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RIGIDITY OF BI-LIPSCHITZ EQUIVALENCE OF WEIGHTED HOMOGENEOUS FUNCTION-GERMS IN THE PLANE

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Abstract. The main goal of this work is to show that if two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are bi-Lipschitz equivalent, in the sense that these function-germs can be included in a strongly bi-Lipschitz trivial family of weighted homogeneous function-germs, then they are analytically equivalent.

1. Introduction

The restriction of the complex polynomial

$$f_t(x, y) = xy(x - y)(x - ty)\ ;\ 0 < |t| < 1$$

defines a family of function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with isolated singularity. In 1965, H. Whitney justified the rigidity of the analytic classification of function-germs by proving that: given $t \neq s$, there is no $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ germ of a bi-analytic map such that $f_t = f_s \circ \phi$; i.e. f_t is not analytically equivalent to f_s. On the other hand, with respect to the topological point of view, this family is not so interesting, since for any $t \neq s$ there exists a germ of homeomorphism $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f_t = f_s \circ \phi$; i.e. f_t is topologically equivalent to f_s. In fact, the topological classification of reduced polynomial function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is well-understood, as was shown in [6]. In the seminal paper [3], Henry and Parusinski considered the bi-Lipschitz equivalence, which is between the analytic and the topological equivalence, of function-germs. This paper motivated other papers about the problem of bi-Lipschitz classification of function-germs. For instance, [4] and [1] showed that, in some sense, for weighted homogeneous real function-germs in two variables the problem of bi-Lipschitz classification is quite close to the problem of analytic classification. The results presented in [3] point out a rigidity of the bi-Lipschitz classification of function-germs. More precisely, they considered the family

$$f_t(x, y) = x^3 + y^6 - 3t^2xy^4\ ;\ 0 < |t| < \frac{1}{2}$$

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and proved that given $t \neq s$, there is no $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ germ of a bi-Lipschitz map such that $f_t = f_s \circ \phi$; i.e. f_t is not bi-Lipschitz equivalent to f_s. The strategy used by them was to introduce a new invariant based on the observation that the bi-Lipschitz homeomorphism does not move much the regions around the relative polar curves. For a single germ f the invariant is given in terms of the leading coefficients of the asymptotic expansions of f along the branches of its generic polar curve. In the case that the bi-Lipschitz triviality of a family of function-germs comes by integrating a Lipschitz vector field, here called strong bi-Lipschitz triviality, the calculations are much easier and very illustrative. The main goal of this work is to show that if two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are bi-Lipschitz equivalent, in the sense that these function-germs can be included in a strongly bi-Lipschitz trivial family of weighted homogeneous function-germs, then they are analytically equivalent.

2. Preliminaries

2.1. **Analytic and bi-Lipschitz equivalences.** Two germs of analytic functions $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are called *analytically equivalent* if there exists a germ of a bi-analytic map $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = g \circ \phi$.

The following two results play an important role in studying the analytic equivalence of function-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

Theorem 2.1 (Mather's Lemma)**.** *Let G be a Lie group, M a smooth manifold and $\alpha\colon G \times M \to M$ a smooth group action. Let V be a connected smooth submanifold of M. Then V is contained in an orbit of α if and only if*

a) *For all $v \in V$, the tangent space of V at v is contained in the tangent space of the orbit $G \cdot v$ at v and*
b) *The dimension of the tangent space of $G \cdot v$ does not depend on $v \in V$.*

For a reference to Mather's Lemma, for example, the reader can see [7].

Theorem 2.2 (Thom-Levine)**.** *Let U be a domain in $\mathbb{C}$, W a neighborhood of 0 in $\mathbb{C}^n$ and $F\colon W \times U \to \mathbb{C}$ such that $F(0, t) = 0$. Let us denote $f_t(x) = F(x, t)$ $\forall\, t \in U$, $\forall\, x \in W$. If there is a family of analytic vector fields $v\colon W \times U \to \mathbb{C}^n$, $v(0, t) = 0$ for all $t \in U$ and*

$$\frac{\partial f_t}{\partial t}(x) = (df_t)(x)(v(x, t))$$

$\forall\, t \in U$, $\forall\, x \in W$, then f_t is analytically equivalent to $f_{t'}$ for any $t, t' \in U$.

For a reference to Thom-Levine's Theorem, the reader can see Proposition 2.2 of [5].

