PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 4, April 2013, Pages 1199–1209
S 0002-9939(2012)10706-3
Article electronically published on August 30, 2012

A NOTE ON $*_w$-NOETHERIAN DOMAINS

CHUL JU HWANG AND JUNG WOOK LIM

(Communicated by Irena Peeva)

ABSTRACT. Let D be an integral domain with quotient field K, $*$ be a star-operation on D, and $GV^*(D)$ be the set of finitely generated ideals J of D such that $J_* = D$. Then the map $*_w$ defined by $I_{*_w} = \{x \in K \mid Jx \subseteq I$ for some $J \in GV^*(D)\}$ for all nonzero fractional ideals I of D is a finite character star-operation on D. In this paper, we study several properties of $*_w$-Noetherian domains. In particular, we prove the Hilbert basis theorem for $*_w$-Noetherian domains.

1. INTRODUCTION

For the sake of clarity, we first review some definitions and notation. Throughout this paper, D denotes an integral domain with quotient field K and $\mathbf{F}(D)$ means the set of nonzero fractional ideals of D. A *star-operation* on D is a mapping $I \mapsto I_*$ from $\mathbf{F}(D)$ into itself which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$:

(1) $(aD)_* = aD$ and $(aI)_* = aI_*$,
(2) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$, and
(3) $(I_*)_* = I_*$.

An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I = I_*$. A $*$-ideal I is said to be of *finite type* if $I = (a_1, \ldots, a_n)_*$ for some $(0) \neq (a_1, \ldots, a_n) \subseteq I$. Given any star-operation $*$ on D, we can construct two new star-operations $*_s$ and $*_w$ induced by $*$. For all $I \in \mathbf{F}(D)$, the $*_s$-operation is defined by $I_{*_s} = \bigcup\{(a_1, \ldots, a_n)_* \mid (0) \neq (a_1, \ldots, a_n) \subseteq I\}$ and the $*_w$-operation is defined by $I_{*_w} = \{x \in K \mid Jx \subseteq I$ for some $J \in GV^*(D)\}$, where $GV^*(D)$ is the set of nonzero finitely generated ideals J of D with $J_* = D$. A star-operation $*$ on D is said to be of *finite character* if $I_* = I_{*_s}$ for each $I \in \mathbf{F}(D)$. It is easy to see that the $*_s$-operation is of finite character. It is known that the $*_w$-operation is also a finite character star-operation on D [AC, Theorem 2.7]. Let $*'$ be a finite character star-operation on D. Recall that each prime ideal minimal over a $*'$-ideal is a $*'$-ideal, and hence each height-one prime ideal is a $*'$-ideal. A prime ideal which is a $*$-ideal is called a *prime $*$-ideal*. A $*$-ideal is called a *maximal $*$-ideal* of D if it is maximal among integral $*$-ideals of D. We denote by $*$-Max(D) the set of maximal $*$-ideals of D. It is well known that a maximal $*$-ideal is a prime ideal and that if D is not a field, then each integral $*'$-ideal is contained in a maximal $*'$-ideal, and hence $*'$-Max$(D) \neq \emptyset$. The $*$-*dimension* of D, denoted by

Received by the editors June 3, 2010 and, in revised form, August 18, 2011.
2010 *Mathematics Subject Classification.* Primary 13A15, 13G05; Secondary 13E99, 13F05, 13F20.
Key words and phrases. $*_w$-operation, $*_w$-Noetherian domain.
The authors thank the referee for valuable suggestions.

$*$-dim(D), is defined to be the supremum of $\{n \in \mathbb{N} \mid P_1 \subsetneq \cdots \subsetneq P_n$ is a chain of prime $*$-ideals of $D\}$. Thus $*$-dim$(D) = 1$ if and only if each maximal $*$-ideal of D has height one. When $* = d$, we write dim(D) rather than d-dim(D). (Recall that the d-operation is the identity map on $\mathbf{F}(D)$, *i.e.*, $I \mapsto I_d = I$.)

Recall that D is called a *$*$-Noetherian domain* if D has the ascending chain condition on integral $*$-ideals of D. It is well known that D is a $*$-Noetherian domain if and only if every integral $*$-ideal of D is of finite type and that if D is a $*$-Noetherian domain, then $* = *_s$ [Z, Theorem 1.1]. We call D a *strong Mori domain* (SM-domain) if $* = w$, a *Mori domain* if $* = v$, and a *Noetherian domain* if $* = d$. (Recall that the v-operation is defined by $I_v = (I^{-1})^{-1}$ for all $I \in \mathbf{F}(D)$ (equivalently, I_v is the intersection of principal fractional ideals of D containing I [G, Theorem 34.1]), where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, and that the t-operation (resp., w-operation) is the $*_s$-operation (resp., $*_w$-operation) when $* = v$.) It is clear that a Noetherian domain is an SM-domain and that an SM-domain is a Mori domain. However, none of the converse statements hold (see [PT, Example 3.5] and [P, Theorems 7 and 10]). Recall that D is an *Artinian domain* if D has the descending chain condition on integral ideals of D. It was shown that D is an Artinian domain if and only if D is a 0-dimensional Noetherian domain [AM, Theorem 8.5].

It is well known that a Noetherian domain possesses good properties. One of them is the Hilbert basis theorem, which says that the polynomial ring over a Noetherian domain is also Noetherian [K, Theorem 69] (or [AM, Theorem 7.5]). Later, this property was generalized to SM-domains by Fanggui and McCasland [FM2, Theorem 1.13]. (Note that the Hilbert basis theorem does not carry over to Mori domains [R2, Proposition 8.3].) So, it might be natural to ask whether a $*_w$-Noetherian domain analogue of this result holds or not. In fact, a $*_w$-Noetherian domain is a generalization of an SM-domain. An important purpose of this paper is to give an affirmative answer to this question.

