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THE DIXMIER APPROXIMATION THEOREM IN ALGEBRAS OF MEASURABLE OPERATORS

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Abstract. In this paper we are concerned with proving versions of the classical Dixmier approximation theorem in the setting of algebras of τ-measurable operators $S(\mathcal{M},\tau)$ and its $\mathcal{M}$-bimodules, where $\mathcal{M}$ is a semi-finite von Neumann algebra equipped with a semi-finite normal faithful trace τ.

1. Introduction

In this paper we study the unitary orbits of τ-measurable operators associated with a semi-finite von Neumann algebra $\mathcal{M}$ equipped with a semi-finite normal faithful trace τ. We are particularly interested in obtaining a version of the Dixmier approximation theorem in this more general setting. We now recall Dixmier's classical result for von Neumann algebras. Given $x \in \mathcal{M}$, the unitary orbit of x is defined by $\mathcal{U}(x) = \{u^*xu : u \in U(\mathcal{M})\}$, where $U(\mathcal{M})$ is the unitary group of the von Neumann algebra $\mathcal{M}$. The convex hull of $\mathcal{U}(x)$ is denoted by $\mathrm{co}\,\mathcal{U}(x)$ and its norm closure in $\mathcal{M}$ is denoted by $\overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x)$. Dixmier's approximation theorem now states that

$$\overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x) \cap \mathcal{Z}(\mathcal{M}) \neq \emptyset,$$

where $\mathcal{Z}(\mathcal{M})$ is the center of $\mathcal{M}$ ([1]; see also [10], Chapter 8).

In the setting of non-commutative integration theory, it is only natural to ask for versions of this approximation theorem in the larger algebra $S(\tau)$ of all τ-measurable operators, with respect to the measure topology, or in symmetric spaces $E \subseteq S(\tau)$, with respect to the norm topology of E. As it turns out, the structure of the underlying von Neumann algebra $\mathcal{M}$ plays an important role in answering these questions (as can be seen by comparing the results of Proposition 7 and Theorem 10).

We would like to point out that previous work on unitary orbits in the setting of non-commutative L^p-spaces was done by F. Hiai and Y. Nakamura in [8] and [9].

After some preliminary information given in Section 2, we consider in Section 3 the approximation theorem in symmetric spaces (or, more generally, in so-called Banach $\mathcal{M}$-bimodules) $E \subseteq S(\tau)$. For instance, it will be shown (see Corollary 5) that if E has order continuous norm and $\mathcal{M} \subseteq E$, then

$$\overline{\mathrm{co}}^{E}\mathcal{U}(x) \cap \mathcal{Z}(S(\tau)) \neq \emptyset,$$

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where $\mathcal{Z}(S(\tau))$ is the center of the algebra $S(\tau)$ and the superscript E indicates that the closure is taken with respect to the norm in E. Furthermore, it will be shown (see Corollary 6) that

$$\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x) \cap \mathcal{Z}(S(\tau)) \neq \emptyset \tag{1}$$

for all $x \in L_1(\tau) + \mathcal{M}$ (where the superscript $\mathcal{T}_m$ indicates that the closure is taken with respect to the measure topology). Without any additional conditions on the von Neumann algebra $\mathcal{M}$, this latter result is in some sense best possible (cf. Proposition 7).

In Section 4, our main interest is the approximation theorem in $S(\tau)$, with respect to the measure topology, for the case that $\mathcal{M}$ is a type II_∞ factor. In this situation, it turns out that (1) holds for all $x \in S(\tau)$ (see Theorem 10). It is somewhat surprising that the analogue of this result fails in the setting when $\mathcal{M}$ is a type II_1-factor. Finally, we apply these results to obtain a version of the approximation theorem in a certain class of symmetric spaces in Corollary 11.

2. Some preliminaries

In this section we collect some of the basic facts and notation that will be used in this paper. For more details we refer the reader to [7], [13] and [15]. Suppose that $\mathcal{M}$ is a semi-finite von Neumann algebra, acting on a Hilbert space $\mathcal{H}$, equipped with a fixed semi-finite normal faithful trace τ. Recall that a closed and densely defined linear operator x in $\mathcal{H}$ is called *τ-measurable* if x is affiliated with $\mathcal{M}$ (that is, $xu = ux$ for all unitary operators in the commutant $\mathcal{M}'$ of $\mathcal{M}$) and there exists $0 < s_0 \in \mathbb{R}$ such that $\tau\left(e^{|x|}(s_0,\infty)\right) < \infty$ (where $e^{|x|}$ denotes the spectral measure of the self-adjoint operator $|x|$). The collection of all τ-measurable operators is denoted by $S(\tau)$. With respect to the strong sum and product, $S(\tau)$ is a $*$-algebra. For $0 < \varepsilon, \delta \in \mathbb{R}$ the set $V(\varepsilon,\delta)$ is defined by

$$V(\varepsilon,\delta) = \left\{x \in S(\tau) : \tau\left(e^{|x|}(\varepsilon,\infty)\right) \leq \delta\right\}.$$

