PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 3, March 2013, Pages 779–789
S 0002-9939(2012)11626-0
Article electronically published on July 12, 2012

COUNTING EQUIVALENCE CLASSES OF VERTEX PAIRS MODULO THE DIHEDRAL ACTION ON THE ASSOCIAHEDRON

DOUGLAS BOWMAN AND ALON REGEV

(Communicated by Jim Haglund)

Abstract. This paper proves explicit formulae for the number of edges, 2-sets and diagonals in the associahedron of dimension n modulo the action of the dihedral group. A generating function for the number of k-sets modulo this action, as well as a formula for the cycle index, is given. A table of values is also provided.

1. Introduction

The distance between two vertices on the associahedron is a function of the symmetry class of the vertices. Such distances are related to the recent conjecture concerning the diameter of these polytopes (see, for example, [5, 1, 13]). This paper enumerates the symmetry classes of various kinds of vertex pairs. Recall that the vertices of P_m, the associahedron of dimension m, can be thought of as the binary bracketings of a word with $m+2$ letters [5, 7, 8]. It is well known that the number of such bracketings is the Catalan number C_{m+1}, where

$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$

Edges are formed between two vertices of P_m if an application of the simple associativity law transforms one bracketing into another. For counting lower-dimensional facets and associativity laws on the associahedron (which are space diagonals, i.e. pairs of vertices that do not belong to any $(m-1)$-dimensional face), see [7, 8].

The associahedron P_m has a dihedral symmetry group. This dihedral structure is conveniently understood by considering the vertices of P_m as triangulations of regular n-gons, where $n = m+3$. Let G_n be the set of all triangulations of a regular n-gon. The dihedral action on a regular n-gon induces an action on G_n equivalent to the dihedral action on P_{n-3}. This in turns induces an action on the set of k-sets of vertices, as well as an action on its set of edges. One of the main results of this paper is the following explicit formula for the number of edges of P_m modulo the dihedral action. Note that throughout this paper, we take $C_n = 0$ when n is not a nonnegative integer.

Received by the editors February 17, 2011 and, in revised form, July 17, 2011.
2010 *Mathematics Subject Classification.* Primary 05C30, 32B25, 52B11, 52B15, 52B05, 05E18.

Theorem 1.1. *Let $g^{(e)}(n)$ be the number of edges of the associahedron P_{n-3} modulo the dihedral action. Then*

$$g^{(e)}(n) = \begin{cases} (\frac{1}{4} - \frac{3}{4n})C_{n-2} + \frac{3}{8}C_{n/2-1} + (1 - \frac{3}{n})C_{n/2-2} + \frac{1}{4}C_{n/4-1}, & \text{if } n \text{ is even,} \\ (\frac{1}{4} - \frac{3}{4n})C_{n-2} + \frac{1}{4}C_{(n-3)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

This sequence $g^{(e)}(n)$ appears in a paper [12] by Read, where it occurs in an array enumerating elements in a class of cellular structures. (There is a natural bijection between these structures and the edges of an associahedron modulo the dihedral action.) This array was later studied by Lisonek in [9]. Lisonek proved that the rows of Read's table are given by polynomials, which he showed how to recursively determine. However, the sequence $g^{(e)}(n)$ occurs as a diagonal in Read's table, so no explicit formula for $g^{(e)}$ follows from Lisonek's work. In a future paper we generalize these results by enumerating the facets of various dimensions of asscia hedra modulo the cyclic and dihedral actions, extending the work of the present paper.

Pairs of distinct vertices that are not edges correspond to paths between vertices of distance at least 2. We call such pairs *diagonals* of P_m. Diagonals fall into equivalence classes under the dihedral action. A further result of the paper is the following explicit formula for the number of diagonals modulo the dihedral action.

