COMMENT ON A RESULT BY ALPIN, CHIEN, AND YEH

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ABSTRACT. Theorem 3 in Alpin, Yu. A.; Chien, Mao-Ting; and Yeh, Lina, The numerical radius and bounds for zeros of a polynomial, Proc. Amer. Math. Soc., 131 (2003), no. 3, 725–730, is incorrect. In this note, we identify the error and present a correctly stated theorem.

1. INTRODUCTION

This note concerns the paper [1] by Alpin, Chien, and Yeh. One of their results, namely, Theorem 3, is erroneous. We provide a counterexample and then identify the flaw in the argument of the proof before stating an amended theorem. Most of our notation will be the same as in [1].

Let
\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & -a_0 \\
1 & 0 & \ldots & 0 & -a_1 \\
0 & 1 & \ldots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -a_{n-1}
\end{pmatrix}
\]
be the companion matrix of the monic polynomial
\[
p(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0 ,
\]
and define for a complex $n \times n$ matrix $A = (a_{ij})$:
\[
R'_i(A) = \sum_{j=1}^{n} |a_{ij}| \quad \text{and} \quad g_i(A) = \frac{1}{2} \left( R'_i(A) + R'_i(A^*) \right) , \quad 1 \leq i \leq n ,
\]
where $A^*$ is the Hermitian conjugate of $A$. We denote the spectrum of $A$ by $\sigma(A)$ and its numerical range by $W(A)$, i.e.,
\[
W(A) = \{ z^* A z : z \in \mathbb{C}^n, |z| = 1 \}.
\]
The set of points $z \in \mathbb{C}$ satisfying $|z - a||z - b| = \rho$ for $a, b \in \mathbb{C}$ and $\rho \geq 0$ is known as an oval of Cassini.

We now state Theorem 3 in [1].

**Theorem 1.1 (1 [1] Theorem 3).** Let $p$ be the polynomial (1.2) and $p(z) = 0$, and let $A$ be the matrix defined by (1.1). Let $\alpha = \max_{1 \leq i < n} g_i(A)$. Then
\[
|z| \leq |a_{n-1}|/2 + \left( |a_{n-1}|^2/4 + (\alpha^2 g_n^2(A) + |a_{n-1}|^2)^{1/2} \right)^{1/2}.
\]
The following claim presents a counterexample to this theorem.

**Claim 1.2.** The upper bound from Theorem 1 is not an upper bound for the modulus of any zero of \(q(t) = t^9 + 0.01t^8 + 0.2\).

**Proof.** For the companion matrix \(A\) of \(q(t)\), we have

\[
R_1'(A) = 0.2, \quad R_2'(A) = R_3'(A) = \cdots = R_8'(A) = 1, \quad R_9'(A) = 1,
\]

\[
R_1'(A^*) = 1, \quad R_2'(A^*) = R_3'(A^*) = \cdots = R_8'(A^*) = 1, \quad R_9'(A^*) = 0.2,
\]

\[
g_1(A) = 0.6, \quad g_2(A) = g_3(A) = \cdots = g_8(A) = 1, \quad g_9(A) = 0.6.
\]

Therefore, \(\alpha = \max_{1 \leq i < 9} g_i(A) = 1\), and the upper bound in Theorem 1 is given by

\[
0.01/2 + \left( \left[ \frac{1}{2} + \left( \frac{\alpha^2}{4} + \frac{\alpha^2 g_n^2(A) + |a_{n-1}|^2}{4} \right)^{1/2} \right] \right) = 0.7797.
\]

However, any zero \(z\) of \(q(t)\) satisfies \(0.8352 \leq |z| \leq 0.8374\), with the entire range being larger than 0.7797. \(\square\)

**2. Corrected Theorem**

The error in the proof of Theorem 3 in [1] was caused by the following incorrect inequality:

\[
(2.1) \quad \alpha \leq |a_{n-1}|/2 + \left( |a_{n-1}|^2/4 + \left( \alpha^2 g_n^2(A) + |a_{n-1}|^2 \right)^{1/2} \right)^{1/2}.
\]

To see why this inequality cannot be correct in general, assume, e.g., that for the coefficients of \(p\) in (1.2): \(a_1 = a_2 = \cdots = a_{n-2} = 0, a_{n-1} = \beta,\) and \(a_0 = \epsilon\) with \(\beta, \epsilon > 0\) and \(\beta, \epsilon \ll 1\). Then \(\alpha = 1\) and \(g_n = (1 + \epsilon)/2\). For \(\beta\) sufficiently small, inequality (2.1) becomes

\[
1 \leq \beta^2/2 + \left( \frac{\beta^2}{4} + \left( \frac{(1 + \epsilon)^2}{4} + \beta^2 \right)^{1/2} \right)^{1/2}
\]

\[
\leq \beta^2/2 + \left( \frac{1 + \epsilon}{2} + O(\beta^2) \right)^{1/2}
\]

\[
= \left( \frac{1 + \epsilon}{2} \right)^{1/2} + O(\beta).
\]

Clearly, as \(\beta, \epsilon \to 0\), this inequality cannot hold. The following theorem corrects Theorem 3 in [1].

**Theorem 2.1.** Let \(p\) be the polynomial (1.2) and \(p(z) = 0\), and let \(A\) be the matrix defined by (1.1). Let \(\alpha_1\) and \(\alpha_2\) be the largest and (possibly equal) second largest of the numbers \(\{g_i(A)\}_{i=1}^{n-1}\). Then

\[
|z| \leq \max \left\{ \sqrt{\alpha_1 \alpha_2}, \quad \frac{|a_{n-1}|}{2} + \left( \frac{|a_{n-1}|^2}{4} + \left( \alpha_1^2 g_n^2(A) + |a_{n-1}|^2 \right)^{1/2} \right)^{1/2} \right\}.
\]
Proof. The proof follows, just as in Theorem 3 in [1], by applying Theorem 2 in [1] to the companion matrix $A$ of $p$. The closed interiors of the ovals of Cassini defined in that theorem are then the collection of points $z \in \mathbb{C}$ for which

$$|z| \leq (g_i(A)g_j(A))^{1/2}, \ 1 \leq i \neq j < n, \quad (2.2)$$

and those for which

$$|z||z + a_{n-1}| \leq (g_i^2(A)g_n^2(A) + |a_{n-1}|^2)^{1/2}, \ 1 \leq i < n. \quad (2.3)$$

The points defined by (2.2) are all contained in the disk defined by $|z| \leq (\alpha_1\alpha_2)^{1/2}$, and the points defined by (2.3) are all contained in the disk defined by

$$|z| \leq \left|\frac{a_{n-1}}{2} + \left(\frac{|a_{n-1}|^2}{4} + (\alpha_1^2g_n^2(A) + |a_{n-1}|^2)^{1/2}\right)^{1/2}\right|.\quad (2.3)$$

This, together with the fact that $\sigma(A) \subset W(A)$, completes the proof. \hfill \Box

For our counterexample with $q(t) = t^9 + 0.01t^8 + 0.2$, we obtain $\alpha_1 = \alpha_2 = 1$, so that Theorem 2 states, correctly, that for any zero $z$ of $q$: $|z| \leq 1$.

References


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