z) = (\sigma(z_{i,j})) = ZM_\sigma$. The map $\sigma \mapsto M_\sigma$ is an isomorphism whose image is furthermore a *linear algebraic group*, that is, a group of invertible matrices whose entries satisfy some fixed set of polynomial equations in n^2 variables. There is a well-developed Galois theory for these groups that describes a correspondence between certain subgroups of G and certain subfields of K as well as associates properties of Equation (6.1) with properties of the group G . The elements of the monodromy group of (6.1) may be identified with elements of this group, and when (6.1) has only regular singular points, it is known that G is the smallest linear algebraic group containing these elements (cf. [19], Theorem 5.8). Further facts about this Galois theory can be found in [13] and [19].

¹Picard-Vessiot extensions and the related Picard-Vessiot theory are developed in a fuller generality in [13] and [19], but we shall restrict ourselves to the above context to be concrete.

Now let

$$(6.2) \quad \frac{dY}{dx} = A(x, t)Y$$

be a parameterized system of linear differential equations where $A(x, t)$ is an $n \times n$ matrix whose entries are rational functions of x with coefficients that are functions of $t = (t_1, \dots, t_r)$, analytic in some domain in \mathbb{C}^r . A differential Galois theory for such equations was developed in [8] and in greater generality in [15]. Let k_0 be a suitably large field² containing $\mathbb{C}(t_1, \dots, t_r)$ and the functions of t appearing as coefficients in the entries of A and such that k_0 is closed under the derivations $\Pi = \{\partial_1, \dots, \partial_r\}$, where each ∂_i restricts to $\frac{\partial}{\partial t_i}$ on $\mathbb{C}(t_1, \dots, t_r)$ and the intersection of the kernels of the ∂_i is \mathbb{C} . As before, existence theorems for solutions of differential equations guarantee the existence of a nonsingular matrix $Z(x, t) = (z_{i,j}(x, t))$ of functions, analytic in some suitable domain in $\mathbb{C} \times \mathbb{C}^r$, such that $\frac{dZ}{dx} = AZ$. We will let $k = k_0(x)$ be the differential field with derivations $\Delta = \{\partial_x, \partial_1, \dots, \partial_r\}$, where $\partial_x(x) = 1$, $\partial_x(z) = 0$ for all $z \in k_0$ and the ∂_i extend the previous ∂_i with $\partial_i(x) = 0$. Finally we will denote by K the smallest field containing k and the $z_{i,j}$ that is closed under the derivations of Δ . This field is called the *parameterized Picard-Vessiot field* or PPV-field of (6.2). The set of field-theoretic automorphisms of K that leave k elementwise fixed and commute with the elements of Δ forms a group G called the *parameterized Picard-Vessiot group* (PPV-group) or *parameterized differential Galois group* of (6.2). One can show that for any $\sigma \in G$, there exists a matrix $M_\sigma \in \mathrm{GL}_n(k_0)$ such that $\sigma(Z) = (\sigma(z_{i,j})) = ZM_\sigma$. Note that ∂_x applied to an entry of such an M_σ is 0 since these entries are elements of k_0 but that such an entry need not be constant with respect to the elements of Π . One may think of these entries as functions of t . In [8], the authors show that the map $\sigma \mapsto M_\sigma$ is an isomorphism whose image is furthermore a *linear differential algebraic group*, that is, a group of invertible matrices whose entries satisfy some fixed set of polynomial *differential* equations (with respect to the derivations $\Pi = \{\partial_1, \dots, \partial_r\}$) in n^2 variables. We say that a set $X \subset \mathrm{GL}_n(k_0)$ is *Kolchin-closed* if it is the zero set of such a set of polynomial differential equations. One can show that the Kolchin-closed sets form the closed sets of a topology, called the *Kolchin topology* on $\mathrm{GL}_n(k_0)$ (cf. [6, 7, 8, 14]). The following result shows how the PPV-group can be used to characterize isomonodromy. As in Section 4, let \mathcal{P} be a simply connected subset of \mathbb{C}^r and \mathcal{D} an open subset of $\mathbb{P}^1(\mathbb{C})$ with $x_0 \in \mathcal{D}$. We assume that $A(x, t)$ in Equation (6.2) is analytic in $\mathcal{D} \times \mathcal{P}$. Assume that $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$ is the union of m disjoint disks D_i and that for each $t \in \mathcal{P}$, Equation (6.2) has a unique singular point in each D_i and that this singular point is a *parameterized regular singular point* in the sense of ([16], Definition 2.3). Note that by ([16], Corollary 2.6) this in particular implies that the singularity is regular singular for each fixed t , in the usual sense. Let $\gamma_i, i = 1, \dots, m$ be the obvious loops generating $\pi_1(\mathcal{D}, x_0)$. We then have that Equation (6.2) is analytic in $\mathcal{D} \times \mathcal{P}$ and we can speak of (parameterized) monodromy matrices $M_i(t)$ corresponding to analytic continuation, for each fixed t , of a fundamental solution matrix along γ_i .