In Section 2, we study the $*_w$-operation on D. We prove that for a given torsion-free D-module N, $N_P = (N_{*_w})_P$ for any prime $*_w$-ideal P of D. As a corollary, we show that a $*_w$-Noetherian domain satisfies the (generalized) principal ideal theorem for $*_w$-Noetherian domains. Moreover, as the star-operation analogue of H-domains, we define a $*$-H-domain whose class contains $*_w$-Noetherian domains, and we give a characterization of $*$-H-domains.

In Section 3, we investigate $*_w$-Noetherian domains. We give the Hilbert basis theorem for $*_w$-Noetherian domains. We also find a sufficient condition for the domain D to be a $*_w$-Noetherian domain.

For any undefined terminology or notation, see [G, K]. The reader can refer to [A1, G] for star-operations on integral domains and to [AC] for $*_w$-operations on integral domains.

2. The $*_w$-operations

We start this section by reviewing two facts about $*_w$-operations on D.

Lemma 2.1. *The following assertions hold.*

(1) *Let $I \in \mathbf{F}(D)$. Then $I \subseteq I_{*_w} \subseteq I_{*_s} \subseteq I_*$ and $I_{*_w} \subseteq I_w$.*
(2) *D is a $*_w$-Noetherian domain if and only if each prime $*_w$-ideal of D is of finite type.*

Proof. (1) The containments $I \subseteq I_{*_w}$ and $I_{*_s} \subseteq I_*$ follow from their definitions. If $x \in I_{*_w}$, then there exists a $J \in GV^*(D)$ such that $xJ \subseteq I$. So $x \in xD = xJ_* = (xJ)_* = (xJ)_{*_s} \subseteq I_{*_s}$. Therefore $I_{*_w} \subseteq I_{*_s}$. Thus $I \subseteq I_{*_w} \subseteq I_{*_s} \subseteq I_*$. The second inclusion $I_{*_w} \subseteq I_w$ follows from the fact that for any $J \in \mathbf{F}(D)$, $J_* \subseteq J_v = (J^{-1})^{-1} \subseteq D$.

(2) This is [C, Theorem 2.6]. □

Let $*_1$ and $*_2$ be finite type star-operations on D. Following [A2], we say that $*_1$ is *coarser* than $*_2$ (denoted by $*_1 \leq *_2$) if $I_{*_1} \subseteq I_{*_2}$ for all $I \in \mathbf{F}(D)$ (equivalently, each $*_2$-ideal is a $*_1$-ideal). Then $\leq$ is a partial order on the star-operations on D. It is clear that if $*_1 \leq *_2$, then a $*_1$-Noetherian domain is a $*_2$-Noetherian domain. Recall that a domain D is called a *Krull domain* if the following three properties hold: (1) $D = \bigcap D_P$, where P runs over the height-one prime ideals of D, (2) for each height-one prime ideal P of D, D_P is a (rank-one) discrete valuation domain and (3) each nonzero element of D is a unit in D_P for almost all height-one prime ideals P of D.

Corollary 2.2. *Let D be a $*_w$-Noetherian domain. Then D is both an SM-domain and a $*$-Noetherian domain. Therefore a completely integrally closed $*_w$-Noetherian domain is a Krull domain.*

Proof. The first assertion is an immediate consequence of Lemma 2.1(1). The second statement follows from the fact that a completely integrally closed SM-domain is a Krull domain [FM1, Theorem 5.4]. □

An ideal J of D is called a *Glaz-Vasconcelos-ideal* (GV-ideal) of D if J is finitely generated and $J^{-1} = D$. For any star-operation $*$, we call an ideal J of D a *$*$-GV-ideal* if J is finitely generated and $J_* = D$. Recall that $GV^*(D)$ is the set of $*$-GV-ideals of D. We give some basic properties of $GV^*(D)$.

Lemma 2.3. *Let $*$ be a star-operation on D. Then the following statements hold.*

(1) *$D \in GV^*(D)$.*
(2) *Let I and J be finitely generated ideals of D with $I \subseteq J$. If $I \in GV^*(D)$, then $J \in GV^*(D)$.*
(3) *If I and J belong to $GV^*(D)$, then so does IJ.*

Proof. (1) This is obvious.

(2) Since $I \in GV^*(D)$, $D = I_* \subseteq J_* \subseteq D$, *i.e.*, $J_* = D$. Since J is finitely generated, $J \in GV^*(D)$.

(3) It is clear that IJ is finitely generated. Note that $D = I_*J_* \subseteq (IJ)_* \subseteq D$. Hence $(IJ)_* = D$, and thus $IJ \in GV^*(D)$. □

Lemma 2.4. *Let J be a finitely generated ideal of D. Then $J \in GV^*(D)$ if and only if $J_{*_w} = D$.*

Proof. ($\Rightarrow$) The containment $J_{*_w} \subseteq D$ is obvious. Note that $J1 \subseteq J$. Since $J \in GV^*(D)$, $1 \in J_{*_w}$. Thus $J_{*_w} = D$.

($\Leftarrow$) If $J_{*_w} = D$, then $D = J_{*_w} \subseteq J_* \subseteq D$ by Lemma 2.1. So $J_* = D$. Since J is finitely generated, $J \in GV^*(D)$. □

Corollary 2.5. *The following statements are equivalent.*

(1) *Every maximal ideal of* D *is a* $*_w$*-ideal.*
(2) $GV^*(D) = \{D\}$.
(3) *Every nonzero ideal of* D *is a* $*_w$*-ideal.*

Proof. (1) $\Rightarrow$ (2) By Lemma 2.3(1), $D \in GV^*(D)$. Let $J \in GV^*(D)$. If $J \subseteq M$ for some maximal ideal M of D, then $D = J_{*_w} \subseteq M_{*_w} = M$ by Lemma 2.4 and (1). This is absurd, and hence $J \nsubseteq M$ for any maximal ideal M of D. Thus $J = D$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) These are obvious. $\square$

Corollary 2.6 (cf. [AC, Theorem 2.15])**.** *Let* Q *be a* P*-primary ideal of* D *for some prime ideal* P *of* D*. Then either* $Q_{*_w} = Q$ *or* $Q_{*_w} = D$.

