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A COTORSION THEORY IN THE HOMOTOPY CATEGORY OF FLAT QUASI-COHERENT SHEAVES

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ABSTRACT. Let X be a Noetherian scheme, $\mathbf{K}(\mathrm{Flat}\,X)$ be the homotopy category of flat quasi-coherent $\mathcal{O}_X$-modules and $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)$ be the homotopy category of all flat complexes. It is shown that the pair $(\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{K}\ (\text{dg-}\mathrm{Cof}X))$ is a complete cotorsion theory in $\mathbf{K}(\mathrm{Flat}\,X)$, where $\mathbf{K}\ (\text{dg-}\mathrm{Cof}X)$ is the essential image of the homotopy category of dg-cotorsion complexes of flat modules. Then we study the homotopy category $\mathbf{K}(\text{dg-}\mathrm{Cof}\,X)$. We show that in the affine case, this homotopy category is equal with the essential image of the embedding functor $j_* : \mathbf{K}(\mathrm{Proj}R) \longrightarrow \mathbf{K}(\mathrm{Flat}R)$ which has been studied by Neeman in his recent papers. Moreover, we present a condition for the inclusion $\mathbf{K}(\text{dg-}\mathrm{Cof}\,X) \subseteq \mathbf{K}(\mathrm{Cof}\,X)$ to be an equality, where $\mathbf{K}(\mathrm{Cof}\,X)$ is the essential image of the homotopy category of complexes of cotorsion flat sheaves.

1. Introduction

The notion of cotorsion pairs in the category of abelian groups was introduced and studied by L. Salce in [S]. But the theory carries over to more general categories and has proved useful in different contexts of mathematics, for example, in the proof of the flat cover conjecture. Let R be an associative ring with identity, $\mathcal{F}$ be the class of the flat R-modules and $\mathcal{C}$ be the class of all cotorsion R-modules. Then the pair $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory which is cogenerated by a set and so it is a complete cotorsion theory; see [ET] and [BEE]. As an effort to study cotorsion theories in the category of complexes of R-modules $\mathbf{C}(\mathrm{Mod}R)$, Enochs and his colleagues [AEGO] showed that $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}R)$ is a covering class and $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}R)^{\perp}$ is an enveloping class in the category $\mathbf{C}(\mathrm{Mod}R)$, where $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}R)$ is the full subcategory of $\mathbf{C}(\mathrm{Mod}R)$ consisting of all flat complexes (pure acyclic complexes of flat R-modules) and the orthogonal is taken in the exact category $\mathbf{C}(\mathrm{Mod}R)$; see §2 for the definitions. More specifically, some of the main results of [B] and [BEIJR] show that complete cotorsion theories in the category of complexes of modules play a significant role in the existence of adjoint functors in the corresponding homotopy categories. This connection to adjoint functors is our motivation to studying complete cotorsion pairs in the category of complexes of flat $\mathcal{O}_X$-modules and its homotopy category.

Let us define cotorsion pairs for a triangulated category. Let $\mathcal{T}$ be a triangulated category. A pair $(\mathcal{S}, \mathcal{C})$ of full subcategories of $\mathcal{T}$ is called a cotorsion theory in

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$\mathcal{T}$ if $\mathcal{S} = {}^{\perp}\mathcal{C}$ and $\mathcal{S}^{\perp} = \mathcal{C}$, where the left orthogonal of $\mathcal{C}$ in $\mathcal{T}$ is defined by ${}^{\perp}\mathcal{C} := \{Y \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(Y, C) = 0, \text{ for all } C \in \mathcal{C}\}$. The right orthogonal of $\mathcal{S}$ in $\mathcal{T}$ is defined similarly. A cotorsion theory $(\mathcal{S}, \mathcal{C})$ is called complete if the inclusion functor $\mathcal{S} \to \mathcal{T}$ has a right adjoint or, equivalently, the inclusion functor $\mathcal{C} \to \mathcal{T}$ has a left adjoint. Let $\mathcal{T}'$ be a triangulated category and $H : \mathcal{T}' \longrightarrow \mathcal{T}$ be a triangulated functor. The essential image of H in $\mathcal{T}$ is the full subcategory of $\mathcal{T}$ formed by all objects which are isomorphic to $H(Y)$ for some $Y \in \mathcal{T}'$.

Throughout, we fix a Noetherian scheme X; $\mathcal{O}_X$-modules are quasi-coherent sheaves. $\mathbf{C}(\mathfrak{Qco}X)$ is the category of complexes of $\mathcal{O}_X$-modules. $\mathbf{C}(\mathrm{Flat}X)$ ($\mathbf{K}(\mathrm{Flat}X)$) is the category (homotopy category) of complexes of flat $\mathcal{O}_X$-modules and $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$ ($\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)$) denotes the full subcategory of $\mathbf{C}(\mathrm{Flat}X)$ ($\mathbf{K}(\mathrm{Flat}X)$) consisting of all pure acyclic complexes of $\mathcal{O}_X$-modules, i.e., those complexes that remain exact after tensoring with any $\mathcal{O}_X$-module. A complex $\mathbf{X}$ is called *dg-cotorsion* if it belongs to $\mathbf{C}_{\mathrm{p}}(\mathrm{FlatX})^{\perp}$, where the orthogonal is taken in the category $\mathbf{C}(\mathfrak{Qco}X)$. We denote by $\mathbf{C}(\text{dg-Cof}X)$ the full subcategory of $\mathbf{C}(\mathrm{Flat}X)$ consisting of dg-cotorsion complexes and by $\mathbf{K}(\text{dg-Cof}X)$ its essential homotopy category in $\mathbf{K}(\mathrm{Flat}X)$. Our first result introduces a complete cotorsion theory in the triangulated category $\mathbf{K}(\mathrm{Flat}X)$.