Theorem 1.2. *Let $d(n)$ be the number of diagonals modulo the dihedral action. Then*

$$d(n) = \begin{cases} \frac{1}{2n}\Big(\frac{1}{2}C_{n-2}^2 + \frac{3}{2}C_{n-2} + (\frac{n^2}{8} + n)C_{n/2-1}^2 - \frac{7n}{4}C_{n/2-1} \\ \qquad + \frac{n^2}{9}C_{n/3-1}^2 - (2n-6)C_{n/2-2} - \frac{n}{2}C_{n/4-1}\Big), & \text{if } 4|n \\ \\ \frac{1}{2n}\Big(\frac{1}{2}C_{n-2}^2 + \frac{3}{2}C_{n-2} + (\frac{n^2}{8} + n)C_{n/2-1}^2 - \frac{9n}{4}C_{n/2-1} \\ \qquad + \frac{n^2}{9}C_{n/3-1}^2 - (2n-6)C_{n/2-2} - \frac{n}{2}C_{n/4-1}\Big), & \text{if } 4|(n-2) \\ \\ \frac{1}{2n}\left(\frac{1}{2}C_{n-2}^2 + C_{n-2} + \frac{n}{2}C_{(n-3)/2}^2 - \frac{3n}{2}C_{(n-3)/2} + \frac{n^2}{9}C_{n/3-1}^2\right), & \text{if } 2 \nmid n. \end{cases}$$

This theorem is built on Theorem 1.1 as well as Theorem 4.3 below. The latter gives the number of 2-sets of triangulations (equivalently, 2-sets of vertices of the associahedron) modulo the dihedral action. More generally, in Theorem 4.2 a generating function for the number of k-sets of triangulations modulo the dihedral action is given. This is based on Proposition 4.1, which gives the cycle index for the dihedral action on G_n. This in turn is based on the elegant paper of Moon and Moser [10].

Enumeration of various forms of polygonal dissections have been considered by many authors; see for example [2, 7, 8, 9, 10, 12]. Brown [3] contains an extensive bibliography on this subject up to 1965.

2. Notation and known results

Throughout this paper, the greatest common divisor of the integers m and n is denoted by (m, n). Let D_{2n} be the dihedral group of order $2n$. The elements of D_{2n} are denoted using ε for the identity element, ρ for rotation by $2\pi/n$, and τ for reflection. In referring to the action of D_{2n} on an n-gon, we choose the convention

that τ fixes one of the vertices of the n-gon (it follows that $\tau\rho^i$ fixes one of the edges if and only if either i is even or n is odd).

Let H be a finite group acting on the set A. The following lemma is a well-known result attributed to Cauchy and Frobenius.

Lemma 2.1 ([4]). *The number $|A/H|$ of orbits of A under the action of H is given by*

$$|A/H| = \frac{1}{|H|}\sum_{h\in H}|\{a \in A | h(a) = a\}|.$$

This lemma will be used to prove Theorem 1.1, which gives a formula for the number of edges of the associahedron modulo the dihedral action. Pólya's Enumeration Theorem will be employed; the following definition is required.

Definition 2.2. Let the group H act on the set A. For $h \in H$, the *cycle index monomial* of h is the product

$$\pi(h; S_1, \ldots, S_{|A|}) = \prod_{k=1}^{|A|} S_k^{j_k(h)}.$$

The *cycle index* is defined by

$$Z(H(A); S_1, \ldots, S_{|A|}) = \frac{1}{|H|}\sum_{h\in H}\prod_{k=1}^{|A|} S_k^{j_k(h)},$$

where $j_k(h)$ is the number of cycles of length k in the disjoint cycle decomposition of h as a permutation of A.

For any $\delta \in D_{2n}$, let $N(\delta) = |\{\Delta \in G_n | \delta(\Delta) = \Delta\}|$. The following result of Moon and Moser will be used to calculate the cycle index of the dihedral action on the set of triangulations.

Proposition 2.3 ([10]).

(1) $N(\varepsilon) = C_{n-2}$.
(2) If $3|n$, then $N(\rho^{n/3}) = N(\rho^{2n/3}) = \frac{n}{3}C_{n/3-1}$.
(3) If n is even, then $N(\rho^{n/2}) = \frac{n}{2}C_{n/2-1}$.
(4) If n and i are even with $0 \le i \le n-2$, then $N(\tau\rho^i) = 2C_{n/2-1}$.
(5) If n is odd and $0 \le i \le n-1$, then $N(\tau\rho^i) = C_{(n-3)/2}$.
(6) $N(\delta) = 0$ for all other $\delta \in D_{2n}$.

Combining Proposition 2.3 with Lemma 2.1, Moon and Moser obtained the following theorem.