Proof. Suppose to the contrary that $Q_{*_w} \neq Q$ and $Q_{*_w} \neq D$. Then there exists an element $x \in Q_{*_w} - Q$; so there exists a $J \in GV^*(D)$ such that $Jx \subseteq Q$. Since $x \notin Q$, $J \subseteq P$. Write $J = (j_1, \dots, j_n)$. Since Q is P-primary, for each $1 \leq i \leq n$, there exists a positive integer m_i such that $j_i^{m_i} \in Q$. Let $m = m_1 + \cdots + m_n$. Then $J^m \subseteq Q$, and hence by Lemma 2.4, $D = (J_{*_w})^m \subseteq (J^m)_{*_w} \subseteq Q_{*_w} \subsetneq D$, which is impossible. Thus $Q_{*_w} = Q$ or $Q_{*_w} = D$. $\square$

In [GV], Glaz and Vasconcelos first introduced the concept of an H-domain. A domain D is called an *H-domain* if every ideal I of D with $I^{-1} = D$ is quasi-finite, *i.e.*, if there exists a $J \in GV^v(D)$ such that $J \subseteq I$. As the star-operation analogue of H-domains, we say that D is a *$*$-H-domain* if every ideal I of D satisfying $I_* = D$ contains a member of $GV^*(D)$. In this case, I is said to be *$*$-quasi-finite*. It is known that a Cohen type theorem holds for H-domains; *i.e.*, we restrict I in the definition of an H-domain to prime ideals [GV, (3.2a)]. Now we prove that the $*$-H-domain analogue of this result holds.

Proposition 2.7. *D is a $*$-H-domain if and only if every prime ideal* P *of* D *with* $P_* = D$ *contains a member of* $GV^*(D)$.

Proof. ($\Rightarrow$) This is clear.

($\Leftarrow$) Let $\mathcal{A} = \{I \mid I$ is an ideal of D such that $I_* = D$ but I does not contain any member of $GV^*(D)\}$ and suppose that $\mathcal{A} \neq \emptyset$. Then $\mathcal{A}$ is partially ordered under inclusion $\subseteq$ and is inductive under this ordering. By Zorn's lemma, $\mathcal{A}$ contains a maximal element P. We claim that P is a prime ideal of D. Assume that $a_1 a_2 \in P, a_1 \notin P$ and $a_2 \notin P$. Then for each $i = 1, 2$, (P, a_i) is not a member of $\mathcal{A}$, and consequently there exists a $J_i \in GV^*(D)$ such that $J_i \subseteq (P, a_i)$. By Lemma 2.3(3), $J_1 J_2 \in GV^*(D)$. Since P contains $J_1 J_2$, this contradicts the choice of P. Thus $\mathcal{A} = \emptyset$, which proves this result. $\square$

Recall that every SM-domain is an H-domain [HZ, Proposition 2.4]. We give the $*_w$-Noetherian domain version of this statement.

Proposition 2.8. *A* $*_w$*-Noetherian domain is a* $*$*-*H*-domain.*

Proof. Let D be a $*_w$-Noetherian domain and I be an ideal of D with $I_* = D$. Then $I_{*_w} = J_{*_w}$ for some finitely generated subideal $J \subseteq I$, and hence $D = I_* = J_*$, *i.e.*, $J \in GV^*(D)$. Therefore D is a $*$-H-domain. $\square$

If $* = v$, then a v-H-domain coincides with an H-domain and a v_w-Noetherian domain is an SM-domain. Thus the next proposition generalizes [FM1, Proposition 5.7].

Proposition 2.9. *D is a $*$-H-domain if and only if every maximal $*_w$-ideal of D is a $*$-ideal.*

Proof. ($\Rightarrow$) Assume that D is a $*$-H-domain and let M be a maximal $*_w$-ideal of D. If $M \neq M_*$, then $M_* = D$ (cf. Corollary 2.6). Since D is a $*$-H-domain, there exists a $J \in GV^*(D)$ such that $J \subseteq M$. Hence by Lemma 2.4, $D = J_{*_w} \subseteq M_{*_w} = M$, which is absurd. Thus $M = M_*$ is a $*$-ideal.

($\Leftarrow$) Assume that every maximal $*_w$-ideal of D is a $*$-ideal and let I be an ideal of D such that $I_* = D$. If I is contained in a maximal $*_w$-ideal M, then $D = I_* \subseteq M_* = M$, which is impossible. So no maximal $*_w$-ideals contain I, and hence $I_{*_w} = D$. Since $1 \in I_{*_w}$, there exists a $J \in GV^*(D)$ such that $J = J1 \subseteq I$. Thus D is a $*$-H-domain. $\square$

Let N be a torsion-free D-module. As in the integral domain case, we can define a star-operation $*_w$ on N to be the set $N_{*_w} = \{x \in N_{D-\{0\}} \mid Jx \subseteq N$ for some $J \in GV^*(D)\}$ [AC, Section 4]. Note that for $* = v$, we have $N_{v_w} = N_w$, the *w-envelope* of N [FM1, Definition 3]. In [AC, Theorem 4.3], Anderson and Cook showed that $(N_{*_w})_M = N_M$ for each $M \in *_w\text{-Max}(D)$. The next proposition extends their result to any prime $*_w$-ideal of D.

Proposition 2.10. *Let N be a torsion-free D-module. Then $N_P = (N_{*_w})_P$ for any prime $*_w$-ideal P of D.*

Proof. By Lemma 2.1(1), we have $N_P \subseteq (N_{*_w})_P$. For the reverse, it suffices to show that $N_{*_w} \subseteq N_P$. Let $d \in N_{*_w}$. Then there exists a $J \in GV^*(D)$ such that $Jd \subseteq N$. Note that $J_{*_w} = D$ by Lemma 2.4. If $J \subseteq P$, then $D = J_{*_w} \subseteq P_{*_w} = P$, which yields a contradiction. This means that $J \nsubseteq P$. Therefore $d \in N_P$, and hence $N_{*_w} \subseteq N_P$. Thus $(N_{*_w})_P = N_P$. $\square$

The next two corollaries are immediate consequences of Proposition 2.10. The proofs are easy, so we omit them.