Two function-germs $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are called *bi-Lipschitz equivalent* if there exists a bi-Lipschitz map-germ $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = g \circ \phi$. In order to study the bi-Lipschitz equivalence of analytic function-germs and motivated by Thom-Levine's Theorem for analytic equivalence of function-germs, we define the following notion of bi-Lipschitz triviality of a family of function-germs. Let f_t ($t \in U$, a domain in $\mathbb{C}$) be a family of analytic function-germs. That is, there is a neighborhood W of 0 in $\mathbb{C}^n$ and an analytic function $F\colon W \times U \to \mathbb{C}$ such that $F(0, t) = 0$ and $f_t(x) = F(x, t)$ $\forall\, t \in U$, $\forall\, x \in W$. We call f_t *strongly bi-Lipschitz* *trivial* when there is a continuous family of Lipschitz vector fields $v_t\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$, where the Lipschitz constant of v_t does not depend on t, such that

$$\frac{\partial f_t}{\partial t}(x) = (df_t)(x)(v_t(x))$$

$\forall\, t \in U$, $\forall\, x \in W$.

The following result shows that if we include two function-germs f and g into a strongly bi-Lipschitz trivial family of function-germs, then the two initial function-germs f and g are bi-Lipschitz equivalent.

Theorem 2.3. *If f_t is strongly bi-Lipschitz trivial, then f_t is bi-Lipschitz equivalent to $f_{t'}$ for any $t \neq t' \in U$.*

To prove Theorem 2.3 it is enough to follow the proof of Thom-Levine's Theorem and observe that the flow of a Lipschitz vector field defines a family of bi-Lipschitz homeomorphisms.

2.2. **Weighted homogeneous functions.** Let $w = (w_1, \ldots, w_n)$ be an n-tuple of positive integer numbers. We say that a polynomial function $f(x_1, \ldots, x_n)$ is *w-homogeneous* of degree d if $f(s^{w_1}x_1, \ldots, s^{w_n}x_n) = s^d f(x_1, \ldots, x_n)$ for all $s \in \mathbb{C}^*$. Let $H^d_w(n,1)$ denote the space of w–homogeneous n-variable polynomials of degree d. Let $\mathcal{O}_n$ be the ring of analytic function-germs at the origin $0 \in \mathbb{C}^n$ and let $\mathcal{M}_n$ be the maximal ideal of $\mathcal{O}_n$.

Proposition 2.4. *Let $F(x_1, \ldots, x_n, t)$ be a polynomial function such that for each $t \in U$, the function $f_t(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, t)$ is w–homogeneous with an isolated singularity at $0 \in \mathbb{C}^n$, where U is a domain of $\mathbb{C}$. If $\frac{\partial F}{\partial t}$ belongs to the ideal of $\mathcal{O}_n$ generated by $\{x_i \frac{\partial F}{\partial x_j} \ : \ i,j = 1, \ldots, n\}$ for each fixed $t \in U$, then f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.*

Proof. Let us denote by TF the ideal of $\mathcal{O}_n$ generated by $\{x_i \frac{\partial F}{\partial x_j} \ : \ i,j = 1, \ldots, n\}$. It is clear that $H^d_w(n,1)$ can be considered a subset of the space of m–jets $J^m(n,1)$, for m large enough. The set

$$A^d_w(n,1) = \{f \in H^d_w(n,1) \ : \ f \text{ has an isolated singularity at the origin}\}$$

is a Zariski open subset of $H^d_w(n,1)$. In particular, $A^d_w(n,1)$ can be seen as a connected submanifold of the m-jets space $J^m(n,1)$. Let $\mathcal{R}(n,n)$ be the group of analytic diffeomorphism-germs $(\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$. We consider the natural action of $G = j^m(\mathcal{R}(n,n))$ on the manifold $M = J^m(n,1)$ given by

$$(j^m(\phi), j^m(f)) \mapsto j^m(f \circ \phi).$$

So, given $f \in A^d_w(n,1)$, we get

$$\begin{aligned}
\operatorname{codim}(G \cdot f) &= \dim(M) - \dim(G \cdot f) \\
&= \dim(M) - \dim(T_f(G \cdot f)) \\
&= \dim\left(\frac{\mathcal{M}_n}{\mathcal{M}_n^{m+1}}\right) - \dim\left(\frac{TF}{\mathcal{M}_n^{m+1}}\right) \\
&= \dim\left(\frac{\mathcal{M}_n}{TF}\right) \\
&= n + \mu(f),
\end{aligned}$$

where $\mu(f)$ is the Milnor number of f. Since the Milnor number of $f \in A_w^d(n,1)$ does not depend on f, we get that the dimension of $T_f(G \cdot f)$ does not depend on $f \in A_w^d(n,1)$. Let

$$P = \{f_t \ : \ t \in U\}.$$

Then,

(1) by the assumptions, $T_f P \subset T_f(G \cdot f)$, for any $f \in P$, and
(2) $\dim(T_f(G \cdot f))$ is constant for $f \in P$, because $P \subset A_w^d(n,1)$.