Theorem 2.4 ([10]). *The number $g(n)$ of triangulations of an n-gon (equivalently, the number of vertices of the associahedron P_{n-3}) modulo the dihedral action is given by*

$$g(n) = \begin{cases} \frac{1}{2n}C_{n-2} + \frac{1}{3}C_{n/3-1} + \frac{3}{4}C_{n/2-1}, & \text{if } n \text{ is even,} \\ \frac{1}{2n}C_{n-2} + \frac{1}{3}C_{n/3-1} + \frac{1}{2}C_{(n-3)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

3. Edges

Triangulations $\Delta_1, \Delta_2 \in G_n$ are said to be *adjacent* if their corresponding vertices are adjacent on P_{n-3}. It is well known (see, for example, [5]) that this is equivalent to one triangulation being obtained from the other by a single "flip" of a diagonal in its associated quadrilateral. Equivalently, each edge of P_{n-3} corresponds to a dissection of an n-gon into $n-4$ triangles and one quadrilateral. Following [5], we call such a dissection an *almost-triangulation*. Let $G_n^{(e)}$ be the set of all almost-triangulations. Since each vertex of the associahedron P_{n-3} has degree $n-3$, the total number of edges is $|G_n^{(e)}| = \frac{n-3}{2}C_{n-2}$.

Using Lemma 2.1 and Proposition 3.1 below, an explicit formula for the number of $g^{(e)}(n)$ of edges of P_{n-3} modulo the dihedral action is obtained. Note that these values have been computed in [12], but no explicit formula was given.

Proposition 3.1. Let $N^{(e)}(\delta)$ be the number of almost-triangulations $\Phi \in G_n^{(e)}$ fixed by $\delta \in D_{2n}$. Then:

(1) $N^{(e)}(\varepsilon) = \frac{n-3}{2}C_{n-2}$.
(2) If $4|n$, then $N^{(e)}(\rho^{n/4}) = N^{(e)}(\rho^{3n/4}) = \frac{n}{2}C_{n/4-1}$.
(3) If n is even, then $N^{(e)}(\rho^{n/2}) = \frac{n}{4}C_{n/2-1}$.
(4) If n and i are even with $0 \le i \le n-2$, then $N^{(e)}(\tau\rho^i) = \frac{n}{2}C_{n/2-1} + \frac{3(n-4)}{2}C_{n/2-2}$.
(5) If n is even and i is odd with $0 \le i \le n-1$, then $N^{(e)}(\tau\rho) = C_{n/2-2}$.
(6) If n is odd and $0 \le i \le n-1$, then $N^{(e)}(\tau) = C_{(n-3)/2}$.
(7) $N^{(e)}(\delta) = 0$ for all other $\delta \in D_{2n}$.

The following lemma is used in the proof of Proposition 3.1.

Lemma 3.2. *For any $n, m \ge 0$,*

$$\sum_{\substack{i_1+\ldots+i_m=n \\ i_1,\ldots,i_m\ge 0}} C_{i_1}\cdots C_{i_m} = \begin{cases} \frac{m(n+1)(n+2)\cdots(n+\frac{m}{2}-1)}{2(n+\frac{m}{2}+2)(n+\frac{m}{2}+3)\cdots(n+m)}C_{n+m/2}, & \text{if } m \text{ is even} \\ \\ \frac{m(n+1)(n+2)\cdots(n+\frac{m-1}{2})}{(n+\frac{m+3}{2})(n+\frac{m+3}{2}+1)\cdots(n+m)}C_{n+(m-1)/2}, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. This follows immediately from [14, (2.5.16), page 54]. □

Proof of Proposition 3.1. Let $\Phi \in G_n^{(e)}$ be an almost-triangulation and let Q be its quadrilateral part. Note that if $\delta \in D_{2n}$ fixes Φ, then in particular it fixes Q. This proves part 7, since a quadrilateral can only be fixed by ε, $\rho^{n/4}$, $\rho^{n/2}$ or $\rho^{3n/4}$, or by reflections. Part 1 is clear since ε fixes all $\Phi \in G_n^{(e)}$.

2. If Q is fixed by rotation by $\rho^{n/4}$ or $\rho^{3n/4}$ (rotations by $\pi/2$), then it must be a square. Thus Φ is determined by choosing one of the $n/4$ possible orientations of Q, and then by choosing one of the $C_{n/4-1}$ triangulations of the $(n/4+1)$-gon formed by one side of Q and $n/4$ consecutive sides of the n-gon (see Figure 1).