Corollary 2.11 ([AC, Theorem 4.3])**.** *Let N be a torsion-free D-module. Then $N_{*_w} = \bigcap_{M \in *_w\text{-Max}(D)} N_M$. In particular, $D = \bigcap_{M \in *_w\text{-Max}(D)} D_M$.*

Corollary 2.12. *The following statements are equivalent for the torsion-free D-modules A and B.*

(1) $A_{*_w} = B_{*_w}$.
(2) $A_P = B_P$ *for any prime $*_w$-ideal P of D.*
(3) $A_M = B_M$ *for any maximal $*_w$-ideal M of D.*

By Proposition 2.10, if N is of $*_w$-finite type, then N_P is a finitely generated D_P-module for any prime $*_w$-ideal P of D. Thus we recover

Corollary 2.13 (cf. [A1, Corollaries 4.2 and 4.3])**.** *If D is a $*_w$-Noetherian domain, then D_M is a Noetherian domain for each $M \in *_w\text{-Max}(D)$.*

Recall that D is said to be of *$*_w$-finite character* if each nonzero nonunit of D is contained in only a finite number of maximal $*_w$-ideals of D, *i.e.*, if the intersection $D = \bigcap_{M \in *_w\text{-Max}(D)} D_M$ has finite character. Let D be a $*_w$-Noetherian domain with $*_w\text{-dim}(D) = 1$. Then D is an SM-domain, and hence D is of w-finite character [FM2, Theorem 1.9]. Since $*_w\text{-dim}(D) = 1$, it is easy to see that $*_w\text{-Max}(D) = w\text{-Max}(D)$. Thus D has $*_w$-finite character. However, if D is a $*_w$-Noetherian domain with $*_w\text{-dim}(D) \geq 2$, then D need not be of $*_w$-finite character (see Example 3.7).

Corollary 2.14. *If D is a one-dimensional $*_w$-Noetherian domain, then D is Noetherian.*

Proof. Note that $*_w$-Max(D) =Max(D) since $\dim(D) = 1$. Thus the result is an immediate consequence of Corollary 2.13 and [K, Section 2.3, Exercise 10]. $\square$

When $* = v$, Corollary 2.14 recovers [FM2, Corollary 1.10].

Corollary 2.15 ([FM2, Corollary 1.10])**.** *If D is a one-dimensional SM-domain, then D is Noetherian.*

The generalized principal ideal theorem (GPIT) states that in a Noetherian domain D, if P is a prime ideal of D minimal over an n-generated ideal, then $\mathrm{ht}(P) \leq n$ [K, Theorem 152]. (Recall that when $n = 1$, this theorem is well known as Krull's principal ideal theorem (PIT) [K, Theorem 142].) This was generalized to SM-domains by Fanggui and McCasland [FM2, Corollary 1.12]. They proved that in an SM-domain D, a prime ideal of D minimal over a w-ideal $(a_1, \ldots, a_n)_w$ has height at most n. (Note that the PIT does not carry over to Mori domains ([BAD, Remark 3.6(c)] and [K, Section 3.2, Exercise 8]).) By Anderson and Cook, it was shown that $*_w$-Noetherian domains also satisfy the GPIT [AC, Corollary 3.7]. By Corollary 2.13, we can revisit the same results as corollaries. Before proving Corollary 2.17, we review the following lemma.

Lemma 2.16 ([C, Corollary 2.7(2)])**.** *Each $*_w$-ideal of a $*_w$-Noetherian domain D has a finite number of minimal prime ideals.*

Corollary 2.17. *Let D be a $*_w$-Noetherian domain.*

(1) (*PIT for $*_w$-Noetherian domains*) *Let a be a nonzero nonunit element of D. If P is a prime ideal of D minimal over (a), then* $\mathrm{ht}(P) \leq 1$.

(2) (*GPIT for $*_w$-Noetherian domains*) *Let $I = (a_1, \ldots, a_n)_{*_w}$ be a $*_w$-finite ideal of D. If P is a prime ideal of D minimal over I, then* $\mathrm{ht}(P) \leq n$.

(3) *Assume that P is a prime $*_w$-ideal of D with* $\mathrm{ht}(P) = n$. *Then P is minimal over an n-generated ideal of D.*

Proof. (1) Let P be a prime ideal minimal over (a). Then P is a $*_w$-ideal of D, and hence there exists a maximal $*_w$-ideal M containing P. By Corollary 2.13, D_M is a Noetherian domain, and note that PD_M is a prime ideal of D_M minimal over aD_M. By PIT, $\mathrm{ht}(PD_M) \leq 1$. Thus $\mathrm{ht}(P) \leq 1$.

(2) Let P be a prime ideal minimal over I. Then P is a $*_w$-ideal of D, and hence there exists a maximal $*_w$-ideal M containing P. By Corollary 2.13, D_M is a Noetherian domain, and note that PD_M is a prime ideal of D_M minimal over ID_M. Thus $\mathrm{ht}(P)$ =$\mathrm{ht}(PD_M) \leq n$ by GPIT.