It follows from Mather's Lemma that P is included in a single G–orbit. □

Remark 2.5. The family of polynomial function-germs $f_t(x,y) = F(x,y,t) = x^3 + (1+t)xy^5 + t^2y^6$ gives an example where the Milnor number of f_0 is 13 and the Milnor number of f_t is 10 for any $t \neq 0$; hence f_0 is not analytically equivalent to f_t for any $t \neq 0$. On the other hand, we show that, $\forall\ t$, $\frac{\partial F}{\partial t}$ belongs to the ideal of $\mathcal{O}_2$ generated by $\{x\frac{\partial F}{\partial x}, x\frac{\partial F}{\partial y}, y\frac{\partial F}{\partial x}, y\frac{\partial F}{\partial y}\}$. In fact, given t, let us denote by J_t the ideal of $\mathcal{O}_2$ generated by

$$\{x\frac{\partial F}{\partial x}(x,y,t), x\frac{\partial F}{\partial y}(x,y,t), y\frac{\partial F}{\partial x}(x,y,t), y\frac{\partial F}{\partial y}(x,y,t)\}.$$

For $t = 0$, it is easy to see that $\frac{\partial F}{\partial t} = xy^5$ belongs to J_0. Now, for $t \neq 0$, we have the following system of equations:

$$\left\{\begin{array}{ccccccccc} 3x^3 & + & (1+t)xy^5 & + & 0y^6 & + & 0x^2y^4 & \equiv & 0 \bmod J_t + \mathcal{M}_2^9 \\ 0x^3 & + & 5(1+t)xy^5 & + & 6t^2y^6 & + & 0x^2y^4 & \equiv & 0 \bmod J_t + \mathcal{M}_2^9 \\ 0x^3 & + & 6t^2xy^5 & + & 0y^6 & + & 5(1+t)x^2y^4 & \equiv & 0 \bmod J_t + \mathcal{M}_2^9 \\ 0x^3 & + & 0xy^5 & + & 0y^6 & + & 3x^2y^4 & \equiv & 0 \bmod J_t + \mathcal{M}_2^9 \end{array}\right.$$

For $t \neq 0$, the following determinant is different from 0:

$$\left|\begin{array}{cccc} 3 & (1+t) & 0 & 0 \\ 0 & 5(1+t) & 6t^2 & 0 \\ 0 & 6t^2 & 0 & 5(1+t) \\ 0 & 0 & 0 & 3 \end{array}\right|;$$

hence we get $x^3, xy^5, y^6, x^2y^4 \in J_t + \mathcal{M}_2^9$. Now, it is easy to see that $\mathcal{M}_2^6 \subset J_t + \mathcal{M}_2^9$ and, by Nakayama's Lemma, $\mathcal{M}_2^6 \subset J_t$. Finally, as $\frac{\partial F}{\partial t}$ belongs to $\mathcal{M}_2^6$, it follows that $\frac{\partial F}{\partial t}$ belongs to J_t, $\forall\ t \neq 0$.

Thus, this example shows that the hypothesis of weighted homogeneity is essential in Proposition 2.4.

3. Results

Proposition 3.1. *Let $F(x,y,t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x,y) = F(x,y,t)$ is w–homogeneous ($w_1 > w_2$) with an isolated singularity at $(0,0) \in \mathbb{C}^2$, where U is an open subset of $\mathbb{C}$. If f_t defines a strongly bi-Lipschitz trivial family of function-germs at the origin $(0,0) \in \mathbb{C}^2$, then there exists a function $k\colon U \to \mathbb{C}$ such that*

$$\frac{\partial F}{\partial t}(x,y,t) - k(t)y\frac{\partial F}{\partial y}(x,y,t)$$

is identically null on the polar set $\{(x,y,t) \ : \ \frac{\partial F}{\partial x}(x,y,t) = 0\}$.