3. If Q is fixed by $\rho^{n/2}$, then each of the diagonals of Q must be fixed by $\rho^{n/2}$; hence this diagonal is a diameter of the n-gon. Thus Φ is determined by choosing one of the $n/2$ possible diameters and by choosing one of the $C_{n/2-1}$ triangulations of the $(n/2+1)$-gon on one side of the diameter. The number of choices is divided

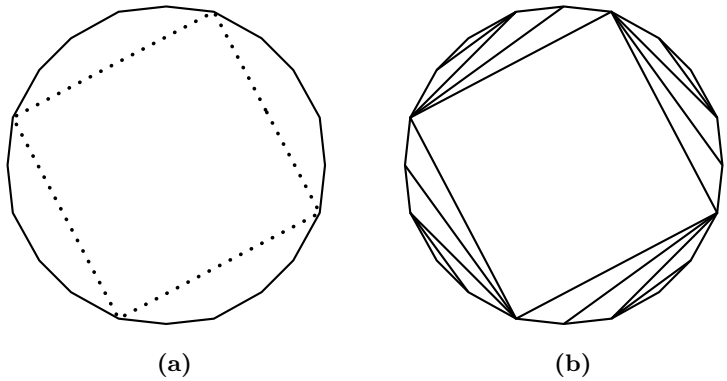


FIGURE 1. Counting the almost-triangulations of a 20-gon fixed by $\rho^{20/4}$. (a) One of the 5 possible positions of the quadrilateral Q, which is denoted by a dotted line. (b) One of the 14 possible almost-triangulations which include this quadrilateral Q.

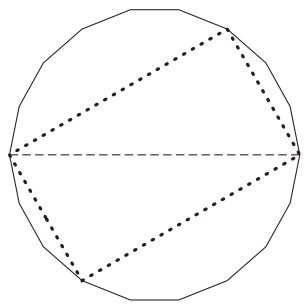
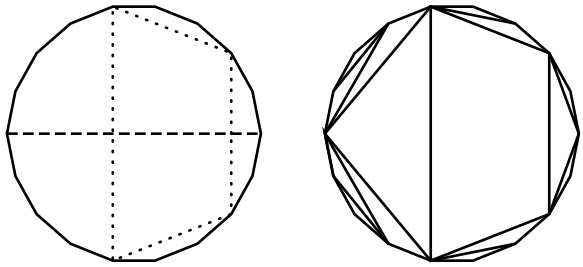


FIGURE 2. Counting the almost-triangulations of an 18-gon fixed by $\rho^{18/2}$. The dashed line is the axis of symmetry, and the dotted line is the quadrilateral Q.

by two since each quadrilateral contains two diameters. Note that in this case Q is determined by the choice of the triangulation of the $(n/2+1)$-gon. (See Figure 2; note that in all the figures, a dashed line represents the axis of symmetry and is not part of the almost-triangulation.)

4. Suppose that Q is fixed by $\tau\rho^m$, where m is even. Then the axis of reflection is a diameter of the polygon connecting opposite vertices. In order for Q to be symmetric about this axis, it must either be an equilateral trapezoid with its bases perpendicular to the diameter or a deltoid with one of its diagonals coinciding with the diameter.

In the first case, Φ is determined by choosing the position of the trapezoid and by choosing three triangulations, one which includes its side and one adjacent to each base. The position of the trapezoid is determined by choosing positive integers i, j, k with $i+j+k=n/2$. Thus, Φ is determined by choosing a triangulation of the $(i+1)$-gon, the $(j+1)$-gon and the $(k+1)$-gon that are formed.



(a) (b)

FIGURE 3. Counting the almost-triangulations of an 18-gon fixed by $\tau\rho^m$, m even. (a) The dashed line is an axis of reflection, and the dotted line is the quadrilateral Q. This example corresponds to the indices $i = 4, j = 3, k = 2$ in (3). (b) One of the $14 \times 2 \times 1$ almost-triangulations which include this quadrilateral Q.

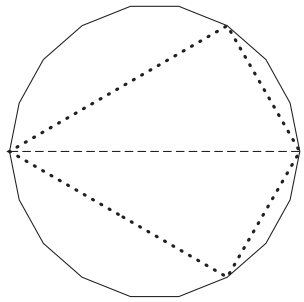


FIGURE 4. Counting the almost-triangulations of an 18-gon fixed by $\tau\rho^m$, m even. The dashed line is the axis of reflection, and the dotted line is the quadrilateral Q. This example corresponds to $i = 6$ and $j = 3$.