The sets $\{V(\varepsilon,\delta) : \varepsilon,\delta > 0\}$ are a neighbourhood base at zero for a complete metrizable vector space topology in $S(\tau)$, which is called the *measure topology* and is denoted by $\mathcal{T}_m$. With respect to $\mathcal{T}_m$, $S(\tau)$ is a topological $*$-algebra. It should be observed that if $x \in S(\tau)$ and $\varepsilon,\delta > 0$, then $x \in V(\varepsilon,\delta)$ if and only if there exists a projection $p \in \mathcal{M}$ such that $\|xp\|_{B(\mathcal{H})} \leq \varepsilon$ and $\tau(\mathbf{1}-p) \leq \delta$.

At some points in the present paper we will also use the *local measure topology.* Recall that the local measure topology $\mathcal{T}_{lm}$ in $S(\tau)$ is the Hausdorff vector space topology for which the sets

$$V(\varepsilon,\delta;e) = \{x \in S(\tau) : exe \in V(\varepsilon,\delta)\},$$

where $\varepsilon,\delta > 0$ and $e \in \mathcal{M}$ a projection with $\tau(e) < \infty$, form a neighbourhood base at zero. This topology $\mathcal{T}_{lm}$ is, in general, not metrizable. If $\{x_\alpha\}$ is a net in $S(\tau)$, then $x_\alpha \overset{\mathcal{T}_{lm}}{\to}_\alpha 0$ if and only if $ex_\alpha e \overset{\mathcal{T}_m}{\to}_\alpha 0$ for all projections $e \in \mathcal{M}$ with $\tau(e) < \infty$. In particular, the local measure topology is weaker than the measure topology. For more information concerning the local measure topology we refer the reader to [5].

For any $x \in S(\tau)$ its *generalized singular value function* $\mu(x) : [0,\infty) \to [0,\infty]$ is defined by setting

$$\mu(t;x) = \inf\left\{s \geq 0 : \tau\left(e^{|x|}(s,\infty)\right) \leq t\right\}, \quad t \geq 0.$$

Note that $\mu(t;x) < \infty$ for all $t > 0$ and that it is decreasing and right continuous. The set $S_0(\tau)$ of all *τ-compact operators* is defined by

$$S_0(\tau) = \{x \in S(\tau) : \lim_{t\to\infty} \mu(t;x) = 0\}.$$

The set $S_0(\tau)$ is a $*$-closed two-sided ideal in $S(\tau)$, which is closed for the measure topology.

The *unitary orbit* of an operator $x \in S(\tau)$ is defined by

$$\mathcal{U}(x) = \{u^* x u : u \in U(\mathcal{M})\},$$

where $U(\mathcal{M})$ denotes the unitary group of the von Neumann algebra $\mathcal{M}$. Furthermore, $\mathrm{co}\,\mathcal{U}(x)$ will denote the convex hull of $\mathcal{U}(x)$. We denote by $\mathcal{D}$ the collection of all linear operators $\pi : S(\tau) \to S(\tau)$ which are of the form

$$\pi x = \sum\nolimits_{j=1}^{n} \lambda_j u_j^* x u_j, \quad x \in S(\tau),$$

where $u_j \in U(\mathcal{M})$, $0 \le \lambda_j \in \mathbb{R}$ with $\sum_{j=1}^{n} \lambda_j = 1$ and $n \in \mathbb{N}$. Note that $\mathcal{D}$ is a convex semi-group of linear operators on $S(\tau)$. It is now clear that

$$\mathrm{co}\,\mathcal{U}(x) = \{\pi x : \pi \in \mathcal{D}\}.$$

The closure of $\mathrm{co}\,\mathcal{U}(x)$ in $S(\tau)$ with respect to the measure topology is denoted by $\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)$. If $x \in \mathcal{M}$, then $\mathrm{co}\,\mathcal{U}(x) \subseteq \mathcal{M}$, in which case the closure of $\mathrm{co}\,\mathcal{U}(x)$ in $\mathcal{M}$ is denoted by $\overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x)$. It should be observed that the classical Dixmier approximation theorem states that

$$\overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x) \cap \mathcal{Z}(\mathcal{M}) \neq \emptyset$$

for all $x \in \mathcal{M}$, where $\mathcal{Z}(\mathcal{M})$ denotes the centre of $\mathcal{M}$.