Theorem 1.1. *The pair* $(\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{K}(\text{dg-Cof}X))$ *is a complete cotorsion theory in the homotopy category of complexes of flat* $\mathcal{O}_X$*-modules.*

Proof. The proof follows from Propositions 3.1, 3.2 and 3.3. □

In the course of the proof of this theorem, by using the fact that quasi-coherent sheaves have flat covers and cotorsion envelopes, we show that the pair $(\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{C}(\text{dg-Cof}X))$ is a complete cotorsion theory in the category $\mathbf{C}(\mathrm{Flat}X)$; see Theorem 2.9. As an attempt to understand the homotopy category $\mathbf{K}(\text{dg-Cof}X)$, we show that the inclusion $\mathbf{K}(\text{dg-Cof}X) \subseteq \mathbf{K}(\mathrm{Cof}X)$ to be equality if and only if every flat complex of cotorsion modules is contractible; see Corollary 3.5. Also, in the affine case, it is shown that this homotopy category equals the essential image of the embedding functor $j_* : \mathbf{K}(\mathrm{Proj}R) \longrightarrow \mathbf{K}(\mathrm{Flat}R)$. To explain this embedding functor, we need to point out some earlier results of Neeman. Namely, in [N1] and [N2], he studied the homotopy category of flat modules $\mathbf{K}(\mathrm{Flat}R)$ and provided a novel relationship between $\mathbf{K}(\mathrm{Flat}R)$ and $\mathbf{K}(\mathrm{Proj}R)$, the homotopy category of projective R-modules. He constructed a right adjoint for the natural embedding functor $j_! : \mathbf{K}(\mathrm{Proj}R) \longrightarrow \mathbf{K}(\mathrm{Flat}R)$ which we denote by j^*. He proved that the functor $j^* : \mathbf{K}(\mathrm{Flat}R) \longrightarrow \mathbf{K}(\mathrm{Proj}R)$ has a right adjoint j_* which is fully faithful. Therefore we have a non-obvious embedding of $\mathbf{K}(\mathrm{Proj}R)$ into $\mathbf{K}(\mathrm{Flat}R)$ by j_*. In the following theorem, we describe this embedding functor.

Theorem 1.2. *The essential image of the embedding functor* $j_* : \mathbf{K}(\mathrm{Proj}R) \longrightarrow \mathbf{K}(\mathrm{Flat}R)$ *is equal to* $\mathbf{K}(\text{dg-Cof}R)$.

2. A cotorsion theory in $\mathbf{C}(\mathrm{Flat}X)$

Let us recall the notion of orthogonality in the category of complexes of (quasi-coherent) $\mathcal{O}_X$-modules $\mathbf{C}(\mathfrak{Qco}X)$. We set $\mathfrak{C} := \mathbf{C}(\mathfrak{Qco}X)$. For each pair of objects $D, C \in \mathfrak{C}$, we have the extension group $\operatorname{Ext}^1_{\mathfrak{C}}(D, C)$ of the equivalence classes of short exact sequences $0 \to C \to P \to D \to 0$ in $\mathfrak{C}$. For a class $\mathcal{C}$ of objects in $\mathfrak{C}$, its right (left) orthogonal class $\mathcal{C}^{\perp}$ (${}^{\perp}\mathcal{C}$) in $\mathbf{C}(\mathfrak{Qco}X)$ is defined as the class of all objects Y such that $\operatorname{Ext}^1_{\mathfrak{C}}(C, Y) = 0$ ($\operatorname{Ext}^1_{\mathfrak{C}}(Y, C) = 0$) for all $C \in \mathcal{C}$.

Following [AEGO] and [EG], a pure acyclic complex of flat $\mathcal{O}_X$-modules is called a *flat complex* in $\mathbf{C}(\mathfrak{Qco}X)$. In other words, an acyclic complex $\mathbf{F}$ of flat $\mathcal{O}_X$-modules is called a *flat complex* if, for any $n \in \mathbb{Z}$, the syzygy $Z^n(\mathbf{F})$ is a flat $\mathcal{O}_X$-module. We denote by $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$ the class of all flat complexes in the category of complexes of $\mathcal{O}_X$-modules. A complex $\mathbf{X}$ is called *dg-cotorsion* if it belongs to $\mathbf{C}_{\mathrm{p}}(\mathrm{FlatX})^{\perp}$. We denote by $\mathbf{C}(\text{dg-Cof}X)$ the full subcategory of $\mathbf{C}(\mathrm{Flat}X)$ consisting of dg-cotorsion complexes. So $\mathbf{C}(\text{dg-Cof}X) := \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp} \cap \mathbf{C}(\mathrm{Flat}X)$. A complex F is called a *dg-cotorsion flat complex* if it belongs to $\mathbf{C}(\text{dg-Cof}X)$. Recall that an $\mathcal{O}_X$-module $\mathcal{G}$ is called cotorsion if $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G}) = 0$, for any flat $\mathcal{O}_X$-module $\mathcal{F}$; it is called cotorsion flat if it is both cotorsion and flat.