Therefore the number of almost-triangulations of this form is

$$\sum_{\substack{i+j+k=\frac{n}{2} \\ i,j,k\geq 1}} C_{i-1}C_{j-1}C_{k-1} = \frac{3(n-4)}{n} C_{n/2-2},$$

which follows from Lemma 3.2, with $m = 3$ and n replaced by $n/2-3$ (see Figure 3).

In the second case, Φ is determined by choosing i, j with $i+j = n/2$ to determine the position of the deltoid Q and then by choosing the triangulations of the resulting $(i+1)$-gon and $(j+1)$-gon. By Lemma 3.2, this gives

$$\sum_{i+j=\frac{n}{2}} C_{i-1}C_{j-1} = C_{n/2-1}$$

almost-triangulations (see Figure 4).

5. In this case, the axis of reflection connects the midpoints of opposite sides of the polygon. Therefore Q must be a rectangle connecting those two sides. Then Φ is determined by choosing one of the $C_{n/2-2}$ triangulations of the $n/2$-gon on one side (see Figure 5).

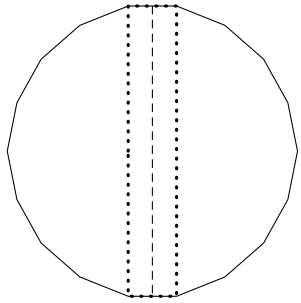


FIGURE 5. Counting the almost-triangulations of an 18-gon fixed by $\tau\rho^m$, m odd. The dashed line is an axis of reflection, and the dotted line is the quadrilateral Q.

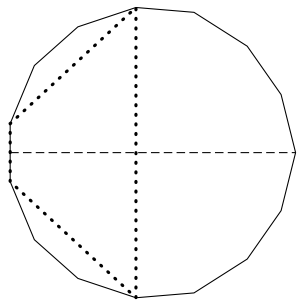


FIGURE 6. Counting the almost-triangulations of a 15-gon fixed by $\tau\rho^m$. The dashed line is an axis of symmetry, and the dotted line is the quadrilateral Q. This example corresponds to $i = 3$ and $j = 4$.

6. In this case, the axis of reflection connects the midpoint of a side of the n-gon with an opposite vertex. Therefore Q must be a trapezoid with one of its bases coinciding with this side. Thus Φ is determined by choosing i, j with $i+j = (n-1)/2$ to determine the position of Q and by choosing the $C_{i/2-1}$ and $C_{j/2-1}$ triangulations of the resulting $(i+1)$-gon and $(j+1)$-gon on one side of the axis. By Lemma 3.2, this gives

$$\sum_{i+j=\frac{n-1}{2}} C_{i-1}C_{j-1} = C_{(n-3)/2}$$

almost-triangulations (see Figure 6). □

Proof of Theorem 1.1. By Lemma 2.1,

$$g^{(e)}(n) = \frac{1}{2n} \sum_{\delta \in D_{2n}} N^{(e)}(\delta).$$

The result then follows from Proposition 3.1. □

4. The number of k-sets modulo the dihedral action

In this section, the result of Theorem 2.4 is generalized to k-sets of triangulations by applying Pólya's Enumeration Theorem [11] to the action of the dihedral group D_{2n} on the set G_n of triangulations of an n-gon.

Proposition 4.1. The cycle index for the action of the dihedral group D_{2n} on the set G_n of triangulations of an n-gon is given for even n by:

$$\begin{aligned}
&Z(D_{2n}(G_n); S_1, \dots, S_n)\\
&= \frac{1}{2n} \sum_{\substack{0\le i\le n-2\\ i \text{ even}}} S_{n/(i,n)}^{(i,n)\left(C_{n-2}-\frac{n}{2}C_{n/2-1}-\frac{n}{3}C_{n/3-1}\right)/n} S_{n/(2i,n)}^{(i,n/2)C_{n/2-1}} S_{n/(3i,n)}^{(i,n/3)C_{n/3-1}}\\
&+ \frac{1}{2n} \sum_{\substack{1\le i\le n-1\\ i \text{ odd}}} S_{n/(i,n)}^{(i,n)\left(C_{n-2}-\frac{n}{2}C_{n/2-1}-\frac{n}{3}C_{n/3-1}\right)/n} S_{n/(2i,n)}^{(i,n/2)C_{n/2-1}} S_{n/(3i,n)}^{(i,n/3)C_{n/3-1}}\\
&\qquad + \frac{1}{2} S_1^{nC_{n/2-1}/2} S_2^{(C_{n-2}-nC_{n/2-1})/2} + \frac{1}{2} S_2^{C_{n-2}/2},
\end{aligned}$$