The centre $\mathcal{Z}(S(\tau))$ of $S(\tau)$ is, as usual, defined by

$$\mathcal{Z}(S(\tau)) = \{x \in S(\tau) : xy = yx \ \forall\ y \in S(\tau)\}.$$

The following simple observation turns out to be useful.

Lemma 1. *The centre of $S(\tau)$ is also given by*

$$\mathcal{Z}(S(\tau)) = \{x \in S(\tau) : xy = yx \ \ \forall\ y \in \mathcal{M}\}. \tag{2}$$

Proof. Denoting the right-hand side of (2) momentarily by $\mathfrak{A}$, it is clear that $\mathcal{Z}(S(\tau)) \subseteq \mathfrak{A}$. Since $\mathfrak{A}$ is a $*$-closed subspace of $S(\tau)$, it follows that $\mathfrak{A} = \mathfrak{A}_h \oplus i\mathfrak{A}_h$ (where $\mathfrak{A}_h$ is the real subpace of all self-adjoint elements of $\mathfrak{A}$), and so it suffices to show that $\mathfrak{A}_h \subseteq \mathcal{Z}(S(\tau))$. Let $a \in \mathfrak{A}_h$ be given. If $b \in S_h(\tau)$, then $e^b(A) \in \mathcal{M}$ and hence, $ae^b(A) = e^b(A)a$ for all Borel sets $A \subseteq \mathbb{R}$. This implies that $ab = ba$. Since $S(\tau) = S_h(\tau) \oplus iS_h(\tau)$, it follows that $a \in \mathcal{Z}(S(\tau))$. The proof is complete. $\square$

It should also be noted that if $\mathcal{M}$ is a factor, that is, $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1}$, then also $\mathcal{Z}(S(\tau)) = \mathbb{C}\mathbf{1}$. Indeed, if $a \in S_h(\tau)$, then $a \in \mathcal{Z}(S(\tau))$ if and only if $e^a(B) \in \mathcal{Z}(\mathcal{M})$ for all Borel sets $B \subseteq \mathbb{R}$.

3. The Dixmier approximation theorem in Banach $\mathcal{M}$-bimodules

In the present section we discuss a version of the Dixmier approximation theorem in a certain class of normed subspaces of $S(\tau)$. As before, we assume that $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra. We recall the following definition.

Definition 2. A linear subspace E of $S(\tau)$, equipped with a norm $\|\cdot\|_E$, is called a *Banach $\mathcal{M}$-bimodule (of τ-measurable operators)* if

(i) $uxv \in E$ and $\|uxv\|_E \leq \|u\|_{B(\mathcal{H})} \|v\|_{B(\mathcal{H})} \|x\|_E$ whenever $x \in E$ and $u, v \in \mathcal{M}$;

(ii) $(E, \|\cdot\|_E)$ is a Banach space.

It should be observed that any Banach $\mathcal{M}$-bimodule is $*$-closed and that $x \in S(\tau)$, $y \in E$ and $|x| \leq |y|$ imply that $x \in E$ and $\|x\|_E \leq \|y\|_E$. The embedding of E into $S(\tau)$ is continuous with respect to the norm topology in E and the local measure topology in $S(\tau)$ (see [6]). An important class of Banach $\mathcal{M}$-bimodules is provided by the symmetric spaces (of τ-measurable operators). Recall that a *symmetric space* is a linear subspace $E \subseteq S(\tau)$, equipped with a norm $\|\cdot\|_E$ such that E is a Banach space and, $x \in S(\tau)$, $y \in E$ and $\mu(x) \leq \mu(y)$ imply that $x \in E$ and $\|x\|_E \leq \|y\|_E$. Examples of symmetric spaces are e.g. non-commutative L_p-spaces ($1 \leq p \leq \infty$), non-commutative Orlicz spaces, etc. It should also be noted that the embedding of any symmetric space E into $S(\tau)$ is actually continuous with respect to the norm topology in E and the measure topology in $S(\tau)$ (see [3]).

Given a Banach $\mathcal{M}$-bimodule $E \subseteq S(\tau)$, we define

$$\mathcal{Z}_E = E \cap \mathcal{Z}(S(\tau)).$$

Observe that it follows from Lemma 1 that $E \cap \mathcal{Z}(\mathcal{M}) \subseteq \mathcal{Z}_E$.