(3) Let $(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P$ be a chain of prime ideals of D. If $P_i \neq (P_i)_{*_w}$ for some $1 \leq i \leq n-1$, then $(P_i)_{*_w} = D$ by Corollary 2.6. So $D = (P_i)_{*_w} \subseteq P$, which is a contradiction. Hence each P_i is also a $*_w$-ideal. Let $0 \neq a_1 \in P_1$. By Lemma 2.16, there exist finitely many prime ideals minimal over (a_1), say $Q_1, \ldots, Q_m$. If $n = 1$, then the statement follows from (1). Suppose that $n \geq 2$ and set $Q = \bigcup_{i=1}^m Q_i$. Then $P_2 \nsubseteq Q$ [K, Theorem 83]. Let $a_2 \in P_2 - Q$. Then P_2 is minimal over (a_1, a_2), and hence $\mathrm{ht}(P_2) \leq 2$ by (2). Since P_2 contains the chain $(0) \subsetneq P_1$, $\mathrm{ht}(P_2) \geq 2$. Therefore $\mathrm{ht}(P_2) = 2$. Repeating this process, we can choose some suitable elements $a_1, \ldots, a_n \in P$ so that P is minimal over $(a_1, \ldots, a_n)$. $\square$

3. Main results

Let $*$ be a star-operation on $D[X]$. Then $*$ induces a star-operation $\overline{*}$ on D defined by $I \mapsto I[X]_* \cap K$ for each $I \in \mathbf{F}(D)$ [M2, Proposition 2.1]. From now on, we refer to this induced star-operation as $\overline{*}$. Note that if $*$ is of finite type, then so is $\overline{*}$.

Lemma 3.1. *Let $*$ be a star-operation on $D[X]$. If $J \in GV^{\overline{*}}(D)$, then $J[X] \in GV^*(D[X])$.*

Proof. Since $J \in GV^{\overline{*}}(D)$, $J_{\overline{*}} = D$. Hence $(J[X])_* = (J_{\overline{*}}[X])_* = D[X]$, where the first equality follows from [M2, Proposition 2.1]. Since J is finitely generated in D, so is $J[X]$ in $D[X]$. Thus $J[X] \in GV^*(D[X])$. □

It is well known as the Hilbert basis theorem that if D is a Noetherian domain, then the polynomial ring $D[X]$ is also a Noetherian domain [K, Theorem 69] (or [AM, Theorem 7.5]). This was generalized to SM-domains, and the statement is that if D is an SM-domain, then so is $D[X]$ [FM2, Theorem 1.13]. Note that if $*$ is the v-operation on $D[X]$, then $\overline{*}$ is the v-operation on D [M2, Remark 2.2]; so $*_w$ (resp., $\overline{*}_w$) is exactly the same as the w-operation on $D[X]$ (resp., D) [HH, Proposition 4.3]. Thus the next theorem is a generalization of the Hilbert basis theorem for SM-domains.

Theorem 3.2 (The Hilbert basis theorem for $*_w$-Noetherian domains)**.** *Let $*$ be a star-operation on $D[X]$. If D is a $\overline{*}_w$-Noetherian domain, then $D[X]$ is a $*_w$-Noetherian domain.*

Proof. Let H be a $*_w$-ideal of $D[X]$ and let I_r be the set of leading coefficients of all polynomials of degree r in H, where r runs over all nonnegative integers. Then it is easy to see that $\{I_r\}_{r\geq 0}$ is an ascending chain of ideals of D. Since D is a $\overline{*}_w$-Noetherian domain, there exists a nonnegative integer m such that $(I_n)_{\overline{*}_w} = (I_m)_{\overline{*}_w}$ for all $n \geq m$. Also, since D is $\overline{*}_w$-Noetherian, for each $0 \leq r \leq m$, I_r is of $\overline{*}_w$-finite type; so we can write $(I_r)_{\overline{*}_w} = (a_{r1}, \ldots, a_{rn_r})_{\overline{*}_w}$, where $a_{r1}, \ldots, a_{rn_r} \in I_r$. Then there exists a polynomial $f_{ri} \in H$ whose leading coefficient is a_{ri}.

Claim. $H = (\{f_{ri} \mid 0 \leq r \leq m \text{ and } 1 \leq i \leq n_r\})_{*_w}$. The containment $(\{f_{ri} \mid 0 \leq r \leq m \text{ and } 1 \leq i \leq n_r\})_{*_w} \subseteq H$ is trivial. For the converse, let $f \in H$. If $f = 0$, then there is nothing to prove. Assume that $f \neq 0$. We use the induction on the degree of f. It is clear when f is a constant. Suppose that this theorem is true for the degree of f less than l. Let f be a polynomial of degree l with leading coefficient a. Assume that $l \geq m$. Then $a \in (I_l)_{\overline{*}_w} = (I_m)_{\overline{*}_w}$, and hence there exists an element $B = (b_1, \ldots, b_k) \in GV^{\overline{*}}(D)$ such that $Ba \subseteq (a_{m1}, \ldots, a_{mn_m})$. So, for each $1 \leq i \leq k$, we can write $b_i a = \sum_{j=1}^{n_m} c_{ij} a_{mj}$, where $c_{ij} \in D$. Set $g_i = b_i f - \sum_{j=1}^{n_m} c_{ij} X^{l-m} f_{mj}$ for each $1 \leq i \leq k$. Then the degree of g_i is less than l. If $l < m$, then $a \in (I_l)_{\overline{*}_w}$, and hence we can construct polynomials g_i whose degrees are less than l by using the similar argument above. In both cases, by the induction hypothesis, $g_i \in (\{f_{ri} \mid 0 \leq r \leq m \text{ and } 1 \leq i \leq n_r\})_{*_w}$. Hence for each $1 \leq i \leq k$, we can find a $J_i \in GV^*(D[X])$ such that $J_i g_i \subseteq (\{f_{ri} \mid 0 \leq r \leq m$ and $1 \leq i \leq n_r\})$. Set $J = J_1 \cdots J_t$. Then $BJf \subseteq (\{f_{ri} \mid 0 \leq r \leq m$ and $1 \leq i \leq n_r\})$, which implies that $B[X]Jf \subseteq (\{f_{ri} \mid 0 \leq r \leq m \text{ and } 1 \leq i \leq n_r\})$. By Lemma 3.1, $B[X] \in GV^*(D[X])$ and by Lemma 2.3(3), $B[X]J \in GV^*(D[X])$.