²To be precise, we need k_0 to be *differentially closed* with respect to Π ; that is, any system of polynomial differential equations in arbitrary unknowns having a solution in an extension field already has a solution in k_0 . See [8] for a discussion of differentially closed fields in the context of this Galois theory.

Proposition 8 ([8, Proposition 5.4]). *Assume that \mathcal{D} , \mathcal{P} and Equation (6.2) are as above. Then this equation is isomonodromic in $\mathcal{D} \times \mathcal{P}'$ for some subset $\mathcal{P}' \subset \mathcal{P}$ if and only if the PPV-group G of this equation over k is conjugate to $G_1(\mathbb{C})$ for some linear algebraic group G_1 defined over \mathbb{C} .*

7. AN ALGEBRAIC CONDITION FOR PROJECTIVE ISOMONODROMY

We now relate the property of projective isomonodromy to properties of the PPV-group. We still assume that Equation (6.2) has (parameterized) regular singularities only, in the sense of ([16], Definition 2.3).

Proposition 9. *Let k, K, A , and G be as above. Equation (6.2) is projectively isomonodromic if and only if its PPV-group G is conjugate to a subgroup of*

$$GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0) \subset GL_n(k_0).$$

Proof. Let $x_i(t)$, $i = 1, \dots, m$ be the singular points of Equation (6.2),

$$\frac{dY}{dx} = A(x, t)Y$$

and let $Y(x, t)$, G_i , c_i be as in the conclusion of Proposition 6. Since \mathcal{P} is simply connected and the $c_i(t)$ are nonzero, there exist analytic $b_i : \mathcal{P} \rightarrow \text{Scal}_n\mathbb{C}$ such that $e^{2\pi\sqrt{-1}b_i} = c_i$ for each i (in fact, we can select $b_i(t) \in \text{Scal}(k_0)$ since k_0 is differentially closed). Consider the system of differential equations

$$\frac{dZ}{dx} = \left(A(x, t) - \sum_{i=1}^m \frac{b_i(t)}{(x - x_i(t))} \right) Z = B(x, t)Z, \tag{7.1}$$

$$\frac{du}{dx} = \left(\sum_{i=1}^m \frac{b_i(t)}{(x - x_i(t))} \right) u = b(t)u.$$

We will consider this as a differential equation with coefficient matrix

$$\begin{pmatrix} B & 0 \\ 0 & b \end{pmatrix} \in \mathfrak{gl}_{2n}(k).$$

Let E be the PPV-extension of k for the system (7.1). Note that the first equation of this system is isomonodromic, hence has a PPV-group that is conjugate in $GL_n(k_0)$ to a constant group over \mathbb{C} . Therefore the PPV-group of (7.1) over k is conjugate in $GL_{2n}(k_0)$ to a subgroup of $GL_n(\mathbb{C}) \times \text{Scal}_n(k_0) \subset GL_{2n}(k_0)$. It is easy to see that $Y \in GL_n(E)$ is a fundamental solution of Equation (6.2) if and only if $Y = Zu$, where