Before we start the proof of Proposition 3.1 we need the following result.

Lemma 3.2. *For each $t \in U$, let $\Gamma_t = \{(x,y,t) \ : \ \frac{\partial F}{\partial x}(x,y,t) = 0\}$. If $y = 0$ is a component of Γ_{t_0} for some t_0, then $y = 0$ is a component of Γ_t for all $t \in U$. Moreover, $F_t(x,y) = y\bar{F}_t(x,y)$, where $\bar{F}_t(x,y)$ is a weighted homogeneous polynomial and $\bar{F}_t(x,0) \neq 0$.*

Proof of Lemma 3.2. If $y = 0$ is a component of Γ_{t_0}, we can write

$$F_t(x,y) = H_t(x,y)\bar{F}_t(x,y)$$

such that H_t and $\bar{F}_t$ are weighted homogeneous of degree a and b respectively and $H_{t_0}(x,y) = y^m$. Since the weighted degree of F_t does not depend on t, a and b do not depend on t. We have $m = 1$, because if $m \geq 2$, then f_{t_0} does not have an isolated singularity at the origin. So, H_t is a weighted homogeneous polynomial of degree w_2 for all t. Since, $w_1 > w_2$, it follows that $H_t(x,y) = a(t)y$ and $a(t) \neq 0$ for all $t \in U$. Hence, we have proved the lemma. □

Proof of Proposition 3.1. By the assumptions, there exists a Lipschitz vector field

$$v(x,y,t) = (v_1(x,y,t), v_2(x,y,t), 1)$$

such that $\frac{\partial F}{\partial v} = 0$. Let us denote

$$\Gamma_t = \{(x,y,t) \ : \ \frac{\partial F}{\partial x}(x,y,t) = 0\}.$$

Let $a_1(t), ..., a_r(t)$ be functions defined in the following way:

Case $\{y = 0\}$ is not a component of Γ_t.

In this case, $a_1(t), ..., a_r(t)$ are the roots of the polynomial equation

$$\frac{\partial F}{\partial x}(x,1,t) = 0.$$

So,

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t), \ i = 1, \dots, r$$

parameterize the branches of Γ_t.

Case $\{y = 0\}$ is a component of Γ_t.

In this case, $a_1(t), ..., a_r(t)$ are the roots of the polynomial equation

$$\frac{\partial \bar{F}}{\partial x}(x,1,t) = 0.$$

So,

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t), \ i = 1, \dots, r$$

parameterize the branches of Γ_t other than $\{y = 0\}$.

Let us define the functions $k_1(t), \dots, k_r(t)$ by

$$k_i(t) = \frac{\frac{\partial F}{\partial t}(a_i(t),1,t)}{\frac{\partial F}{\partial y}(a_i(t),1,t)}.$$

We claim that $k_i(t) = k_j(t)$ for any $i \neq j$. In fact, since $v_2(x,y,t)$ is a Lipschitz function, there exists a constant $\lambda > 0$ such that

$$|v_2(a_i(t)s^{w_1}, s^{w_2}, t) - v_2(a_j(t)s^{w_1}, s^{w_2}, t)| \leq \lambda |a_i(t) - a_j(t)||s|^{w_1}.$$

On the other hand,

$$\begin{aligned}&|v_2(a_i(t)s^{w_1},s^{w_2},t)-v_2(a_j(t)s^{w_1},s^{w_2},t)|\\&=|\frac{\frac{\partial F}{\partial t}(a_i(t)s^{w_1},s^{w_2},t)}{\frac{\partial F}{\partial y}(a_i(t)s^{w_1},s^{w_2},t)}-\frac{\frac{\partial F}{\partial t}(a_j(t)s^{w_1},s^{w_2})}{\frac{\partial F}{\partial y}(a_j(t)s^{w_1},s^{w_2})}|\\&=|k_i(t)-k_j(t)||s|^{w_2}.\end{aligned}$$

Since $w_1 > w_2$, we have $k_i(t) = k_j(t)$.