In this section, we will show that the pair $(\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{C}(\text{dg-Cof}X))$ is a complete cotorsion theory in the category $\mathbf{C}(\mathrm{Flat}X)$, i.e., $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X) = {}^{\perp}\mathbf{C}(\text{dg-Cof}X)$ and any complex of flat $\mathcal{O}_X$-modules admits a flat precover and a dg-cotorsion flat preenvelope; see Theorem 2.9. The argument provides a short way in proving flat covers in the category of complexes of R-modules (R is an associative ring with identity); see [AEGO].

Let us fix some notation. The class of all affine open subsets of X is denoted by $\mathcal{U}$ and the cardinality of an $\mathcal{O}_X$-module $\mathcal{F}$ is defined as $|\mathcal{F}| = |\coprod_{U\in\mathcal{U}} \mathcal{F}(U)|$. In this subsection κ is a cardinal number such that $\kappa \geq \max\{|\mathcal{O}_X|, |\mathcal{U}|, \aleph_0\}$. Recall that a submodule $\mathcal{F}$ of an $\mathcal{O}_X$-module $\mathcal{G}$ is called pure if, for each $U \in \mathcal{U}$, $\mathcal{F}(U)$ is a pure submodule of $\mathcal{G}(U)$ as an $\mathcal{O}_X(U)$-module.

By the same method that was used in the proof of [EG, 3.4], one can deduce the following proposition.

Proposition 2.1. *A complex $\mathbf{C} = (C^i, \delta^i)$ of $\mathcal{O}_X$-modules is dg-cotorsion if and only if*

(i) *C^i is a cotorsion $\mathcal{O}_X$-module, for all $i \in \mathbb{Z}$,*
(ii) *for any flat complex F, $\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{F}, \mathbf{C})$ is an acyclic complex.*

The following lemma plays a significant role in our arguments in this section.

Lemma 2.2. *Let $0 \longrightarrow \mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \longrightarrow 0$ be an exact sequence of flat $\mathcal{O}_X$-modules. For any $U \in \mathcal{U}$, let Y_U be a subset of $\mathcal{F}(U)$ with $|Y_U| \leq \kappa$. Then there exist flat and pure submodules $\mathcal{K}', \mathcal{K}$ and $\mathcal{K}''$ respectively of $\mathcal{F}', \mathcal{F}$ and $\mathcal{F}''$ such that the induced sequence $0 \longrightarrow \mathcal{K}' \xrightarrow{f|_{\mathcal{K}'}} \mathcal{K} \xrightarrow{g|_{\mathcal{K}}} \mathcal{K}'' \longrightarrow 0$ is exact. Furthermore, $Y_U \subseteq \mathcal{K}(U)$, for all $U \in \mathcal{U}$, and $\max\{|\mathcal{K}'|, |\mathcal{K}|, |\mathcal{K}''|\} \leq \kappa$.*

Proof. For any $U \in \mathcal{U}$, let $X_U = g(Y_U)$. By [EE, 3.3], there is a pure submodule $\mathcal{K}''$ of $\mathcal{F}''$ such that, for any $U \in \mathcal{U}$, $X_U \subseteq \mathcal{K}''(U)$ and $|\mathcal{K}''| \leq \kappa$. Consider the pullback diagram of maps i and g

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{P} \arrow[r, "g'"] \arrow[d, "e"] & \mathcal{K}'' \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r, "g"] & \mathcal{F}'' \arrow[r] & 0
\end{tikzcd}$$

which is pure exact in rows and columns. One can see easily that, for any $U \in \mathcal{U}$, $Y_U \subseteq \mathcal{P}(U)$. For any $U \in \mathcal{U}$, we consider a subset L_U of $\mathcal{P}(U)$ such that $Y_U \subseteq L_U$, $g'(L_U) = \mathcal{K}''(U)$ and $|L_U| \leq \kappa$. By using [EE, 3.3], we have a pure submodule $\mathcal{K}$ of $\mathcal{P}$ such that, for any $U \in \mathcal{U}$, $L_U \subseteq \mathcal{K}(U)$ and $|\mathcal{K}| \leq \kappa$. Set $\mathcal{K}' := \mathrm{Ker} g'|_{\mathcal{K}}$. $\square$

The argument given above may also be applied to prove the following lemma.