Therefore $f \in (\{f_{ri} \mid 0 \leq r \leq m \text{ and } 1 \leq i \leq n_r\})_{*_w}$, *i.e.*, $H \subseteq (\{f_{ri} \mid 0 \leq r \leq m$ and $1 \leq i \leq n_r\})_{*_w}$. Hence the claim is proved.

Since each $*_w$-ideal of $D[X]$ is of finite type, we conclude that $D[X]$ is a $*_w$-Noetherian domain. □

Lemma 3.3. *For any nonzero integral ideal I of D, $I_{\overline{*_w}} = I_{\bar{*}_w}$.*

Proof. Let $a \in I_{\overline{*_w}}$. Then $a \in (I[X])_{*_w} \cap K$; so there exists a $J \in GV^*(D[X])$ such that $Ja \subseteq I[X]$. Let C be the ideal of D generated by coefficients of generators of J. Then $C_{\bar{*}} = (C[X])_* \cap K \supseteq J_* \cap K = D$, and hence $C_{\bar{*}} = D$. Clearly, C is finitely generated. Therefore $C \in GV^{\bar{*}}(D)$. Since $Ca \subseteq I$, we have $a \in I_{\bar{*}_w}$. Thus $I_{\overline{*_w}} \subseteq I_{\bar{*}_w}$. Conversely, if $b \in I_{\bar{*}_w}$, then there exists a $J \in GV^{\bar{*}}(D)$ such that $Jb \subseteq I$; so $bJ[X] \subseteq I[X]$. By Lemma 3.1, $J[X] \in GV^*(D[X])$, which indicates that $b \in (I[X])_{*_w} \cap K = I_{\overline{*_w}}$. Hence $I_{\bar{*}_w} \subseteq I_{\overline{*_w}}$, and thus the equality holds. □

By Lemma 3.3, the concept of a $\bar{*}_w$-Noetherian domain is the same as that of a $\overline{*_w}$-Noetherian domain. Thus we have

Corollary 3.4. *For a star-operation $*$ on $D[X]$, if D is a $\overline{*_w}$-Noetherian domain, then $D[X]$ is a $*_w$-Noetherian domain.*

Remark 3.5. It is natural to ask whether the Hilbert basis theorem for $*_s$-Noetherian domains holds or not. However, it was already shown that the answer is negative. When $* = v$, Roitman proved that there exists a domain D containing a countable field such that D is Mori but $D[X]$ is not Mori [R2, Theorem 8.4]. For the interested readers, we also mention that if D is an integrally closed Mori domain, then $D[X]$ is a Mori domain [Q, §3, Théorème 5] and that if D is a Mori domain containing an uncountable field, then $D[X]$ is a Mori domain [R1, Theorem 3.15].

Now, we would like to characterize $*_w$-Noetherian domains. It is well known that D is a Noetherian domain (resp., SM-domain) if and only if D_M is Noetherian for all $M \in \mathrm{Max}(D)$ (resp., $M \in w\text{-}\mathrm{Max}(D)$) and any nonzero element of D lies in only finitely many maximal ideals (resp., maximal w-ideals) [K, Section 2.3, Exercise 10] (resp., [FM2, Theorem 1.9]). Motivated by these results, we study the $*_w$-Noetherian domain analogue.

Theorem 3.6 (cf. Corollary 2.13)**.** *Assume that D_M is a Noetherian domain for each maximal $*_w$-ideal M of D and that D is of $*_w$-finite character. Then D is a $*_w$-Noetherian domain.*

Proof. Assume that D_M is a Noetherian domain for every $M \in *_w\text{-}\max(D)$ and let I be a prime $*_w$-ideal of D. Choose any nonzero element $a \in I$. Since D is of $*_w$-finite character, there exists only a finite number of maximal $*_w$-ideals of D containing a, say $M_1, \dots, M_n$. Since D_{M_i} is Noetherian for each $1 \leq i \leq n$, $ID_{M_i} = (a_{i1}, \dots, a_{im_i})D_{M_i}$ for some $a_{i1}, \dots, a_{im_i} \in I$. Let C be the ideal of D generated by a and all a_{ij}. Then C is a finitely generated ideal of D which is contained in I. Hence $CD_{M_i} = ID_{M_i}$ for each $1 \leq i \leq n$. Let M' be a maximal $*_w$-ideal such that $M' \neq M_i$ for all $1 \leq i \leq n$. Then we have $a \notin M'$, and hence $CD_{M'} = D_{M'} = ID_{M'}$. Thus $CD_M = ID_M$ for all $M \in *_w\text{-}\max(D)$. It follows from Corollary 2.12 that $I = C_{*_w}$. This means that every prime $*_w$-ideal of D is of finite type. Thus D is a $*_w$-Noetherian domain by Lemma 2.1(2). □

It is worth remarking at this point that Noetherian domains (resp., SM-domains) have finite character (resp., w-finite character), and this property plays a significant role when many mathematicians verify some famous theorems (for example, Matijevic's theorem [M1, Corollary] and the Krull-Akizuki theorem [K, Theorem 93] (or [N, Theorem 33.2])). The next example shows that a $*_w$-Noetherian domain need not have $*_w$-finite character. Therefore the converse of Theorem 3.6 is not true in general.

Example 3.7. This example is due to [C, Example 4.5]. Let K be a field, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ be a set of indeterminates over K, $D = K[\mathbf{X}]$, and $\mathcal{P}_n$ be the set of prime ideals P of D with $\mathrm{ht}(P) = n$. For each $n \geq 1$, let $*_n$ be the star-operation on D defined by $I_{*_n} = \bigcap_{P \in \mathcal{P}_n} ID_P$ for all $I \in \mathbf{F}(D)$. Then D is a $(*_n)_w$-Noetherian domain with $(*_n)_w$-dim$(D) = n$. We also note that $(*_n)_w$-Max$(D) = \mathcal{P}_n$. Fix $n \geq 2$. For each $i \geq n$, set $P_i = (X_1, \ldots, X_{n-1}, X_i)$. Then $\mathrm{ht}(P_i) = n$, and hence P_i is a maximal $(*_n)_w$-ideal of D containing X_1, *i.e.*, X_1 belongs to infinitely many maximal $(*_n)_w$-ideals $(X_1, \ldots, X_{n-1}, X_i)$ of D, where $i \geq n$. Thus if $(*_n)_w$-dim$(D) \geq 2$, then D does not have $(*_n)_w$-finite character.