Lemma 3. *If $E \subseteq S(\tau)$ is a Banach $\mathcal{M}$-bimodule, then $\mathcal{Z}_E$ is a norm closed and $*$-closed linear subspace of E.*

Proof. Only the norm closedness of $\mathcal{Z}_E$ needs a proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{Z}_E$ and that $x \in E$ is such that $\|x_n - x\|_E \to 0$ as $n \to \infty$. If $y \in \mathcal{M}$, then it follows from property (i) in Definition 2 that $x_n y \to xy$ and $yx_n \to yx$ in norm as $n \to \infty$. Since $x_n y = yx_n$ for all n, this implies that $xy = yx$. By Lemma 1, this suffices to show that $x \in \mathcal{Z}(S(\tau))$, and we may conclude that $x \in \mathcal{Z}_E$. $\square$

Given a Banach $\mathcal{M}$-bimodule $E \subseteq S(\tau)$, the unitary orbit $\mathcal{U}(x)$ of $x \in E$ is clearly contained in E. The norm closure of $\mathrm{co}\,\mathcal{U}(x)$ will be denoted by $\overline{\mathrm{co}}^E \mathcal{U}(x)$. Furthermore, each $\pi \in \mathcal{D}$ defines a contraction $\pi : E \to E$. Note that $\pi z = z$ for all $z \in \mathcal{Z}_E$ and $\pi \in \mathcal{D}$.

Theorem 4. *If $E \subseteq S(\tau)$ is a Banach $\mathcal{M}$-bimodule such that $\mathcal{M} \subseteq E$, then*

$$\overline{\mathrm{co}}^E \mathcal{U}(x) \cap \mathcal{Z}_E \neq \emptyset$$

for all $x \in \overline{\mathcal{M}}^E$ (where $\overline{\mathcal{M}}^E$ denotes the closure of $\mathcal{M}$ in E).

Proof. First observe that, since both $\mathcal{M}$ and E are continuously embedded in $S(\tau)$, equipped with the local measure topology, it follows from the closed graph theorem that there exists a constant $C > 0$ such that $\|y\|_E \leq C \|y\|_{B(H)}$ for all $y \in \mathcal{M}$.

Let $x \in \overline{\mathcal{M}}^E$ be given and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{M}$ such that $\|x - x_n\|_E \to 0$ as $n \to \infty$. Using the Dixmier approximation theorem in $\mathcal{M}$, we may inductively construct sequences $\{\pi_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ and $\{z_n\}_{n=1}^{\infty}$ in $\mathcal{Z}(\mathcal{M})$ satisfying

$$\|\pi_n \pi_{n-1} \cdots \pi_1(x_n) - z_n\|_{B(H)} \leq 2^{-n}, \quad n = 1, 2, \ldots .$$

We claim that $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Indeed, for $n > m$ we have

$$\|z_n - z_m\|_E \leq \|z_n - \pi_n \cdots \pi_1(x_n)\|_E + \|\pi_n \cdots \pi_1(x_n) - z_m\|_E$$

and

$$\|z_n - \pi_n \cdots \pi_1(x_n)\|_E \leq C2^{-n} \leq C2^{-m}.$$

Using that all $\pi \in \mathcal{D}$ are contractions in E and that $\pi z = z$ for all $z \in \mathcal{Z}(\mathcal{M})$ and $\pi \in \mathcal{D}$, we also find that

$$\begin{aligned} \|\pi_n \cdots \pi_1(x_n) - z_m\|_E &= \|\pi_n \cdots \pi_1(x_n) - \pi_n \cdots \pi_{m+1} z_m\|_E \\ &\leq \|\pi_m \cdots \pi_1(x_n) - z_m\|_E \\ &\leq \|\pi_m \cdots \pi_1(x_n - x_m)\|_E + \|\pi_m \cdots \pi_1(x_m) - z_m\|_E \\ &\leq \|x_n - x_m\|_E + C2^{-m}. \end{aligned}$$

Consequently,

$$\|z_n - z_m\|_E \leq \|x_n - x_m\|_E + 2C2^{-m},$$

whenever $n > m$, which proves our claim.

Since E is a Banach space, it follows that there exists $z \in E$ such that $\|z - z_n\|_E \to 0$ as $n \to \infty$. Since $\mathcal{M} \subseteq E$, it follows from Lemma 1 that $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{Z}_E$, and so it follows from Lemma 3 that $z \in \mathcal{Z}_E$.

Finally, we show that $z \in \overline{\mathrm{co}}^E \mathcal{U}(x)$. Indeed,

$$\begin{aligned} \|\pi_1 \cdots \pi_n(x) - z\|_E &\leq \|\pi_1 \cdots \pi_n(x - x_n)\|_E + \|\pi_1 \cdots \pi_n(x_n) - z_n\|_E \\ &\quad + \|z_n - z\|_E \\ &\leq \|x - x_n\|_E + C2^{-n} + \|z_n - z\|_E \end{aligned}$$

and hence, $\|\pi_1 \cdots \pi_n(x) - z\|_E \to 0$ as $n \to \infty$. The proof is complete. □

We mention some consequences of the above theorem. Recall that the norm $\|\cdot\|_E$ in a Banach $\mathcal{M}$-bimodule $E \subseteq S(\tau)$ is called *order continuous* if it follows from $x_\alpha \downarrow_\alpha 0$ in E that $\|x_\alpha\|_E \downarrow_\alpha 0$. If E has order continuous norm, then $E \cap \mathcal{M}$ is dense in E (indeed, if $a \in E^+$, then $a_n = ae^a[0, n] \uparrow_n a$ and $a_n \in E \cap \mathcal{M}$ for all $n \in \mathbb{N}$).