Lemma 2.3. *Let* $0 \longrightarrow \mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \longrightarrow 0$ *be an exact sequence of flat* $\mathcal{O}_X$*-modules with* $|\mathcal{F}''| \leq \kappa$*. Then there exist flat and pure submodules* $\mathcal{K}', \mathcal{K}$ *respectively of* $\mathcal{F}', \mathcal{F}$ *such that the induced sequence* $0 \longrightarrow \mathcal{K}' \xrightarrow{f|_{\mathcal{K}'}} \mathcal{K} \xrightarrow{g|_{\mathcal{K}}} \mathcal{F}'' \longrightarrow 0$ *is exact and* $\max\{|\mathcal{K}'|, |\mathcal{K}|\} \leq \kappa$*.*

Theorem 2.4. *Let* $\mathbf{F} = (\mathcal{F}^i, \delta^i)$ *be a flat complex and* $n \in \mathbb{Z}$*. Assume that, for each* $U \in \mathcal{U}$*,* T_U *is a subset of* $\mathcal{F}^n(U)$ *with* $|T_U| < \kappa$*. Then there exists a flat subcomplex* $\mathbf{F}_0 = (\mathcal{F}^i_0, \delta^i_0)$ *of* $\mathcal{F}$ *such that, for any* $U \in \mathcal{U}$*,* $T_U \subseteq \mathcal{F}^n_0(U)$*,* $|\mathbf{F}_0| \leq \kappa$ *and* $\frac{\mathbf{F}}{\mathbf{F}_0}$ *is a flat complex.*

Proof. Without loss of generality we may assume that $n = 0$. Consider the short exact sequence $0 \longrightarrow \operatorname{Ker}(\delta^0) \longrightarrow \mathcal{F}^0 \longrightarrow \operatorname{Im}(\delta^0) \longrightarrow 0$. By Lemma 2.2, we have the following diagram of flat modules which is pure exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}^0_0 & \longrightarrow & \mathcal{F}^0_0 & \longrightarrow & \mathcal{K}^1_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}(\delta^0) & \longrightarrow & \mathcal{F}^0 & \longrightarrow & \operatorname{Ker}(\delta^1) & \longrightarrow & 0,
\end{array}$$

such that, for each affine open subset U, $T_U \subseteq \mathcal{F}^0_0(U)$ and $\max\{|\mathcal{F}^0_0|, |\mathcal{K}^0_0|, |\mathcal{K}^1_0|\} \leq \kappa$. We use Lemma 2.3 and an inductive procedure to obtain, for every $i \leq 0$, a pure exact sequence $0 \longrightarrow \mathcal{K}^{i-1}_0 \longrightarrow \mathcal{F}^{i-1}_0 \longrightarrow \mathcal{K}^i_0 \longrightarrow 0$, such that $\max\{|\mathcal{F}^{i-1}_0|, |\mathcal{K}^{i-1}_0|, |\mathcal{K}^i_0|\} \leq \kappa$. Set $\mathcal{F}^1_0 := \mathcal{K}^1_0$, $\mathcal{F}^i_0 := 0$, for all $i > 1$, and $\delta^i_0 := \delta^i|_{\mathcal{F}^i_0}$, for all i. The complex $\mathbf{F}_0 = (\mathcal{F}^i_0, \delta^i_0)$ provides the required complex. □

Theorem 2.5. *The pair* $(\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp})$ *is cogenerated by a set in the category of complexes of* $\mathcal{O}_X$*-modules; that is, there exists a set* Y *of objects in* $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$ *such that* $Y^{\perp} = \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp}$*.*

Proof. Using Theorem 2.4, we see that, for a flat complex $\mathbf{F}$, we can construct a continuous chain $\{\mathbf{F}_\alpha \mid \alpha \leq \gamma\}$ of flat pure subcomplexes of $\mathbf{F}$ with $\mathbf{F} = \bigcup_{\alpha \leq \gamma} \mathbf{F}_\alpha$, and such that $|\mathbf{F}_0| \leq \kappa$ and, for all $\alpha \leq \gamma$, $|\frac{\mathbf{F}_{\alpha+1}}{\mathbf{F}_\alpha}| \leq \kappa$ and $\frac{\mathbf{F}_{\alpha+1}}{\mathbf{F}_\alpha}$ is a flat complex. Let Y be a representative set of flat complexes $\mathbf{F}$ with $|\mathbf{F}| \leq \kappa$. Since $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$ is closed under extensions and direct limits, by the same argument as that used in [ET, Lemma 1], one can deduce that $Y^{\perp} = \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp}$. □

Proposition 2.6. *For any complex* $\mathbf{X}$ *of* $\mathcal{O}_X$*-modules, there is an exact sequence* $0 \longrightarrow \mathbf{X} \longrightarrow \mathcal{C} \longrightarrow \mathcal{P} \longrightarrow 0$ *of complexes of* $\mathcal{O}_X$*-modules with* $\mathcal{C} \in \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp}$ *and* $\mathcal{P} \in \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$*.*

Proof. The subcategory $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$ of the Grothendieck category $\mathbf{C}(\mathfrak{Qco}X)$ is closed under extensions and direct limits. Moreover, by Theorem 2.5, the pair $(\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$, $\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp})$ is cogenerated by a set. Now, the result follows by the same argument as that used in [ET, Theorem 2]. □

By [E, 3.2] (see also, [EE, 4.1]), one can deduce the following corollary.

Corollary 2.7. *The category of complexes of (quasi-coherent)* $\mathcal{O}_X$*-modules* $\mathbf{C}(\mathfrak{Qco}X)$ *admits flat covers and dg-cotorsion envelopes.*

Proposition 2.8. *For any complex $\mathbf{X}$ of flat $\mathcal{O}_X$-modules, there is an exact sequence* $0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{P} \longrightarrow \mathbf{X} \longrightarrow 0$ *of complexes of flat $\mathcal{O}_X$-modules with* $\mathcal{C} \in \mathbf{C}(\text{dg-Cof}X)$ *and* $\mathcal{P} \in \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$.

