SIGNED SUMS OF TERMS OF A SEQUENCE

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ABSTRACT. We give a sufficient and necessary condition on the sequence $\{a_n\}$ of integers that for any integer $l \geq 1$, every integer can be represented in the form $\varepsilon_l a_l + \varepsilon_{l+1} a_{l+1} + \cdots + \varepsilon_k a_k$, where $\varepsilon_i \in \{-1, 1\}$ $(i = l, l+1, \ldots, k)$. This generalizes the known result on integral-valued polynomial values. Moreover, we show that such sequences exist with any growth rate. This answers two problems posed by Bleicher. We also pose several problems for further research.

1. INTRODUCTION

In [1], Bleicher proved the following interesting result: for any given integer $k \ge 2$, every integer can be represented in the form

$$n = \varepsilon_1 1^k + \varepsilon_2 2^k + \dots + \varepsilon_t t^k, \quad \varepsilon_i \in \{-1, 1\}, \quad i = 1, 2, \dots, t.$$

Yu [11] generalized the result to integral-valued polynomials with fixed divisor equal to 1. Boulanger and Chabert [2] considered the problem in the ring of algebraic integers O_K of a cyclotomic field K. For related research, one may refer to [3], [4], [5], [7], [8] and [9]. Starting from the result on kth powers, Bleicher [1] posed the following two problems.

Problem 1. Does there exist a constant c > 1 and an increasing sequence of integers $\{a_i\}$ with $a_i > c^i$ for every positive integer i such that for every positive integer n, there is a positive integer m and a choice of $\varepsilon_i = \pm 1$ for which $n = \sum_{i=1}^{m} \varepsilon_i a_i$?

Problem 2. On the assumption that Problem 1 is answered affirmatively, is there an upper bound for possible choices of c?

For sequences, Erdős and Surányi [6] proved the following result: Let $0 < a_1 < a_2 < \cdots$ be a sequence of integers. If (a) the sequence contains infinitely many odd numbers; (b) for some m, all positive integers > m can be represented as the sum of different elements of the sequence; (c) $a_{n+1} < 2a_n - m$ for $n \ge n_0$, then every integer can be represented in the form $\varepsilon_1 a_1 + \varepsilon_2 a_2 + \cdots + \varepsilon_t a_t$, where $\varepsilon_i \in \{-1, 1\} (i = 1, 2, \ldots, t)$.

In this note we prove the following result. This generalizes the result on integralvalued polynomials (Bleicher [1], Yu [11]). Moreover, we answer Problem 1 affirmatively and Problem 2 negatively.

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Theorem 1. Let a_1, a_2, \ldots be a sequence of integers. Then for any integers $l \ge 1$ and n, there exists an integer $t \ge l$ and a choice of $\varepsilon_i = \pm 1$ such that

$$n = \varepsilon_l a_l + \varepsilon_{l+1} a_{l+1} + \dots + \varepsilon_t a_t$$

if and only if the following two conditions hold:

(i) there exists a nonzero integer M such that for any $l \ge 1$ there exists an integer $r = r_l \ge l$ and $\varepsilon_i \in \{-1, 1\} (l \le i \le r)$ with $M = \varepsilon_l a_l + \cdots + \varepsilon_r a_r$;

(ii) for any $l \geq 1$ we have $gcd(a_l, a_{l+1}, \ldots) = 1$.

Remark. For any $l \geq 1$, letting $r = r_l$ and $s = r_{r+1}$, we have

$$0 = M - M = \varepsilon_l a_l + \dots + \varepsilon_r a_r - \varepsilon_{r+1} a_{r+1} - \dots - \varepsilon_s a_s$$

So, for a fixed $l \ge 1$ and a fixed integer n, the t in Theorem 1 can take infinitely many values.

Remark. Let $a_1 = 2, a_2 = 3$ and $a_{i+2} = a_{i+1} + a_i + 1$ (i = 1, 2, ...). Since $1 = -a_i - a_{i+1} + a_{i+2}$, (i) is true for M = 1 and (ii) is true by $(a_i, a_{i+1}, a_{i+2}) = 1$ for all $i \ge 1$. This verifies Theorem 1. One may easily prove that $a_i > (\sqrt{2})^i$ for all $i \ge 1$. Thus Problem 1 is answered affirmatively.