It was shown that D is an SM-domain with w-dim$(D) = 1$ if and only if for every nonzero w-ideal I of D, every descending chain of w-ideals of D containing I is stationary [FM2, Theorem 3.2]. We extend this result to the $*_w$-Noetherian domain.

Theorem 3.8. *The following assertions are equivalent.*

(1) *D is a $*_w$-Noetherian domain with $*_w$-dim$(D) = 1$.*

(2) *For any nonzero $*_w$-ideal I of D, every descending chain of $*_w$-ideals of D containing I stabilizes.*

Proof. As mentioned before Corollary 2.14, we note again that a $*_w$-Noetherian domain with $*_w$-dim$(D) = 1$ has $*_w$-finite character.

(1) $\Rightarrow$ (2) Let $\{I_n\}_{n \in \mathbb{N}}$ be a descending chain of $*_w$-ideals of D containing I. Since D is a $*_w$-Noetherian domain with $*_w$-dim$(D) = 1$, there exist only finitely many maximal $*_w$-ideals containing I, say $M_1, \ldots, M_n$. Also, for each $1 \leq i \leq n$, D_{M_i} is a Noetherian domain with $\dim(D_{M_i}/ID_{M_i}) = 0$ by Corollary 2.13. Therefore for each $1 \leq i \leq n$, D_{M_i}/ID_{M_i} is an Artinian domain [AM, Theorem 8.5], and hence we can find a positive integer m_i such that $I_k D_{M_i} = I_{m_i} D_{M_i}$ for all $k \geq m_i$. Set $m = \max\{m_1, \ldots, m_n\}$, and then we have $I_k D_{M_i} = I_m D_{M_i}$ for all $k \geq m$ and all $1 \leq i \leq n$. Let M' be a maximal $*_w$-ideal such that $M' \neq M_i$ for all $i = 1, \ldots, n$. Since $I \nsubseteq M'$, we have $I_k D_{M'} = D_{M'} = I_m D_{M'}$ for every $k \geq 1$. By Corollary 2.12, $I_k = I_m$ for all $k \geq m$. Thus the chain $\{I_n\}_{n \in \mathbb{N}}$ is stationary.

(2) $\Rightarrow$ (1) First, we show that D is a $*_w$-Noetherian domain. Let M be a maximal $*_w$-ideal of D and I be a $*_w$-ideal of D contained in M.

Claim 1. D_M/ID_M is an Artinian domain. Let $\{J_n\}_{n \in \mathbb{N}}$ be a descending chain of ideals of D_M which contain ID_M and set $I_n = J_n \cap D$ for each $n \geq 1$. Then $\{(I_n)_{*_w}\}_{n \in \mathbb{N}}$ is a descending chain of $*_w$-ideals of D containing I. By (2), this chain stabilizes, and hence there exists a positive integer m such that $(I_n)_{*_w} = (I_m)_{*_w}$ for all $n \geq m$. By Proposition 2.10, $J_n = I_n D_M = (I_n)_{*_w} D_M = (I_m)_{*_w} D_M = I_m D_M = J_m$ for all $n \geq m$. Therefore the descending chain $\{J_n\}_{n \in \mathbb{N}}$ stabilizes, and hence D_M/ID_M is an Artinian domain.

Claim 2. D_M is a Noetherian domain. Let $\{C_n\}_{n\in\mathbb{N}}$ be an ascending chain of nonzero ideals of D_M. Set $I = C_1 \cap D$. Then I is a nonzero ideal of D and by Proposition 2.10, $I_{*_w}D_M = ID_M = C_1$. It follows from Claim 1 that D_M/C_1 is an Artinian domain; so D_M/C_1 is Noetherian [AM, Theorem 8.5]. Thus D_M is a Noetherian domain [E, Lemma 4].

Let d be a nonzero nonunit element of D. If D does not have $*_w$-finite character, then there is an infinite set $\{M_n\}_{n\in\mathbb{N}}$ of maximal $*_w$-ideals of D containing d; so $\{Q_n = \bigcap_{i=1}^n M_i\}_{n\in\mathbb{N}}$ is a descending chain of $*_w$-ideals of D containing (d). By (2), there exists an $m \geq 1$ such that $Q_m = Q_{m+1}$. Since each M_n is a maximal $*_w$-ideal of D, $M_{m+1} = M_i$ for some $1 \leq i \leq m$. This is a contradiction. Therefore d belongs to only a finite number of maximal $*_w$-ideals of D. Thus Theorem 3.6 indicates that D is a $*_w$-Noetherian domain.

Next, we show that $*_w\text{-dim}(D) = 1$. Let M be a maximal $*_w$-ideal of D and choose any nonzero element $a \in M$. By Claim 1, D_M/aD_M is Artinian, whence $\dim(D_M/aD_M)$=0 [AM, Theorem 8.5]. Therefore MD_M is the only minimal prime ideal over aD_M, and hence $\text{ht}(MD_M) = 1$. This means that $\text{ht}(M) = 1$, and thus we conclude that $*_w\text{-dim}(D) = 1$. □

Corollary 3.9. *Let D be a $*_w$-Noetherian domain with* $*_w\text{-dim}(D) = 1$ *and let I be a nonzero $*_w$-ideal of D. Then D_M/ID_M is an Artinian domain for each maximal $*_w$-ideal M which contains I.*

Proof. The proof comes from (1) ⇒ (2) in Theorem 3.8. □

In [C, Section 3], the $*$-*global transform* of D is defined to be the set $D^{*g} = \{x \in K \mid M_1 \cdots M_n x \subseteq D \text{ for some } M_i \in *_s\text{-Max}(D)\}$. Then D^{*g} is an overring of D and $D^{*g} = D^{*_s g} = D^{*_w g}$, so the concept of $*_w$-global transform coincides with that of $*$-global transform. We are closing this article with a simple result about the $*_w$-global transform of D.