Proof. By Proposition 2.6, there is an exact sequence $0 \longrightarrow \mathbf{X} \xrightarrow{f} \mathcal{C} \longrightarrow \mathcal{P} \longrightarrow 0$ of complexes of $\mathcal{O}_X$-modules with $\mathcal{P} \in \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$. Since $\mathbf{X}$ is a complex of flat modules, $\mathcal{C} \in \mathbf{C}(\text{dg-Cof}X)$. Since $\mathcal{P}$ is a flat complex, this sequence is degree-wise pure exact. Therefore, for any $\mathcal{O}_X$-module $\mathcal{G}$, the following sequence of complexes is exact:

$$0 \longrightarrow \mathbf{X} \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{f \otimes_{\mathcal{O}_X} 1_{\mathcal{G}}} \mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow 0 \ .$$

Since $\mathcal{P}$ is pure acyclic, $\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an exact complex. Therefore the mapping cone of $(f \otimes_{\mathcal{O}_X} 1_{\mathcal{G}})$ is acyclic. This implies that the mapping cone of f, $\mathrm{cone}(f)$, is a flat complex. Now consider the exact sequence

$$0 \longrightarrow \Sigma^{-1}\mathcal{C} \xrightarrow{\lambda} \mathrm{cone}(f) \longrightarrow \mathbf{X} \longrightarrow 0 \ .$$

By using Proposition 2.6, we have an exact sequence $0 \longrightarrow \Sigma^{-1}\mathcal{C} \xrightarrow{\gamma} \mathcal{S} \longrightarrow \mathcal{P}' \longrightarrow 0$ of complexes of flat $\mathcal{O}_X$-modules with $\mathcal{S} \in \mathcal{C}(\text{dg-Cof}X)$ and $\mathcal{P}' \in \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$. Now the exact sequence $0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{L} \longrightarrow \mathbf{X} \longrightarrow 0$ in the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{-1}\mathcal{C} \arrow[r, "\lambda"] \arrow[d, "\gamma"] & \mathrm{cone}(f) \arrow[r] \arrow[d] & \mathbf{X} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathbf{X} \arrow[r] & 0 \\
 & \mathcal{P}' \arrow[r, equal] & \mathcal{P}' & &
\end{tikzcd}$$

provides the required sequence. □

Theorem 2.9. $(\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{C}(\text{dg-Cof}X))$ *is a complete cotorsion theory in the category of complexes of flat $\mathcal{O}_X$-modules* $\mathbf{C}(\mathrm{Flat}X)$.

Proof. In view of the above two propositions, we only need to show that ${}^{\perp}\mathbf{C}(\text{dg-Cof}X) \subseteq \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$, where the orthogonal is taken in $\mathbf{C}(\mathrm{Flat}X)$. Let $\mathbf{Y} \in {}^{\perp}\mathbf{C}(\text{dg-Cof}X)$. By Proposition 2.8, there is an exact sequence $0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{P} \longrightarrow \mathbf{Y} \longrightarrow 0$ such that $\mathbf{C} \in \mathbf{C}(\text{dg-Cof}X)$ and $\mathbf{P}$ is a flat complex. This sequence is split, and hence $\mathbf{Y}$ is a flat complex. □

3. A cotorsion theory in $\mathbf{K}(\mathrm{Flat}X)$

In the previous section we showed that the pair $(\mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{C}(\text{dg-Cof}X))$ is a complete cotorsion theory in the category $\mathbf{C}(\mathrm{Flat}X)$. By using Proposition 2.1, one can deduce that $\mathbf{C}(\text{dg-Cof}X)$ is closed under suspension and for any chain map f in $\mathbf{C}(\text{dg-Cof}X)$, the mapping cone of f is a dg-cotorsion complex of flat modules. Therefore we can consider the essential subcategory $\mathbf{K}(\text{dg-Cof}X)$ of $\mathbf{K}(\mathrm{Flat}X)$; i.e. $\mathbf{K}(\text{dg-Cof}X) = \{\mathcal{F} \in \mathbf{K}(\mathrm{Flat}X) |\ \mathcal{F} \text{ is isomorphic to some } \mathcal{Y} \in \mathbf{C}(\text{dg-Cof}X)\}$ which is a full subcategory of $\mathbf{K}(\mathrm{Flat}X)$. In this section we will show that the pair $(\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{K}(\text{dg-Cof}X))$ is a complete cotorsion theory in $\mathbf{K}(\mathrm{Flat}X)$. Then we study the homotopy category $\mathbf{K}(\text{dg-Cof}X)$ and investigate its relation with the homotopy categories $\mathbf{K}(\mathrm{Cof}X)$ and $\mathbf{K}(\mathrm{Proj}R)$. Note that $\mathbf{K}(\mathrm{Cof}X)$ is the essential homotopy category of complexes of cotorsion flat $\mathcal{O}_X$-modules as a full subcategory of $\mathbf{K}(\mathrm{Flat}X)$.

Proposition 3.1. *Let* $\mathbf{X}$ *be a complex of flat* $\mathcal{O}_X$*-modules. Then there exists a triangle* $\mathbf{X} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{S} \longrightarrow \Sigma\mathbf{X}$ *in* $\mathbf{K}(\mathrm{Flat}X)$ *such that* $\mathbf{Q} \in \mathbf{K}(\text{dg-Cof}X)$ *and* $\mathbf{S} \in \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)$.