and for odd n by:

$$\begin{aligned}
&Z(D_{2n}(G_n); S_1, \dots, S_n)\\
&= \frac{1}{2n} \sum_{i=0}^{n-1} S_{n/(i,n)}^{(i,n)\left(C_{n-2}-\frac{n}{3}C_{n/3-1}\right)/n} S_{n/(3i,n)}^{(i,n/3)C_{n/3-1}} + \frac{1}{2} S_1^{C_{(n-3)/2}} S_2^{(C_{n-2}-C_{(n-3)/2})/2}.
\end{aligned}$$

Proof. Note first that all elements of D_{2n} have order at most n, so that the indeterminates $S_{n+1}, \dots, S_{|G_n|}$ can indeed be omitted. It follows from Proposition 2.3 that the cycle index monomial of ρ is

$$\pi(\rho; S_1, \dots, S_n) = S_n^{(C_{n-2}-N(\rho^{n/2})-N(\rho^{n/3}))/n} S_{n/2}^{N(\rho^{n/2})} S_{n/3}^{N(\rho^{n/3})}.$$

The expressions for $\pi(\rho^i; S_1, \dots, S_n)$ follow. Also by Proposition 2.3, the cycle index monomial of $\tau\rho^i$ for all $i = 0, \dots, n-1$ is given by

$$\pi(\tau\rho^i; S_1, \dots, S_n) = S_1^{N(\tau\rho^i)} S_2^{(C_{n-2}-N(\tau\rho^i))/2},$$

yielding the result. □

Theorem 4.2. *The number of k-sets of triangulations of an n-gon modulo the dihedral action is equal to the coefficient of X^k in the polynomial given for even n by:*

$$\begin{aligned}
&Z(D_{2n}(G_n); 1+X, \dots, 1+X^n)\\
&\quad= \frac{1}{2n} \sum_{\substack{0\le i\le n-2\\ i \text{ even}}} (1+X^{n/(i,n)})^{(i,n)\left(C_{n-2}-\frac{n}{2}C_{n/2-1}-\frac{n}{3}C_{n/3-1}\right)/n}\\
&\qquad\qquad \times (1+X^{n/(i,n)})^{(i,n/2)C_{n/2-1}} (1+X^{n/(3i,n)})^{(i,n/3)C_{n/3-1}}\\
&\quad+ \frac{1}{2n} \sum_{\substack{1\le i\le n-1\\ i \text{ odd}}} (1+X^{n/(i,n)})^{(i,n)\left(C_{n-2}-\frac{n}{2}C_{n/2-1}-\frac{n}{3}C_{n/3-1}\right)/n}\\
&\qquad\qquad \times (1+X^{n/(2i,n)})^{(i,n/2)C_{n/2-1}} (1+X^{n/(3i,n)})^{(i,n/3)C_{n/3-1}}\\
&\quad+ \frac{1}{2}(1+X)^{nC_{n/2-1}/2} (1+X^2)^{(C_{n-2}-nC_{n/2-1})/2} + \frac{1}{2}(1+X^2)^{C_{n-2}/2},
\end{aligned}$$

and for odd n by:

(4.1)

$$Z(D_{2n}(G_n); 1+X, \dots, 1+X^n)$$
$$= \frac{1}{2n}\sum_{i=0}^{n-1}((1+X)^{n/(i,n)})^{(i,n)\left(C_{n-2}-\frac{n}{3}C_{n/3-1}\right)/n}((1+X)^{n/(3i,n)})^{(i,n/3)C_{n/3-1}}$$
$$+\frac{1}{2}(1+X)^{C_{(n-3)/2}}(1+X^2)^{(C_{n-2}-C_{(n-3)/2})/2}.$$

Proof. This follows from Proposition 4.1 and [6], Corollary 2.5.1. □

Note that the coefficient of X in (4.1) agrees with the values of $g(n)$ given by Theorem 2.4, as expected. Extracting the coefficient of X^2 in (4.1) yields the following result.