$$\begin{pmatrix} Z & 0 \\ 0 & u \end{pmatrix}$$

is a fundamental solution of (7.1). Let us choose Z such that the corresponding representation H of the PPV-group is a subgroup of $GL_n(\mathbb{C}) \times \text{Scal}_n(k_0)$ and let $Y = Zu$ be a fundamental solution of (6.2). We clearly have $K \subset E$, where K is the PPV-extension of k for (6.2). The action of H on E gives

$$(\sigma, \tau)(Y) = \sigma(Z)\tau(u)$$

for all $(\sigma, \tau) \in H$, and it induces a homomorphism Φ of H onto the PPV-group of K over k . In terms of matrices, Φ is given by $\Phi(\sigma, \tau) = \sigma \cdot \tau$ and its image is in $GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0)$.

The converse is clear since the (parameterized) monodromy matrices $M_i(t)$ all belong to the PPV-group (see Theorem 3.5 of [16]). \square

One easy consequence of Proposition 9 is

Corollary 10. *Let k, K, A , and G be as above. If Equation (6.2) is projectively isomonodromic, then the commutator subgroup (G, G) is conjugate to a subgroup of $GL_n(\mathbb{C})$.*

This corollary yields a simple test to show that (6.2) is not projectively isomonodromic: If the eigenvalues of the commutators of the monodromy matrices (with respect to any fundamental solution matrix) are not constant, then (6.2) is not projectively isomonodromic. In particular, if the determinant or trace of any of these matrices is not constant, then (6.2) is not projectively isomonodromic. The converse of the corollary is not true in general (see Remark 7.1 below), but it is true if (6.2) is absolutely irreducible, that is, when (6.2) does not factor over \bar{k} , the algebraic closure of k . Before we prove this, we will discuss some group-theoretic facts.

In the following, we say that a subgroup $H \subset GL_n(k_0)$ is *irreducible* if the only H -invariant subspaces of k_0^n are $\{0\}$ and k_0^n . A differential equation with coefficients in k is *absolutely irreducible* if it is irreducible over any finite extension of k .

Lemma 11. *Let H be an irreducible subgroup of $GL_n(\mathbb{C})$ and let $g \in GL_n(k_0)$ normalize H . Then $g \in GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0)$.*

Proof. For any $h \in H$ and $g \in GL_n(k_0)$ normalizing H , we have that

$$0 = \partial_i(g^{-1}hg) = -g^{-1}\partial_i(g)g^{-1}hg + g^{-1}h\partial_i(g)$$

for all $\partial_i \in \Pi$. Therefore,

$$\partial_i(g)g^{-1}h = h\partial_i(g)g^{-1}.$$

Since H is irreducible, Schur’s Lemma implies that $\partial_i(g)g^{-1} = z_i \in \text{Scal}_n(k_0)$. One can check that the z_i satisfy the integrability conditions $\partial_i z_j = \partial_j z_i$, so there exists a nonzero $u \in \text{Scal}_n(k_0)$ such that $\partial_i u = z_i u$ for all i . This implies that $\partial_i(u^{-1}g) = 0$ for all i and so $g = uh$ for some $h \in GL_n(\mathbb{C})$. \square

It is well known that if G and H are linear algebraic groups with H normal in G , then G/H is also a linear algebraic group. For $\text{Scal}_n(k_0) \triangleleft GL_n(k_0)$, we will denote by ρ the canonical map $\rho : GL_n(k_0) \rightarrow GL_n(k_0)/\text{Scal}_n(k_0)$.

Lemma 12. *Let $H \subset GL_n(k_0)$ be a Kolchin-connected linear differential algebraic group and let \overline{H} be its Zariski-closure in $GL_n(k_0)$. Assume that \overline{H} is irreducible. Then*

$$H \subset (H, H)_\Pi \cdot \text{Scal}_n(k_0),$$

where $(H, H)_\Pi$ is the Kolchin-closure of (H, H) .