Proof. By Proposition 2.8, there is an exact sequence $0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{P} \longrightarrow \mathbf{X} \longrightarrow 0$ with $\mathbf{C} \in \mathbf{C}(\text{dg-Cof}X)$ and $\mathbf{P} \in \mathbf{C}_{\mathrm{p}}(\mathrm{Flat}X)$. Since, by Proposition 2.1, $\mathbf{C}$ is a complex of cotorsion modules this sequence is degree-wise split. Thus there exists a canonical morphism $u : \mathbf{X} \to \Sigma\mathbf{C}$ such that $\mathbf{C} \longrightarrow \mathbf{P} \longrightarrow \mathbf{X} \xrightarrow{u} \Sigma\mathbf{C}$ is a triangle in $\mathbf{K}(\mathrm{Flat}X)$. This finishes the proof. $\square$

Proposition 3.2. $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp} = \mathbf{K}(\text{dg-Cof}X)$*, where the orthogonal is taken in* $\mathbf{K}(\mathrm{Flat}X)$.

Proof. By Proposition 2.1, $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp} \supseteq \mathbf{K}(\text{dg-CofX})$. Let $\mathbf{X} \in \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp}$. Then, by Proposition 3.1, there is a triangle $\mathbf{X} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{S} \longrightarrow \Sigma\mathbf{X}$ in $\mathbf{K}(\mathrm{Flat}X)$ such that $\mathbf{Q} \in \mathbf{K}(\text{dg-Cof}X)$ and $\mathbf{S} \in \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)$. Now, by applying the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\mathrm{Flat}X)}(\mathbf{S}, -)$, one can deduce that $\mathrm{Hom}_{\mathbf{K}(\mathrm{Flat}X)}(\mathbf{S}, \mathbf{S}) = 0$, which implies that $\mathbf{S}$ is contractible. $\square$

By the same argument as that used in the proof of the above proposition, one can settle the following statement.

Proposition 3.3. $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X) = {}^{\perp}\mathbf{K}(\text{dg-Cof}X)$*, where the orthogonal is taken in* $\mathbf{K}(\mathrm{Flat}X)$.

Remark 3.4. The above results and the arguments work as well for the category of R-modules whenever R is an associative ring.

Proof of Theorem 1.2. Let $\mathbf{P} \in \mathbf{K}(\mathrm{Proj}R)$. By Proposition 3.1, we have a triangle

$$j_*(\mathbf{P}) \longrightarrow \mathbf{C} \longrightarrow \mathbf{L} \rightsquigarrow$$

in $\mathbf{K}(\mathrm{Flat}R)$ such that $\mathbf{C} \in \mathbf{K}(\text{dg-Cof}R)$ and $\mathbf{L} \in \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}R)$. By the adjoint property, $\mathrm{Hom}_{\mathbf{K}(\mathrm{FlatR})}(\mathbf{L}, \mathrm{j}_*(\mathbf{P})) = \mathrm{Hom}_{\mathbf{K}(\mathrm{ProjR})}(\mathrm{j}^*(\mathbf{L}), \mathbf{P})$, which is zero because $j^*(\mathbf{L})$ is contractible; see [N1, 2.12 and 8.6]. By Proposition 3.3, $\mathrm{Hom}_{\mathbf{K}(\mathrm{FlatR})}(\mathbf{L}, \mathbf{C}) = 0$ and this implies that $\mathrm{Hom}_{\mathbf{K}(\mathrm{FlatR})}(\mathbf{L}, \mathbf{L}) = 0$. Therefore $\mathbf{L}$ is contractible and so $j_*(\mathbf{P}) = \mathbf{C}$ in $\mathbf{K}(\mathrm{Flat}R)$.

Now consider the diagram

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Proj}R) \arrow[r, "j^*"] \arrow[dr, bend right, "U"'] & \mathbf{K}(\mathrm{dg} - \mathrm{Cof}R) \arrow[r, "\mathrm{i}"] & \mathbf{K}(\mathrm{Flat}R) \arrow[dl, bend left, "\pi"] \\
 & \frac{\mathbf{K}(\mathrm{Flat}R)}{\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}R)} &
\end{tikzcd}$$

where i is the inclusion functor, π is the Verdier quotient functor and $U = \pi \mathrm{i} j_*$. Since, by Theorem 1.1, the inclusion functor $\mathrm{i} : \mathbf{K}(\text{dg-Cof}R) \to \mathbf{K}(\mathrm{Flat}R)$ has a left adjoint and ${}^{\perp}\mathbf{K}(\text{dg-Cof}R) = \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}R)$, one can deduce that the composite functor $\pi\mathrm{i}$ is an equivalence. Morover, by [N1, 2.12 and 8.6], U is also an equivalence. Now the result follows.