Theorem 4.3. (1) *The number of 2-sets $\{\Delta_1, \Delta_2\} \subset G_n$ (with $\Delta_1 \neq \Delta_2$) of triangulations of an n-gon modulo the dihedral action is*

(4.2)

$$g^{(2)}(n) = \begin{cases} \frac{1}{2n}\big(\frac{1}{2}C_{n-2}^2 + \frac{n}{2}C_{n-2} \\ \qquad +(\frac{n^2}{8}+n)C_{n/2-1}^2 - nC_{n/2-1} + \frac{n^2}{9}C_{n/3-1}^2\big) & \text{if } 4|n, \\ \frac{1}{2n}\big(\frac{1}{2}C_{n-2}^2 + \frac{n}{2}C_{n-2} \\ \qquad +(\frac{n^2}{8}+n)C_{n/2-1}^2 - \frac{3n}{2}C_{n/2-1} + \frac{n^2}{9}C_{n/3-1}^2\big) & \text{if } 4|(n-2), \\ \frac{1}{2n}\big(\frac{1}{2}C_{n-2}^2 + \frac{n-1}{2}C_{n-2} + \frac{n}{2}C_{(n-3)/2}^2 - nC_{(n-3)/2} \\ \qquad +\frac{n^2}{9}C_{n/3-1}^2 - \frac{n}{3}C_{n/3-1}\big) & \text{if } 2 \nmid n. \end{cases}$$

(2) *The number $g^{(\leq 2)}(n)$ of unordered pairs of triangulations of an n-gon modulo the dihedral action is*

$$g^{(\leq 2)}(n) = g^{(2)}(n) + g(n),$$

with $g^{(2)}(n)$ as in (4.2) and $g(n)$ as in Theorem 2.4.

Proof.

(1) Suppose that $4|n$, and let $y_i = \pi(\rho^i; 1+X, 1+X^2, \dots, 1+X^n)$. Then the only terms y_i that may contribute to the coefficient of X^2 are y_0, $y_{n/2}$, $y_{n/4}$, $y_{3n/4}$, $y_{n/3}$, $y_{2n/3}$, $y_{n/6}$ and $y_{5n/6}$.

Since $y_0 = (1+X)^{C_{n-2}}$, its X^2-coefficient is $\binom{C_{n-2}}{2}$. Next,

$$y_{n/2} = (1+X^2)^{\left(C_{n-2}-\frac{n}{2}C_{n/2-1}-\frac{n}{3}C_{n/3-1}\right)/2}(1+X)^{\frac{n}{2}C_{n/2-1}}(1+X^2)^{\frac{n}{6}C_{n/3-1}}$$
$$= (1+X^2)^{\left(C_{n-2}/2-\frac{n}{4}C_{n/2-1}\right)}(1+X)^{\frac{n}{2}C_{n/2-1}},$$

so its X^2-coefficient is $\left(\frac{1}{2}C_{n-2} - \frac{n}{4}C_{n/2-1}\right) + \binom{\frac{n}{2}C_{n/2-1}}{2}$.

For $i = n/4$, note that $(1+X^2)^{\frac{n}{4}C_{n/2-1}}$ are the only factors in $y_{n/4}$ contributing to the coefficient of X^2; hence this coefficient is $\frac{n}{4}C_{n/2-1}$. The X^2-coefficients of the other y_i are computed similarly.

Now let $z_i = \pi(\tau\rho^m; 1+X, \dots, 1+X^n)$. Since n is even, $N(\tau\rho^i) = 2C_{n/2-1}$ if i is even, and $N(\tau\rho^i) = 0$ otherwise. If i is odd, then $z_i = (1+X^2)^{C_{n-2}/2}$, which has an X^2-coefficient $C_{n-2}/2$. For i even,

$$z_i = (1+X)^{2C_{n/2-1}}(1+X^2)^{(C_{n-2}-2C_{n/2-1})/2},$$

which has an X^2-coefficient $\binom{2C_{n/2-1}}{2} + (C_{n-2} - 2C_{n/2-1})/2$. The remaining cases are computed similarly. By Theorem 4.2, $g^{(2)}(n)$ is the sum of the X^2-coefficients of all the y_i and z_i, divided by $2n$.