Corollary 5. *If $E \subseteq S(\tau)$ is a Banach $\mathcal{M}$-bimodule with order continuous norm, then*

$$\overline{\mathrm{co}}^{E+\mathcal{M}} \mathcal{U}(x) \cap \mathcal{Z}_{E+\mathcal{M}} \neq \emptyset$$

for all $x \in E + \mathcal{M}$. In particular, if E has order continuous norm and $\mathcal{M} \subseteq E$, then $\overline{\mathrm{co}}^E \mathcal{U}(x) \cap \mathcal{Z}_E \neq \emptyset$ for all $x \in E$.

Proof. First observe that, since both E and $\mathcal{M}$ are continuously embedded in the Hausdorff topological vector space $S(\tau)$, equipped with the local measure topology, the sum $E + \mathcal{M}$ is a Banach space (with respect to the usual norm). Furthermore, it is easy to see that $E + \mathcal{M}$ is a Banach $\mathcal{M}$-bimodule.

As observed above, since E has order continuous norm, for every $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty} \in \mathcal{M}$ such that $\|x - x_n\|_E \to 0$ as $n \to \infty$. Since $\|x - x_n\|_{E+\mathcal{M}} \leq \|x - x_n\|_E$, this shows that $E \subseteq \overline{\mathcal{M}}^{E+\mathcal{M}}$. Hence, $\overline{\mathcal{M}}^{E+\mathcal{M}} = E + \mathcal{M}$. The result now follows from applying Theorem 4 to the Banach $\mathcal{M}$-bimodule $E + \mathcal{M}$. □

We wish to point out the following special case of the above corollary.

Corollary 6. *If $(\mathcal{M},\tau)$ is any semi-finite von Neumann algebra, then*

$$\overline{\mathrm{co}}^{L_1+\mathcal{M}}\mathcal{U}(x)\cap\mathcal{Z}_{L_1+\mathcal{M}}\neq\emptyset$$

for all $x\in L_1(\tau)+\mathcal{M}$. This implies, in particular, that $\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}_{L_1+\mathcal{M}}\neq\emptyset$ for all $x\in L_1(\tau)+\mathcal{M}$.

Proof. It only needs to be observed that the norm in $L_1(\tau)$ is order continuous and that the embedding of $L_1(\tau)+\mathcal{M}$ into $S(\tau)$ is continuous for the measure topology, as $L_1(\tau)+\mathcal{M}$ is a symmetric space. □

It is of some interest to point out that the above result is, in a certain sense, the best possible in the general situation. This is illustrated by the following proposition.

Proposition 7. *Suppose that $(\mathcal{M},\tau)$ is a II_1 factor with $\tau(\mathbf{1})=1$. If $0\leq x\in S(\tau)$ and $x\notin L_1(\tau)$, then $\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}(S(\tau))=\emptyset$.*

Proof. As observed in the comments after Lemma 1, $\mathcal{Z}(\mathcal{M})=\mathbb{C}\mathbf{1}$ implies that also $\mathcal{Z}(S(\tau))=\mathbb{C}\mathbf{1}$. If

$$z\in\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}(S(\tau)),$$

then there exists a sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ such that $\pi_n x\overset{\mathcal{T}_m}{\to}z$. Note that $z\geq 0$. Given $\varepsilon>0$, there exists a projection $e\in\mathcal{M}$ such that $\tau(\mathbf{1}-e)\leq\varepsilon$ and, by passing to a subsequence of $\{\pi_n\}_{n=1}^{\infty}$ if necessary,

$$\|e(\pi_n x)e-eze\|_{B(\mathcal{H})}\to 0,\quad n\to\infty$$

(see e.g. [14]), which implies that $\tau(e(\pi_n x)e)\to\tau(eze)$. Therefore, there exists $n\in\mathbb{N}$ such that

$$\tau(eze)\geq\tau(e(\pi_n x)e)-\varepsilon.$$

Let $u_j\in U(\mathcal{M})$ and $0\leq\lambda_j\in\mathbb{R}$ $(1\leq j\leq K)$ be such that $\sum_{j=1}^{K}\lambda_j=1$ and

$$\pi_n x=\sum\nolimits_{j=1}^{K}\lambda_j u_j^* x u_j.$$

Using Lemma 8.2 of [11], it follows that

$$\tau\left(e\left(u_j^* x u_j\right)e\right)\geq\int_{\varepsilon}^{1}\mu\left(t;u_j^* x u_j\right)dt=\int_{\varepsilon}^{1}\mu(t;x)\,dt,\quad 1\leq j\leq K,$$

and hence,

$$\tau(e(\pi_n x)e)=\sum\nolimits_{j=1}^{K}\lambda_j\tau\left(e\left(u_j^* x u_j\right)e\right)\geq\int_{\varepsilon}^{1}\mu(t;x)\,dt.$$