Corollary 3.5. *The pair* $(\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{K}(\mathrm{CofX}))$ *is a complete cotorsion theory in* $\mathbf{K}(\mathrm{Flat}X)$ *if and only if every flat complex of cotorsion modules is contractible.*

Proof. By Theorem 1.1, $(\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{K}(\text{dg-Cof}X))$ is a complete cotorsion theory in $\mathbf{K}(\mathrm{Flat}X)$ and, by Proposition 2.1, $\mathbf{K}(\text{dg-Cof}X) \subseteq \mathbf{K}(\mathrm{CofX})$. So we should show that $\mathbf{K}(\mathrm{Cof}X) \subseteq \mathbf{K}(\text{dg-Cof}X)$ if and only if every flat complex of cotorsion $\mathcal{O}_X$-modules is contractible. Let $\mathbf{X} \in \mathbf{K}(\mathrm{Cof}X)$. By Proposition 3.1, there is a triangle $\mathbf{X} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{S} \longrightarrow \Sigma\mathbf{X}$ in $\mathbf{K}(\mathrm{Flat}X)$ such that $\mathbf{Q} \in \mathbf{K}(\text{dg-Cof}X)$ and $\mathbf{S} \in \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)$. Up to isomorphisms in $\mathbf{K}(\text{dg-Cof}X)$, we can consider $\mathbf{Q}$ as a complex of cotorsion flat modules. Also, by assumption, $\mathbf{X}$ is a complex of cotorsion flat modules. Therefore we can replace $\mathbf{S}$ by a complex of cotorsion flat modules in $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)$ which is contractible by the assumption. Hence $\mathbf{X} \simeq \mathbf{Q}$ in $\mathbf{K}(\mathrm{Flat}X)$. Now assume that $\mathbf{K}(\mathrm{Cof}X) = \mathbf{K}(\text{dg-Cof}X)$. Since, by Proposition 2.1, $\mathbf{K}(\text{dg-Cof}X) \subseteq \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X)^{\perp}$ then, for every flat complex $\mathbf{F}$ of cotorsion modules, we have $\mathrm{Hom}_{\mathbf{K}(\mathrm{FlatR})}(\mathbf{F}, \mathbf{F}) = 0$. Therefore $\mathbf{F}$ is contractible. $\square$

Here we fix some notation. For any quasi-coherent sheaf $\mathcal{F}$ consider the colimit preserving functor $- \otimes_{\mathcal{O}_X} \mathcal{F} \ : \mathfrak{Qco}X \to \mathfrak{Qco}X$. This functor has a right adjoint (by the adjoint functor theorem), which we denote by $\mathcal{H}om_{\mathrm{qc}}(\mathcal{F}, -) : \mathfrak{Qco}X \to \mathfrak{Qco}X$. This defines the internal structure $\mathcal{H}om(-,-)$ on $\mathfrak{Qco}X$. Note that, when $\mathcal{F}$ is coherent and $\mathcal{G}$ is a (quasi-coherent) $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent and there is a canonical isomorphism $\mathcal{H}om_{\mathrm{qc}}(\mathcal{F}, \mathcal{G}) \approx \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, where $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf home of $\mathcal{F}$ into $\mathcal{G}$.

Lemma 3.6. *Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then, for any $\mathcal{O}_X$-module $\mathcal{K}$, $\mathcal{H}om_{\mathrm{qc}}(\mathcal{K}, \mathcal{I})$ is cotorsion.*

Proof. Set $\mathcal{N} := \mathcal{H}om_{\mathrm{qc}}(\mathcal{K}, \mathcal{I})$. Let $(*) : \quad 0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{F} \to 0$ be an exact sequence with $\mathcal{F}$ flat module. Hence the sequence $0 \to \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{K} \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K} \to 0$ is exact. By applying the exact functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$, we get the exact sequence $0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{K}, \mathcal{I}) \to 0$.

Now, by using the adjoint property, we obtain the exact sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}) \longrightarrow 0$, which implies that the sequence $(*)$ is split. Thus $\mathrm{Ext}^1(\mathcal{F}, \mathcal{H}om_{\mathrm{qc}}(\mathcal{K}, \mathcal{I})) = 0$, for any flat module $\mathcal{F}$. $\square$

Lemma 3.7. *Let the flat precovers be epimorphisms in the category $\mathfrak{Qco}X$ and $\mathcal{C}$ be a cotorsion $\mathcal{O}_X$-module. Then the cohomology functors $\mathrm{Ext}^i_X(-, \mathcal{C})$ vanish over flat modules, for all $i > 0$.*

Proof. Let $\mathcal{F}$ be a flat module and $[\xi] : \quad 0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{N} \xrightarrow{e} \mathcal{M} \longrightarrow \mathcal{F} \longrightarrow 0$ be a representation of an element in $\mathrm{Ext}^2_X(\mathcal{F}, \mathcal{C})$. Let $\alpha : \mathcal{F}' \to \mathcal{M}$ be a flat precover of $\mathcal{M}$. Consider the pullback diagram of the maps α and e:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{C} & \xrightarrow{t} & \mathcal{N}' & \xrightarrow{u} & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow 0 \\
& \| & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \| & \\
0 \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{N} & \xrightarrow{e} & \mathcal{M} & \longrightarrow & \mathcal{F} & \longrightarrow 0.
\end{array}$$

Since $\mathcal{F}$ and $\mathcal{F}'$ are flat, $\mathrm{Im}(u)$ is flat. We set $T := \mathrm{Im}(u)$. Let $\beta : T \to \mathcal{C}'$ be the cotorsion envelope of T. Consider the pushout diagram formed by β and the inclusion map $i : T \to \mathcal{F}'$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{i} & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array}$$