Let us denote $k(t) = k_1(t) = \cdots = k_r(t)$. Now, let us fix t. We see that the function

$$\frac{\partial F}{\partial t}(x,y,t)-k(t)y\frac{\partial F}{\partial y}(x,y,t)$$

is identically null on each branch of Γ_t at $(0,0,t)$, even when $\{y=0\}$ is a component of Γ_t. □

Remark 3.3. As a consequence of the proof of Proposition 3.1, we see that $k(t)$ satisfies additional properties. Since $k(t)$ is an algebraic expression of the functions $a_i(t)$ and, since we can develop $a_i(t)$ as a Puiseux series in a neighborhood of each t_0, we get that for any $t_0 \in U$ there exist two open sets $0 \in V \subset \mathbb{C}$ and $t_0 \in U' \subset U$ such that

$$s \mapsto s^N + t_0$$

maps V onto U' and the function

$$s \mapsto k(s^N + t_0)$$

is analytic for some $N \in \mathbb{N}$.

Theorem 3.4. *Let $F(x,y,t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x,y) = F(x,y,t)$ is w–homogeneous ($w_1 > w_2$) with an isolated singularity at $(0,0) \in \mathbb{C}^2$, where $U \subset \mathbb{C}$ is a domain. If $\{f_t \ : \ t \in U\}$, as a family of function-germs at $(0,0) \in \mathbb{C}^2$, is strongly bi-Lipschitz trivial, then f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.*

Proof. Let $k(t)$ be given by Proposition 3.1; hence we have a function $k\colon U \to \mathbb{C}$ such that

$$\frac{\partial F}{\partial t}(x,y,t)-k(t)y\frac{\partial F}{\partial y}(x,y,t)$$

is identically null on the polar set $\{(x,y,t) \ : \ \frac{\partial F}{\partial x}(x,y,t)=0\}$.

Let $t_0 \in U$. According to Remark 3.3, there exist two open sets $0 \in V \subset \mathbb{C}$ and $t_0 \in U' \subset U$ such that

$$s \mapsto s^N + t_0$$

maps V onto U' and the function

$$s \mapsto k(s^N + t_0)$$

is analytic for some $N \in \mathbb{N}$. Let us denote $G(x,y,s) = F(x,y,s^N+t_0)$, $g_s(x,y) = G(x,y,s)$ and $\tilde{k}(s) = k(s^n + t_0)$. Thus, we have an analytic funtion $\tilde{k}\colon V \to \mathbb{C}$ such that

$$\frac{\partial G}{\partial s}(x,y,s)-Ns^{N-1}\tilde{k}(s)y\frac{\partial G}{\partial y}(x,y,s)$$

is identically null on the polar set $\{(x,y,s) \ : \ \frac{\partial G}{\partial x}(x,y,s)=0\}$. Let

$$P_1(x,y,s),\ldots,P_r(x,y,s)$$

be such that they define the analytic irreducible factors of $\frac{\partial G}{\partial x}(x,y,s)$ in $\mathcal{O}_3$. Let $j(s) = Ns^{N-1}\tilde{k}(s)$. Let

$$\alpha_i = \max\{\alpha \in \mathbb{N} \ : P_i^{\alpha_i} \text{ divides } \frac{\partial G}{\partial x} \text{ in } \mathcal{O}_3\}.$$

By hypothesis, we have integer positive numbers $\beta_1, \ldots, \beta_r$ such that

$$\frac{\partial G}{\partial s}(x,y,s) = u(x,y,s)P_1^{\beta_1}(x,y,s)\cdots P_r^{\beta_r}(x,y,s) + j(s)y\frac{\partial G}{\partial y}(x,y,s)$$

with $u = u(x,y,s) \in \mathcal{O}_3$. Moreover, we can suppose that P_i does not divide u in $\mathcal{O}_3$. Let us denote

$$u_i(x,y,s) = u(x,y,s)P_1^{\beta_1}(x,y,s)\cdots \widehat{P_i^{\beta_i}(x,y,s)}\cdots P_r^{\beta_r}(x,y,s).$$

Thus, we have the equation

$$\frac{\partial G}{\partial s}(x,y,s) = u_i(x,y,s)P_i^{\beta_i}(x,y,s) + j(s)y\frac{\partial G}{\partial y}(x,y,t)m,$$

where $u_i = u_i(x,y,s) \in \mathcal{O}_3$ and P_i does not divide u_i in $\mathcal{O}_3$.