Consequently,

$$\tau(z)\geq\tau(eze)\geq\int_{\varepsilon}^{1}\mu(t;x)\,dt-\varepsilon.$$

Since this holds for all $\varepsilon>0$ and $\int_0^1\mu(t;x)\,dt=\tau(x)=\infty$, this implies that $\tau(z)=\infty$. However, since $z\in\mathcal{Z}(S(\tau))=\mathbb{C}\mathbf{1}$, this is a contradiction. The proof is complete. □

In the next section we shall see that the situation in II_∞ factors is different.

4. The Dixmier approximation theorem for the measure topology

In the present section we will discuss the closures of the convex hulls of unitary orbits in $S(\tau)$ with respect to the measure topology. As is clear from Proposition 7, in order to obtain versions of Dixmier's approximation theorem in this setting, we need to impose some assumptions on the von Neumann algebra $\mathcal{M}$. Actually, most of the results in this section are of interest only for II_∞ factors (the only exclusion is Corollary 11, which retains some interest also for the I_∞ case).

Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra on a Hilbert space $\mathcal{H}$. Recall that if $\mathcal{M}$ is a factor, then $\mathcal{Z}(S(\tau)) = \mathbb{C}\mathbf{1}$.

Lemma 8. *Suppose that $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1}$ and $\tau(\mathbf{1}) = \infty$. If $x \in \mathcal{M} \cap S_0(\tau)$, then*

$$\overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x) \cap \mathcal{Z}(\mathcal{M}) = \{0\}.$$

Proof. It follows from Dixmier's approximation theorem that $\overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x) \cap \mathcal{Z}(\mathcal{M}) \neq \emptyset$. Given $z \in \overline{\mathrm{co}}^{\mathcal{M}}\mathcal{U}(x) \cap \mathcal{Z}(\mathcal{M})$, there is a sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ such that $\|\pi_n x - z\|_{B(\mathcal{H})} \to 0$ as $n \to \infty$. This implies that $\pi_n x \overset{\mathcal{T}_m}{\to} z$. Furthermore, $x \in S_0(\tau)$ and so $\pi_n x \in S_0(\tau)$ for all n. Since $S_0(\tau)$ is closed for the measure topology (see e.g. [3]), it follows that $z \in S_0(\tau)$. By hypothesis, $z = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{C}$. Since $\tau(\mathbf{1}) = \infty$, this implies that $\lambda = 0$, that is, $z = 0$. The proof is complete. $\square$

Theorem 9. *Suppose that $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1}$ and $\tau(\mathbf{1}) = \infty$. If $x \in S_0(\tau)$, then*

$$\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x) \cap \mathcal{Z}(\mathcal{M}) = \{0\}.$$

Proof. Let $x \in S_0(\tau)$ be fixed. Given $\varepsilon, \delta > 0$, let $\alpha > 0$ be such that $\tau\left(e^{|x|}(\alpha,\infty)\right) \leq \delta$ and define

$$e = \mathbf{1} - e^{|x|}(\alpha,\infty) = e^{|x|}[0,\alpha].$$

If $x = v|x|$ is the polar decomposition of x, then $xe = v|x|e^{|x|}[0,\alpha] \in \mathcal{M}$ and so, $exe \in e\mathcal{M}e$. Furthermore, $x \in S_0(\tau)$ implies that $exe \in S_0(e\mathcal{M}e,\tau)$. Since $\mathcal{Z}(e\mathcal{M}e) = e\mathcal{Z}(\mathcal{M})e = \mathbb{C}e$ and $\tau(e) = \infty$, it follows from Lemma 8 that there exist $v_1,\ldots,v_n \in U(e\mathcal{M}e)$ and $0 \leq \lambda_j \in \mathbb{R}$ with $\sum_{j=1}^n \lambda_j = 1$ such that

$$\left\|\sum\nolimits_{j=1}^n \lambda_j v_j^*(exe)v_j\right\|_{B(\mathcal{H})} \leq \varepsilon. \tag{3}$$

Note that $ev_j = v_je = v_j$ and $v_j^*v_j = v_jv_j^* = e$ for all j.