Remark. For an integral-valued polynomial f(x) of degree h with fixed divisor equal to 1, let $a_i = f(i)(i = 1, 2, ...)$. For any integers l and n, by the Lagrange interpolation formula we know that f(n) is a combination of f(l), f(l+1), ..., f(l+h) with integral coefficients. Since f(x) has no fixed factors, we have

$$gcd(f(l), f(l+1), \dots, f(l+h)) = 1.$$

On the other hand, let $f_1(x) = f(x+1) - f(x)$, $f_{i+1}(x) = f_i(x+2^i) - f_i(x)(i = 1, 2, ...)$. Then $f_i(x)$ is a polynomial of degree h - i $(1 \le i \le h)$. So $f_h(x)$ is a nonzero constant. We also have

$$f_h(x) = \sum_{i=0}^{2^h - 1} \varepsilon_i f(x+i), \quad \varepsilon_i \in \{-1, 1\}.$$

Since f(x) is integral-valued, $f_h(x) = f_h(0)$ is an integer. For related information, one may refer to [10]. By Theorem 1, for any $l \ge 1$, there exists $t \ge l$ such that every integer can be represented in the form $n = \varepsilon_l f(l) + \varepsilon_{l+1} f(l+1) + \cdots + \varepsilon_t f(t)$, where $\varepsilon_i \in \{-1, 1\} (i = l, l+1, \dots, t)$. That is the main result in [11].

For Problems 1 and 2, we have the following general result. This means that sequences exist with any growth rate.

Theorem 2. For any sequence $1 < c_1 < c_2 < \cdots$, there exists a sequence $1 < a_1 < a_2 < \cdots$ of integers with $a_i > c_i$ for every positive integer i such that for any integers $l \ge 1$ and n, there are infinitely many positive integers $m \ge l$ for which there is a choice of $\varepsilon_i = \pm 1$ with $n = \sum_{i=1}^m \varepsilon_i a_i$.

Remark. For any given C > 1, let $c_i = C^i$. Problems 1 and 2 immediately follow from Theorem 2.

In the proof of Theorem 2, some consecutive terms are "near". If we require that consecutive terms are not "near", i.e. $a_{k+1} \ge \alpha a_k$ for all $k \ge 1$, what is the largest possible value of α ? We have the following precise result.

Theorem 3. (i) If $\{a_i\}$ is a sequence of positive integers such that, for some integer n_0 and infinitely many positive integers l, there is an integer m > l and a choice of $\varepsilon_i = \pm 1$ with

$$\sum_{i=l}^{m} \varepsilon_i a_i = n_0,$$

then there are infinitely many positive integers k such that $a_{k+1} \leq 2a_k - 1$.

(ii) Let a_1 be any positive integer, and define the sequence $\{a_i\}$ by $a_{k+1} = 2a_k - 1$ for all $k \ge 1$. Then for any integers $l \ge 1$ and n, there are infinitely many positive integers m > l for which there is a choice of $\varepsilon_i = \pm 1$ with

$$n = \sum_{i=l}^{m} \varepsilon_i a_i.$$

Now we consider only representations of the form

$$n = \sum_{i=1}^{m} \varepsilon_i a_i.$$

For this purpose we introduce the following definition.

Definition 1. Let M be a positive integer. The subsequence $\{a_l, a_{l+1}, \ldots, a_k\}$ $(k \ge l)$ is said to be M-coprime if $gcd(M, a_l, a_{l+1}, \ldots, a_k) = 1$.

The condition (ii) in Theorem 1 implies that there are infinitely many disjoint M-coprime subsequences.