We should show that $\beta_i \geq \alpha_i$. If $\alpha_i = 1$, we have nothing to do. Thus, let us consider $\alpha_i > 1$. It follows from

$$\frac{\partial G}{\partial s}(x,y,s) = u_i(x,y,s)P_i^{\beta_i}(x,y,s) + j(s)y\frac{\partial G}{\partial y}(x,y,s)$$

that

$$\frac{\partial^2 G}{\partial x \partial s} = \frac{\partial u_i}{\partial x}P_i^{\beta_i} + \beta_i u_i P_i^{\beta_i - 1}\frac{\partial P_i}{\partial x} + jy\frac{\partial^2 G}{\partial x \partial y}. \tag{3.1}$$

Now, since $P_i^{\alpha_i}$ divides $\frac{\partial G}{\partial x}$, we have that $P_i^{\alpha_i - 1}$ divides $\frac{\partial^2 G}{\partial s \partial x}$ and $\frac{\partial^2 G}{\partial y \partial x}$; hence, by using (3.1), we get that

$$P_i^{\alpha_i - 1} \text{ divides } P^{\beta_i - 1}(\frac{\partial u_i}{\partial x}P_i + \beta_i u_i\frac{\partial P_i}{\partial x}). \tag{3.2}$$

Since P_i does not divide either u_i or $\frac{\partial P_i}{\partial x}$, it follows from (3.2) that $\beta_i \geq \alpha_i$.

Once $\beta_i \geq \alpha_i$, we have that $\frac{\partial G}{\partial s}$ can be written in the following way:

$$\frac{\partial G}{\partial s}(x,y,s) = q(x,y,s)\frac{\partial G}{\partial x}(x,y,s) + j(s)y\frac{\partial G}{\partial y}(x,y,s).$$

Now, $\frac{\partial G}{\partial s}(x,y,s)$ and $y\frac{\partial G}{\partial y}(x,y,s)$, fixed s, are w-homogeneous polynomial functions of degree d, in the variables x and y, and $\frac{\partial G}{\partial x}(x,y,s)$ is w-homogeneous of degree $d - w_1$; hence $q(x,y,s)$ should be w-homogeneous of degree w_1. In particular, $w(0,0,s) = 0$ and $\frac{\partial G}{\partial s}$ belongs to the ideal of $\mathcal{O}_3$ generated by

$$\{x\frac{\partial G}{\partial x}, x\frac{\partial G}{\partial y}, y\frac{\partial G}{\partial x}, y\frac{\partial G}{\partial y}\}.$$

Thus, it comes from Mather's Lemma or the analytic version of the Thom-Levine result that G_{s_1} is analytically equivalent to G_{s_2} for any $s_1, s_2 \in V$. This means that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U'$. Finally, it follows from the connectivity of U that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$. □

If $\frac{\partial F}{\partial x}$ were analytically reduced, this theorem could be proved in the following simpler way. Let $k(t)$ be given by Proposition 3.1. We fix $t \in U$, and since $\frac{\partial F}{\partial x}(x,y,t)$ is analytically reduced, there exists an analytic function $u(x,y)$ such that

$$\frac{\partial F}{\partial t}(x,y,t) - k(t)y\frac{\partial F}{\partial y}(x,y,t) = u(x,y)\frac{\partial F}{\partial x}(x,y,t).$$

It follows from this equation that $u(x,y)$ is w–homogeneous of degree w_1; in particular $u(0,0)=0$. Thus, $\frac{\partial F}{\partial t}$ belongs to the ideal of $\mathcal{O}_2$ generated by

$$\{x\frac{\partial F}{\partial x}(x,y,t), x\frac{\partial F}{\partial y}(x,y,t), y\frac{\partial F}{\partial x}(x,y,t), y\frac{\partial F}{\partial y}(x,y,t)\}.$$

Finally, since $f_t(x,y) = F(x,y,t)$ is w–homogeneous for all $t \in U$, it follows from Proposition 2.4 that $\{f_t \ : \ t \in U\}$ defines a family of function-germs at the origin $(0,0) \in \mathbb{C}^2$ such that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.

The above argument allows us to extend Theorem 3.4 to w-homogeneous polynomials in n variables ($w_1 > \cdots > w_n$) with the additional hypothesis that the ideal

$$\{\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n-1}}\}$$

is radical. However, the following example shows that, for $n \geq 3$ variables, if we remove some hypothesis above this result is not true at all.

Example 3.5. $f_t(x,y,z) = x^4 + y^4 + z^k + tx^2y^2$ is strongly bi-Lipschitz trivial (for t close to 0), but f_{t_1} is not analytically equivalent to f_{t_2} when $t_1 \neq t_2$.

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