Proof. Since \overline{H} is irreducible, it must be reductive ([21], p. 37). Since H is Kolchin-connected, \overline{H} is Zariski-connected, so we can write $\overline{H} = Z(\overline{H}) \cdot (\overline{H}, \overline{H})$, where $Z(\overline{H})$ is the center of \overline{H} ([11], Ch. 27.5). Using the irreducibility again, Schur’s Lemma implies that $Z(\overline{H}) \subset \text{Scal}_n(k_0)$. Using the map ρ above, we have that $\rho(\overline{H})$ is isomorphic to $(\overline{H}, \overline{H})/(Z(\overline{H}) \cap (\overline{H}, \overline{H}))$ and so is a connected semisimple linear algebraic group. Furthermore, $\rho(H)$ is a Zariski-dense, Kolchin-connected, subgroup of $\rho(\overline{H})$. Propositions 11 and 13 and those of [7] imply

that $\rho(H)$ equals $(\rho(H), \rho(H))_{\Pi}$, the Kolchin-closure of its commutator subgroup $(\rho(H), \rho(H))$. Since $(H, H)_{\Pi}$ is a linear differential algebraic group, we have that $\rho((H, H)_{\Pi})$ is a linear differential algebraic group containing $(\rho(H), \rho(H))$ and therefore contains $(\rho(H), \rho(H))_{\Pi}$. Since $(H, H)_{\Pi} \subset H$ we have that $\rho((H, H)_{\Pi}) = \rho(H)$. Therefore $H \subset (H, H)_{\Pi} \cdot \text{Scal}_n(k_0)$. \square

Lemma 13. *Let $G \subset GL_n(k_0)$ be a linear differential group and assume that*

- (1) $(G, G) \subset GL_n(\mathbb{C})$ and
- (2) *the identity component \overline{G}^0 of \overline{G} , the Zariski-closure of G in $GL_n(k_0)$, is irreducible.*

Then $G \subset GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0)$.

Proof. We first note that the Zariski-closure of G^0 , the Kolchin-component of the identity of G , is Zariski-connected and of finite index in \overline{G} . Therefore \overline{G}^0 is the Zariski-closure $\overline{G^0}$ of G^0 . We now apply Lemma 12 to $H = G^0$ and conclude that $G^0 \subset (G^0, G^0)_{\Pi} \cdot \text{Scal}_n(k_0)$. Since $(G, G) \subset GL_n(\mathbb{C})$ we have that $(G^0, G^0)_{\Pi} \subset GL_n(\mathbb{C})$. Furthermore, since \overline{G}^0 is irreducible and is the Zariski-closure of G^0 , we have that G^0 is irreducible. Therefore $(G^0, G^0)_{\Pi}$ is an irreducible subgroup of $GL_n(\mathbb{C})$. Any $g \in G$ normalizes G^0 and therefore normalizes $(G^0, G^0)_{\Pi}$. Applying Lemma 11 to $H = (G^0, G^0)_{\Pi}$, we have that $G \subset GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0)$. \square

Remark 7.1. Simple examples (e.g., $G = \text{Diag}_n(k_0)$, the group of diagonal matrices) show that the condition $(G, G) \subset GL_n(\mathbb{C})$ does not imply $G \subset GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0)$ without some additional hypotheses.

Proposition 14. *Let k, K, A, G be as in Proposition 9. If Equation (6.2) is absolutely irreducible and (G, G) is conjugate to a subgroup of $GL_n(\mathbb{C})$, then (6.2) is projectively isomonodromic.*

Proof. As noted above, \overline{G} is the usual Picard-Vessiot group of (6.2) over k . If (6.2) is absolutely irreducible, then \overline{G}^0 is an irreducible subgroup of $GL_n(k_0)$. Lemma 13 implies that $G \subset GL_n(\mathbb{C}) \cdot \text{Scal}_n(k_0)$. \square

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