Theorem 4. Let a_1, a_2, \ldots be a sequence of integers satisfying (i) there exists a positive integer M such that for any $l \ge 1$ there exists an integer $r = r_l \ge l$ and $\varepsilon_i \in \{-1, 1\} (l \le i \le r)$ with

$$M = \varepsilon_l a_l + \dots + \varepsilon_r a_r;$$

(ii) there are at least M disjoint M-coprime subsequences of $\{a_1, a_2, \ldots\}$. Then every integer can be represented in the form

$$n = \varepsilon_1 a_1 + \varepsilon_2 a_2 + \dots + \varepsilon_t a_t, \quad \varepsilon_i \in \{-1, 1\}, \quad i = 1, 2, \dots, t.$$

2. Proof of theorems

Proof of Theorem 1. First we assume that for any integers $l \ge 1$ and n, there exists an integer $t \ge l$ and a choice of $\varepsilon_i = \pm 1$ such that

$$n = \varepsilon_l a_l + \varepsilon_{l+1} a_{l+1} + \dots + \varepsilon_t a_t.$$

We take n = 1. Then (i) holds with M = 1. (ii) follows from

$$gcd(a_l, a_{l+1}, \ldots, a_t) = 1.$$

Now we assume that both (i) and (ii) hold.

For any $l \ge 1$, let $r = r_l$ and $s = r_{r+1}$. We have

$$2M = \varepsilon_l a_l + \dots + \varepsilon_r a_r + \varepsilon_{r+1} a_{r+1} + \dots + \varepsilon_s a_s.$$

So we may assume that M is a positive even number.

Fix an integer $l \ge 1$. Let

$$A_k = \{\varepsilon_l a_l + \varepsilon_{l+1} a_{l+1} + \dots + \varepsilon_k a_k : \varepsilon_i \in \{-1, 1\} (l \le i \le k)\}, \quad k = l, l+1, \dots$$

and

$$A_k = \{b : 0 \le b \le M - 1, b \equiv a \pmod{M}, a \in A_k\}, k = l, l + 1, \dots$$

Let $m_k = |A_k|$. Since the numbers in A_k have the same parity, we know that the numbers in \overline{A}_k have the same parity. Thus $1 \leq m_k \leq M/2$ for all $k \geq l$. It is clear that $m_l \leq m_{l+1} \leq \cdots$. So there exist two positive integers m and $i_0 > l$ such that $m_i = m$ for all $i \geq i_0$. Let

$$A_i = \{b_{i1}, b_{i2}, \dots, b_{im}\}, \quad i \ge i_0.$$

By $|\bar{A}_{i+1}| = m$, we have

$$(b_{i1} - a_{i+1}) + (b_{i2} - a_{i+1}) + \dots + (b_{im} - a_{i+1})$$

$$\equiv (b_{i1} + a_{i+1}) + (b_{i2} + a_{i+1}) + \dots + (b_{im} + a_{i+1}) \pmod{M}.$$

Thus $2ma_{i+1} \equiv 0 \pmod{M}$ for all $i \geq i_0$. By (ii) we have $2m \equiv 0 \pmod{M}$. Since $1 \leq m \leq M/2$, we have m = M/2. Again, by (ii), there exists an integer $j \geq i_0$ such that a_{j+1} is odd. Hence

$$\bar{A}_{j} \cup \bar{A}_{j+1} = \{0, 1, \dots, M-1\}$$

Thus, for any integer n, there exists k = j or j + 1, $\varepsilon_i \in \{-1, 1\} (i = l, l + 1, ..., k)$ and an integer u such that

$$n = \varepsilon_l a_l + \varepsilon_{l+1} a_{l+1} + \dots + \varepsilon_k a_k + uM.$$

By (i) we obtain a proof of Theorem 1.

Proof of Theorem 2. Let

$$a_{2i-1} = [c_{2i}] + 2i + 1, \quad a_{2i} = [c_{2i}] + 2i + 2, \quad i = 1, 2, \dots$$

Then $a_{2i+1} > a_{2i} > a_{2i-1} > [c_{2i}] + 1 > c_{2i} > c_{2i-1}$ for all $i \ge 1$.