(2) This follows at once since there are $g^{(2)}(n)$ distinct unordered pairs and $g(n)$ nondistinct unordered pairs. □

The proof of Theorem 1.2 now follows.

Proof of Theorem 1.2. The result is obtained by subtracting the values $g_e(n)$ found in Theorem 1.1 from the values $g^{(2)}(n)$ found in Theorem 4.3. □

Note that for any $k > 2$, the coefficients of X^k in (4.1) can be extracted by a similar method as the one used in Theorem 4.2. However, such a calculation becomes tedious.

5. Table of values

We conclude this paper by providing a table of values for the counting sequences studied. Note that the values of $g^{(e)}(n)$ up to $n = 15$ agree with diagonal values given by Read in table 5 of [12]. The values in Table 1 were computed in Matlab using the formulas obtained in this paper. A number of initial values were independently verified by using lists, table structures and relations in computer memory.

Table 1. Values of $g^{(e)}$, $g^{(2)}$, $g^{(\leq 2)}$ and d, giving the number of edges, 2-sets of vertices, unordered pairs of vertices and diagonal paths, respectively, modulo the dihedral group, for the associahedron of dimension $n-3$.

n	$g^{(e)}(n)$	$g^{(2)}(n)$	$g^{(\leq 2)}(n)$	$d(n)$
3	0	0	1	0
4	1	1	2	0
5	1	2	3	1
6	3	14	17	11
7	7	72	76	65
8	24	600	612	576
9	74	5213	5240	5139
10	259	51690	51772	51431
11	891	538398	538626	537507
12	3176	5883600	5884333	5880424
13	11326	66471559	66473841	66460233
14	40942	772736436	772743964	772695494
15	148646	9198517961	9198542795	9198369315
16	543515	111760777482	111760861380	111760233967
17	1996212	1382208497142	1382208782499	1382206500930
18	7367075	17363412549584	221154923761750	17363405182509
19	27294355	221154920349330	221154923761750	221154893054975
20	101501266	2851734257496696	2851734269441310	2851734155995430

References

1. L. Bowen, J. A. De Loera, M. Develin, and F. Santos, The Gromov norm of the product of two surfaces, Topology 44 (2005), no. 2, 321–339. MR2114710 (2005h:57023)
2. W. G. Brown, Enumeration of triangulations of the disk, Proc. London Math. Soc. (3) 14 (1964) 746-768. MR0168485 (29:5747)
3. W. G. Brown, Historical note on a recurrent combinatorial problem, Amer. Math. Monthly 72 (1965), 973–977. MR1533479
4. W. Burnside, Theory of Groups of Finite Order, 2nd ed., Cambridge University Press, Cambridge, 1911, p. 191.
5. J. A. De Loera, J. Rambau, and F. Santos, Triangulations: Structures for Algorithms and Applications, Springer-Verlag, Berlin, 2010. MR2743368 (2011j:52037)
6. F. Harary and E. M. Palmer, Graphical Enumeration, Academic Press, NY, 1973. MR0357214 (50:9682)
7. D. Huguet and D. Tamari, La structure polyédrale des complexes de parenthésages, J. Combin. Inform. System Sci. 3 (1978) 69-81. MR538214 (80h:05016)
8. C. Lee, The associahedron and triangulations of the n-gon, European J. Combin. 10 (1989), no. 6, 551-560. MR1022776 (90i:52010)
9. P. Lisonek, Closed forms for the number of polygon dissections. Journal of Symbolic Computation 20 (1995), 595-601. MR1395415 (97j:52019)
10. J. W. Moon and L. Moser, Triangular dissections of n-gons, Canad. Math. Bull. 6 (1963), 175-178. MR0154821 (27:4765)
11. G. Pòlya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, Acta Math. 68 (1937), 145–254.
12. R. C. Read, On general dissections of a polygon, Aequat. Math. 18 (1978), 370-388. MR522523 (80e:05069)
13. D. D. Sleator, R. E. Tarjan, and W. P. Thurston, Rotation Distance, Triangulations, and Hyperbolic Geometry, Journal of the American Mathematical Society, Vol. 1, No. 3, 1988. MR928904 (90h:68026)
14. H. S. Wilf, Generatingfunctionology, 2nd ed., Academic Press, Boston, MA, 1994. MR1277813 (95a:05002)

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

E-mail address: bowman@math.niu.edu

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

E-mail address: regev@math.niu.edu