Proposition 3.10. *If D is a $*_w$-Noetherian domain with* $*_w\text{-dim}(D) = 1$, *then $D^{*_w g} = K$.*

Proof. By the definition of $D^{*_w g}$, it is clear that $D^{*_w g} \subseteq K$, and so it remains to show that $K \subseteq D^{*_w g}$. Let $x \in K$. If $x \in D$, then there is nothing to prove. Assume that $x \notin D$ and set $I = \{y \in D \mid xy \in D\}$. Note that $I \neq D$ because $x \notin D$, and hence I is a $*_w$-ideal of D. Since D has $*_w$-finite character, there are only a finite number of maximal $*_w$-ideals containing I, say $M_1, \ldots, M_n$. Since D is a $*_w$-Noetherian domain, for each $1 \leq i \leq n$, we can find a finitely generated ideal $J_i \subseteq M_i$ of D such that $M_i = (J_i)_{*_w}$. Then $J_1 \cdots J_n \subseteq M_1 \cdots M_n \subseteq \bigcap_{i=1}^n M_i = \sqrt{I}$, where the equality holds because $*_w\text{-dim}(D) = 1$. It is obvious that $J_1 \cdots J_n$ is finitely generated, and hence there exists a positive integer m such that $(J_1 \cdots J_n)^m \subseteq I$. Therefore $((M_1 \cdots M_n)^m)_{*_w} = ((J_1 \cdots J_n)^m)_{*_w} \subseteq I_{*_w} = I$. Thus we conclude from the definition of I that $x \in D^{*_w g}$. □

References

[A1] D.D. Anderson, *Star-operations induced by overrings*, Comm. Algebra 16 (1988) 2535-2553. MR955324 (89f:13032)

[A2] D.F. Anderson, *A general theory of class groups*, Comm. Algebra 16 (1988) 805-847. MR932636 (89f:13023)

[AC] D.D. Anderson and S.J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra 28 (2000) 2461-2475. MR1757473 (2001c:13033)

[AM] M.F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Series in Math., Westview Press, 1969. MR0242802 (39:4129)

[BAD] V. Barucci, D.F. Anderson, and D. Dobbs, *Coherent Mori domains and the principal ideal theorem*, Comm. Algebra 15 (1987) 1119-1156. MR882945 (88c:13015)

[C] G.W. Chang, *$*$-Noetherian domains and the ring $D[\mathbf{X}]_{N_*}$*, J. Algebra 297 (2006) 216-233. MR2206856 (2006k:13002)

[E] P.M. Eakin, Jr., *The converse to a well known theorem on Noetherian rings*, Math. Annalen 177 (1968) 278-282. MR0225767 (37:1360)

[FM1] W. Fanggui and R.L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra 25 (1997) 1285-1306. MR1437672 (98g:13025)

[FM2] W. Fanggui and R.L. McCasland, *On strong Mori domains*, J. Pure Appl. Algebra 135 (1999) 155-165. MR1667555 (99m:13044)

[G] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Appl. Math., vol. 90, Queen's University, Kingston, Ontario, Canada, 1992. MR1204267 (93j:13001)

[GV] S. Glaz and W.V. Vasconcelos, *Flat ideals II*, Manuscripta Math. 22 (1977) 325-341. MR0472797 (57:12487)

[HH] J.R. Hedstrom and E.G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra 18 (1980) 37-44. MR578564 (81m:13008)

[HZ] E. Houston and M. Zafrullah, *Integral domains in which each t-ideal is divisorial*, Michigan Math. J. 35 (1988) 291-300. MR959276 (89i:13027)

[K] I. Kaplansky, *Commutative Rings*, University of Chicago Press, 1974. MR0345945 (49:10674)

[M1] J. R. Matijevic, *Maximal ideal transforms of Noetherian rings*, Proc. Amer. Math. Soc. 54 (1976) 49-52. MR0387269 (52:8112)

[M2] A. Mimouni, *Note on star operations over polynomial rings*, Comm. Algebra 36 (2008) 4249-4256. MR2460413 (2009i:13004)

[N] M. Nagata, *Local Rings*, Interscience Publishers, A division of John Wiley & Sons, New York, London, 1962. MR0155856 (27:5790)

[P] M.H. Park, *Power series rings over strong Mori domains*, J. Algebra 270 (2003) 361-368. MR2016667 (2004h:13023)

[PT] M.H. Park and F. Tartarone, *Strong Mori and Noetherian properties of integer-valued polynomial rings*, J. Pure Appl. Algebra 186 (2004) 297-309. MR2025591 (2004m:13054)

[Q] J. Querré, *Idéaux divisoriels d'un anneau de polynômes*, J. Algebra 64 (1980) 270-284. MR575795 (83b:13007)

[R1] M. Roitman, *On Mori domains and commutative rings with $CC^{\perp}I$*, J. Pure Appl. Algebra 56 (1989) 247-268. MR982638 (90c:13019)

[R2] M. Roitman, *On polynomial extensions of Mori domains over countable fields*, J. Pure Appl. Algebra 64 (1990) 315-328. MR1061306 (91i:13021)

[Z] M. Zafrullah, *Ascending chain conditions and star operations*, Comm. Algebra 17 (1989) 1523-1533. MR997154 (90b:13023)

Department of Mathematics Education, Silla University, Pusan 617-736, Republic of Korea

E-mail address: cjhwang@silla.ac.kr

Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea

E-mail address: lovemath@postech.ac.kr