Let l, n be two integers with $l \ge 1$. For any integer j > l + |n| we have $n + a_l + \cdots + a_{2j} > n + 2j > 0$. Let $t = n + \sum_{i=l}^{2j} a_i$. Then

$$n = \sum_{i=l}^{2j} (-a_i) + \sum_{i=j+1}^{j+t} (-a_{2i-1} + a_{2i}).$$

Hence, for every integer *n* there are infinitely many positive integers m > l for which there is a choice of $\varepsilon_i = \pm 1$ with $n = \sum_{i=l}^{m} \varepsilon_i a_i$. This completes the proof of Theorem 2.

Proof of Theorem 3. (i) Suppose that $\{a_n\}$ is a sequence of positive integers satisfying the condition, but $a_{k+1} \ge 2a_k$ for all $k \ge k_0$. Then $a_k \to +\infty$ as $k \to +\infty$. Take an integer $l \ge k_0$ with $a_l > |n_0|$ for which there is an integer m > l and a choice of $\varepsilon_i = \pm 1$ with

$$\sum_{i=l}^{m} \varepsilon_i a_i = n_0$$

Thus

$$a_{m} = \varepsilon_{m} n_{0} + \sum_{i=l}^{m-1} -\varepsilon_{m} \varepsilon_{i} a_{i} \leq |n_{0}| + \sum_{i=l}^{m-1} a_{i}$$

$$< a_{l} + \sum_{i=l}^{m-1} a_{i} = 2a_{l} + \sum_{i=l+1}^{m-1} a_{i}$$

$$\leq a_{l+1} + \sum_{i=l+1}^{m-1} a_{i} = 2a_{l+1} + \sum_{i=l+2}^{m-1} a_{i}$$

$$\leq \cdots$$

$$\leq 2a_{m-2} + a_{m-1} \leq 2a_{m-1},$$

a contradiction with $a_{k+1} \ge 2a_k$ for all $k \ge k_0$. This completes the proof of Theorem 3 (i).

(ii) Let $l \ge 1$ and n be two integers. For any integer t > l + |n|, let $s = n + a_l + a_{l+1} + \cdots + a_t$. Then by $a_i \ge 1$ for all $i \ge 1$ we have $s \ge n + t - l + 1 > 0$. By $a_{k+1} = 2a_k - 1$ we have

$$a_{t+s} = a_{t+s-1} + a_{t+s-1} - 1 = a_{t+s-1} + a_{t+s-2} + a_{t+s-2} - 2$$

= ...
= $a_{t+s-1} + a_{t+s-2} + \dots + a_t + a_t - s$
= $a_{t+s-1} + a_{t+s-2} + \dots + a_t - a_{t-1} - a_{t-2} - \dots - a_l - n.$

Thus

$$n = -\sum_{i=l}^{t-1} a_i + \sum_{i=t}^{t+s-1} a_i - a_{t+s}$$

This completes the proof of Theorem 3 (ii).

Proof of Theorem 4. If M = 1, then Theorem 4 is clear. Now we assume that M > 1. Let the notation be as in the proof of Theorem 1 (the fixed l is equal to 1 and we do not assume that M is even). Let $i_0 = 0$ and let $\{a_{i_{j-1}+1}, \ldots, a_{i_j}\}(j = 1, 2, \ldots, M)$ be M disjoint M-coprime subsequences of the sequence $\{a_i\}$. It is clear that $m_{i_1} \ge 1$. If $m_{i_2} = m_{i_1}$, then, similarly to the proof of Theorem 1, we have

$$2m_{i_1}a_j \equiv 0 \pmod{M}, \quad j = i_1 + 1, \dots, i_2$$

Since $\{a_{i_1+1}, \ldots, a_{i_2}\}$ is an M-coprime subsequence, we have $2m_{i_1} \equiv 0 \pmod{M}$. If M is odd, then $m_{i_1} = M$. If M is even, then, similarly to the proof of Theorem 1, we have $m_{i_1} = M/2$. Hence, if $m_{i_1} \neq M, M/2$, then $m_{i_2} > m_{i_1}$. Continuing these arguments, we have that if M is odd, then $m_{i_M} = M$. Thus $\bar{A}_{i_M} = \{0, 1, \ldots, M-1\}$. If M is even, then $m_{i_{M/2}} = M/2$ and $m_i \leq M/2$ for all i. Since $\{a_{i_{M/2}+1}, \ldots, a_{i_{(M/2)+1}}\}$ is an M-coprime subsequence, there exists $j_0 \in \{i_{M/2}+1, \ldots, i_{(M/2)+1}\}$ such that a_{j_0} is odd. Since $m_{i_{M/2}} \leq m_{j_0-1} \leq m_{j_0} \leq M/2$, we have $m_{j_0-1} = m_{j_0} = M/2$. Since the parities of the numbers in \bar{A}_{j_0-1} and in \bar{A}_{j_0} are distinct, we have