Define $u_j \in U(\mathcal{M})$ by setting

$$u_j = (\mathbf{1} - e) + v_j, \quad 1 \leq j \leq n,$$

and let $\pi \in \mathcal{D}$ be given by

$$\pi y = \sum\nolimits_{j=1}^n \lambda_j u_j^* y u_j, \quad y \in S(\tau).$$

For $1 \leq j \leq n$ we have

$$\begin{aligned} u_j^*(\mathbf{1}-e)xu_j &= (\mathbf{1}-e)xu_j, \quad u_j^*ex(\mathbf{1}-e)u_j = u_j^*ex(\mathbf{1}-e), \\ u_j^*exeu_j &= v_j^*(exe)v_j. \end{aligned}$$

Hence,

$$\pi((\mathbf{1}-e)x) = \sum\nolimits_{j=1}^n \lambda_j(\mathbf{1}-e)xu_j = (\mathbf{1}-e)x\sum\nolimits_{j=1}^n \lambda_j u_j.$$

Since $\tau(\mathbf{1}-e)=\tau\left(e^{|x|}(\alpha,\infty)\right)\leq\delta$, this shows that $\pi((\mathbf{1}-e)x)\in V(\varepsilon,\delta)$ (recall that the set $V(\varepsilon,\delta)$ is $*$-closed). Similarly, we find that

$$\begin{aligned}\pi(ex(\mathbf{1}-e)) &= \sum\nolimits_{j=1}^{n}\lambda_j u_j^* ex(\mathbf{1}-e)\\ &= \left(\sum\nolimits_{j=1}^{n}\lambda_j u_j^*\right)ex(\mathbf{1}-e),\end{aligned}$$

which shows that $\pi(ex(\mathbf{1}-e))\in V(\varepsilon,\delta)$, as $\tau(\mathbf{1}-e)\leq\delta$.

Furthermore, observe that

$$\pi(exe)=\sum\nolimits_{j=1}^{n}\lambda_j u_j^* exeu_j=\sum\nolimits_{j=1}^{n}\lambda_j v_j^*(exe)v_j.$$

Hence, it follows from (3) that

$$\|\pi(exe)\|_{B(\mathcal{H})}\leq\varepsilon,$$

and so $\pi(exe)\in V(\varepsilon,\delta)$. Collecting the above, we find that

$$\begin{aligned}\pi x &= \pi((\mathbf{1}-e)x)+\pi(ex(\mathbf{1}-e))+\pi(exe)\\ &\in V(\varepsilon,\delta)+V(\varepsilon,\delta)+V(\varepsilon,\delta)\subseteq V(3\varepsilon,3\delta).\end{aligned}$$

Since $\varepsilon,\delta>0$ are arbitrary, we may conclude that

$$0\in\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x).$$

On the other hand, if $z\in\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}(\mathcal{M})$, then there exists a sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ such that $\pi_n x\overset{\mathcal{T}_m}{\to}z$. Since $x\in S_0(\tau)$, it follows that $\pi_n x\in S_0(\tau)$ for all n. The set $S_0(\tau)$ being closed for the measure topology implies that $z\in S_0(\tau)$. However, $z=\lambda\mathbf{1}$ for some $\lambda\in\mathbb{C}$ and $\tau(\mathbf{1})=\infty$, and so $\lambda=0$. We may conclude that $\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}(\mathcal{M})=\{0\}$. □

Using the above theorem, we may now obtain the following result for arbitrary $x\in S(\tau)$.

Theorem 10. *Suppose that $\mathcal{Z}(\mathcal{M})=\mathbb{C}\mathbf{1}$ and $\tau(\mathbf{1})=\infty$. If $x\in S(\tau)$, then*

$$\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}(\mathcal{M})\neq\emptyset.$$

Proof. Given $x\in S(\tau)$, there exists $0<s_0\in\mathbb{R}$ such that $\tau\left(e^{|x|}(s_0,\infty)\right)<\infty$. Define $x_1=xe^{|x|}(s_0,\infty)$ and $x_2=xe^{|x|}[0,s_0]$. Since $\mu\left(|x|e^{|x|}(s,\infty)\right)=\mu(x)\chi_{[0,t_0)}$, with $t_0=\tau\left(e^{|x|}(s_0,\infty)\right)$, it follows that $x_1\in S_0(\tau)$. It is also clear that $x_2\in\mathcal{M}$.