Since $\mathcal{C}$ is cotorsion and T is flat, the sequence $0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{N} \longrightarrow T \longrightarrow 0$ is split. This implies that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{C} & \xrightarrow{t} & \mathcal{N}' & \xrightarrow{u} & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C} \oplus \mathcal{C}' & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array}$$

Since $\mathcal{C}'$ is cotorsion and $\mathcal{F}$ is flat, the sequence $0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow 0$ is split, which implies that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{C} & \xrightarrow{t} & \mathcal{N}' & \xrightarrow{u} & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C} \oplus \mathcal{C}' & \longrightarrow & \mathcal{C}' \oplus \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array}$$

Obviously, the last row is a zero representation which is equivalent with the representation $[\xi]$. For $i > 2$ the proof is similar. □

Example 3.8. By [EE, 4.2], the category of quasi-coherent sheaves admits flat covers and cotorsion envelopes. But we do not know whether flat covers are epimorphisms. In [M, 3.12], Murfet has shown that flat precovers are epimorphisms in the category $\mathfrak{Qco}X$ whenever the scheme X is semi-separated. Recall that the scheme X is called *semi-separated* if there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that for all $\lambda \in \Lambda$, U_λ and all the finite intersections of $\{U_\lambda\}_{\lambda \in \Lambda}$ are affine; see [TT] and [AJPV] for more details.

3.1. **Dualizing complex.** In this subsection we assume that the scheme X admits a dualizing complex $\mathcal{J}$. That is, $\mathcal{J}$ is an object of $\mathbf{D}^b(\mathrm{Coh}X)$ so that it has a bounded injective resolution and that the natural functor

$$\mathcal{H}om(-, \mathcal{J}) : \mathbf{D}^b(\mathrm{Coh}X)^{\mathrm{op}} \longrightarrow \mathbf{D}^b(\mathrm{Coh}X)$$

is an equivalence of categories, where $\mathbf{D}^b(\mathrm{Coh}X)$ is the bounded derived category of coherent sheaves on X. Let $\xi : \mathcal{J} \to \mathcal{D}$ be a quasi-isomorphism, where $\mathcal{D}$ is a bounded complex of injective modules. If $\mathcal{F}$ is a complex of flat $\mathcal{O}_X$-modules, then ξ induces a morphism

$$\alpha : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

Composing α with the canonical morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ we have a natural morphism $\delta' : \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F})$.

Proposition 3.1.1. *Let flat precovers be epimorphisms in the category $\mathfrak{Qco}X$ and $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. Then the cohomology functors $\mathrm{Ext}^i_X(-, \mathcal{F})$ vanish over flat modules, for all $i \gg 0$.*

Proof. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. By a localizing method and using [IK, 1.1], one can deduce that $\delta' : \mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{J}, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F})$ is a quasi-isomorphism. Let

$$\mathrm{cone}(\delta') : \qquad \cdots \to \mathcal{T}^{-1} \to \mathcal{T}^0 \to \mathcal{T}^1 \to \mathcal{T}^2 \to \cdots$$

be the mapping cone of δ' which is exact and bounded. We can assume that $\mathcal{T}^1 = \mathcal{F} \oplus \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F})^{-1}$. By using Lemma 3.6, $\mathcal{T}^i$ is cotorsion, for all $i \neq 1$. Now by breaking the above exact sequence into the short exact sequences and using the long exact sequences induced by Ext, we deduce that the functor $\mathrm{Ext}^i(-, \mathcal{F})$ vanishes over flat $\mathcal{O}_X$-modules, for all $i \gg 0$. □

Corollary 3.1.2. *Let flat precovers be epimorphisms in the category $\mathfrak{Qco}X$. Then any complex of cotorsion flat $\mathcal{O}_X$-modules is contractible and so $\mathbf{K}(\mathrm{Cof}X) = \mathbf{K}(\text{dg-}\mathrm{Cof}X)$.*

Proof. Let $\mathcal{T}$ be a flat complex of cotorsion modules. Since for all integers i, $\mathrm{Ker}\partial^i_{\mathcal{T}}$ is flat, using Proposition 3.1.1, one can deduce that the complex $\mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Ker}\partial^i_{\mathcal{T}}, \mathcal{T})$ is exact. This implies that $\mathcal{T}$ is contractible. □

Example 3.1.3. Let X be a semi-separated Noetherian scheme of finite Krull dimension. Then, by [AS, 6.9], there is an integer d such that for any flat (quasi-coherent) sheaf $\mathcal{F}$ the relative cohomology functors $\mathrm{Ext}^i_X(\mathcal{F}, \ )$ vanish, for all $i > d$. Now by the same argument as that used in the above corollary, any complex of cotorsion flat $\mathcal{O}_X$-modules is contractible and so $\mathbf{K}(\mathrm{Cof}X) = \mathbf{K}(\text{dg-}\mathrm{Cof}X)$. Therefore, by Corollary 3.5, the pair $(\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}X), \mathbf{K}(\mathrm{Cof}X))$ is a cotorsion theory in the homotopy category of complexes of flat $\mathcal{O}_X$-modules $\mathbf{K}(\mathrm{Flat}X)$.

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