$$\bar{A}_{j_0-1} \cup \bar{A}_{j_0} = \{0, 1, \dots, M-1\}.$$

Now the following arguments are similar to those of Theorem 1. This completes the proof of Theorem 4. $\hfill \Box$

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3. FINAL REMARKS

For a sequence $A = \{a_i\}$ (finite or infinite) and an integer n, let $\sigma_A(n)$ be the ways of representation of $n = \varepsilon_1 a_1 + \varepsilon_2 a_2 + \cdots + \varepsilon_t a_t$, $\varepsilon_i \in \{-1, 1\}$, $i = 1, 2, \ldots, t$. In the previous arguments all A satisfy $\sigma_A(n) = +\infty$. It is natural to ask: *is there a sequence* $A = \{a_i\}$ such that $1 \leq \sigma_A(n) < +\infty$ for all integers n? Now we construct such a sequence.

For $A_k = \{a_1, a_2, \dots, a_k\}$, let

$$\Delta(A_k) = \{\varepsilon_1 a_1 + \varepsilon_2 a_2 + \dots + \varepsilon_k a_k : \varepsilon_i \in \{-1, 1\}, \quad i = 1, 2, \dots, k\}.$$

Let $a_1 = 1, a_2 = 3, a_3 = 4$. Then $\Delta(A_1) = \{\pm 1\}, \ \Delta(A_2) = \{\pm 2, \pm 4\}$ and $\Delta(A_3) = \{0, \pm 2, \pm 6, \pm 8\}$. Suppose that we have $a_1, \ldots, a_k \ (k \ge 3)$. Let n_k be the least positive integer which is not in $\Delta(A_1) \cup \cdots \cup \Delta(A_k)$, and let $a_{k+1} = a_1 + \cdots + a_k + n_k$. Then $n_k \in \Delta(A_{k+1})$ and each positive integer in $\Delta(A_{k+1})$ is at least $a_{k+1} - (a_1 + \cdots + a_k) = n_k$. Thus we obtain two sequences, $A = \{a_i\}_{i=1}^{\infty}$ and $\{n_i\}_{i=3}^{\infty}$, such that $3 = n_3 < n_4 < \cdots$ and for every $k \ge 3$ we have $\sigma_A(n) = \sigma_{A_k}(n) \ge 1$ for all $n < n_k$.

Now we have proved the following theorem.

Theorem 5. There exists a strictly increasing sequence $A = \{a_i\}_{i=1}^{\infty}$ of positive integers such that $1 \leq \sigma_A(n) < +\infty$ for all integers n.

We pose the following problems here for further research.

Problem 3. Is there any (strictly increasing) sequence $A = \{a_i\}$ of positive integers such that $\sigma_A(0) = 2$ and $\sigma_A(n) = 1$ for all integers n?

Problem 4. Is there a constant c > 1 and a (strictly increasing) sequence $A = \{a_i\}$ of positive integers such that $1 \le \sigma_A(n) \le c$ for all integers n?

If $A = \{1, 3, 3^2, ...\}$, then $\sigma_A(n) \in \{0, 1\}$ for all integers n. We pose the following problem.

Problem 5. Is there any (strictly increasing) sequence $A = \{a_i\}$ of positive integers such that $\sigma_A(n) \ge 1$ for all integers n and $\sigma_A(n) = 1$ for infinitely many integers n?

Added in proof. We have known that Problem 3 is negative and that Problems 4 and 5 are affirmative.

Added after posting. We find that Problem 4 is still open.

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