By Dixmier's approximation theorem, there exist $z\in\mathcal{Z}(\mathcal{M})$ and a sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ such that $\|\pi_n x_2-z\|_{B(\mathcal{H})}\leq 2^{-n}$ for all $n\in\mathbb{N}$. Since $x_1\in S_0(\tau)$ implies that $\pi_n x_1\in S_0(\tau)$, it follows from Theorem 9 that there exists a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ such that $\sigma_n\pi_n x_1\in V(2^{-n},2^{-n})$ for all n. Since

$$\|\sigma_n\pi_n x_2-z\|_{B(\mathcal{H})}=\|\sigma_n(\pi_n x_2-z)\|_{B(\mathcal{H})}\leq\|\pi_n x_2-z\|_{B(\mathcal{H})}\leq 2^{-n},$$

it follows that $\sigma_n\pi_n x_2-z\in V(2^{-n},2^{-n})$, $n\geq 1$. As a consequence,

$$\sigma_n\pi_n x-z=\sigma_n\pi_n x_1+(\sigma_n\pi_n x_2-z)\in V\left(2^{-n+1},2^{-n+1}\right)$$

for all $n\in\mathbb{N}$. Hence, $\sigma_n\pi_n x\overset{\mathcal{T}_m}{\to}z$, which completes the proof. □

We end this paper with an application of Theorem 9 to a certain class of symmetric spaces. We consider the case that $E(0,\infty)$ is a symmetric Banach function space on $(0,\infty)$ with order continuous norm (see e.g. [12]). Given a semi-finite von Neumann algebra $(\mathcal{M},\tau)$, the corresponding non-commutative symmetric space is defined by

$$E(\tau)=\{x\in S(\tau):\mu(x)\in E(0,\infty)\},$$

equipped with the norm given by $\|x\|_{E(\tau)}=\|\mu(x)\|_{E(0,\infty)}$, $x\in E(\tau)$ (see [2], [11]). It should be observed that the space $E(\tau)$ is actually fully symmetric; that is, if $x\in S(\tau)$, $y\in E(\tau)$ and $x\prec\prec y$, then $x\in E(\tau)$ and $\|x\|_{E(\tau)}\leq\|y\|_{E(\tau)}$. Here, $x\prec\prec y$ denotes submajorization, that is,

$$\int_0^t\mu(s;x)\,ds\leq\int_0^t\mu(s;y)\,ds,\quad t\geq 0.$$

The Köthe dual of $E(0,\infty)$ will be denoted by $E^{\times}(0,\infty)$ (see e.g. [3]). It should be observed that the condition $E^{\times}(0,\infty)\subseteq S_0(0,\infty)$ is equivalent to saying that $E(0,\infty)$ is not contained in $L_1(0,\infty)$.

Corollary 11. *Suppose that $\mathcal{Z}(\mathcal{M})=\mathbb{C}\mathbf{1}$ and $\tau(\mathbf{1})=\infty$. If $E(0,\infty)$ is a symmetric Banach function space with order continuous norm and $E^{\times}(0,\infty)\subseteq S_0(0,\infty)$, then*

$$\overline{\mathrm{co}}^{E(\tau)}\mathcal{U}(x)\cap\mathcal{Z}(\mathcal{M})=\{0\}$$

for all $x\in E(\tau)$.

Proof. It should be observed first that if $\pi\in\mathcal{D}$, then $\pi x\prec\prec x$ for all $x\in S(\tau)$. Indeed, if $u_j\in U(\mathcal{M})$ and $0\leq\lambda_j\in\mathbb{R}$ $(1\leq j\leq n)$ are such that $\pi x=\sum_{j=1}^n\lambda_j u_j^*xu_j$, then

$$\mu(\pi x)\prec\prec\sum\nolimits_{j=1}^n\mu\left(\lambda_j u_j^*xu_j\right)=\sum\nolimits_{j=1}^n\lambda_j\mu(x)=\mu(x).$$

The order continuity of the norm of $E(0,\infty)$ implies that $E(0,\infty)\subseteq S_0(0,\infty)$ (see [3]) and hence, $E(\tau)\subseteq S_0(\tau)$. Given $x\in E(\tau)$, it follows from Theorem 9 that there exists a sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ such that $\pi_n x\overset{\mathcal{T}_m}{\to}0$ as $n\to\infty$. Since $\pi_n x\prec\prec x$ for all n and $E^{\times}(0,\infty)\subseteq S_0(0,\infty)$, it now follows from Proposition 2.2 in [4] that $\|\pi_n x\|_{E(\tau)}\to 0$ as $n\to\infty$. This shows that $0\in\overline{\mathrm{co}}^{E(\tau)}\mathcal{U}(x)$.

On the other hand, since the embedding of $E(\tau)$ into $S(\tau)$ is continuous for the measure topology it follows that

$$\overline{\mathrm{co}}^{E(\tau)}\mathcal{U}(x)\cap\mathcal{Z}(\mathcal{M})\subseteq\overline{\mathrm{co}}^{\mathcal{T}_m}\mathcal{U}(x)\cap\mathcal{Z}(\mathcal{M})=\{0\}.$$

The proof is complete. □

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