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ON THE X-RANK OF A CURVE $X \subset \mathbb{P}^n$: AN EXTREMAL CASE

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ABSTRACT. Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a set $S \subset Y$ such that P is in the linear span of S. Landsberg and Teitler proved that $r_X(P) \leq n$ for any X and any P. Here we classify the pairs (X, Q), $Q \in X_{reg}$, such that all points of the tangent line T_QX (except Q) have X-rank n: $X \cong \mathbb{P}^1$ and T_QX has order of contact $\deg(X) + 2 - n$ with X at Q.

Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denotes the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-rank is defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. If $\mathrm{char}(\mathbb{K}) = 0$, then use of Bertini's theorem for base point free linear systems gives $r_X(P) \leq n + 1 - \dim(X)$ for all $P \in \mathbb{P}^n$ ([4], 5.1). When X is a Veronese embedding $\nu_d(\mathbb{P}^m)$ of a projective space $\mathbb{P}^m$, then the X-rank of a point P is called the symmetric tensor rank of P. The study of the symmetric tensor rank is an active topic of research in which the main motivations come from engineering and applied mathematics ([2], [4], [1], [3] and the references therein). Inside this large area a small chapter is dedicated to the X-rank for arbitrary X. In our opinion a good motivation for this chapter comes from the fact that in some cases the computation of $r_{\nu_d(\mathbb{P}^m)}(P)$ requires the computation of $r_X(P)$ for some curve $X \subset \nu_d(\mathbb{P}^m)$ (see the cases of border rank ≤ 3 studied in [3]). Here we prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve and $Q \in X_{reg}$. Set $d := \deg(X)$. We have $r_X(P) = n$ for all $P \in T_QX \setminus \{Q\}$ if and only if X is smooth and rational, $(T_Q \cap X)_{red} = \{Q\}$ and the scheme $T_QX \cap X$ has length $d + 2 - n$.*

In the case $d = n$ the "if" part is a consequence of the complete description of the function r_X when X is a rational normal curve proved by G. Comas and M. Seiguer ([2], Theorem 2, [4], 4.1).

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Example 1. Fix integers $d > n \geq 3$. All pairs (X, Q) as in the statement of Theorem 1 are constructed in the following way. Let $Y \subset \mathbb{P}^d$ be a rational normal curve. Fix $O \in Y$. For any integer $t > 0$ let $tO \subset Y$ denote the effective Cartier divisor tO of Y seen as a degree t zero-dimensional subscheme of $\mathbb{P}^d$ and let $\langle tO \rangle \subseteq \mathbb{P}^d$ be its linear span. We have $\dim(\langle tO \rangle) = \min\{t-1, d\}$, $\langle 1O \rangle = \{O\}$ and $\langle 2O \rangle = T_OY$. Fix a $(d-n-1)$-dimensional linear subspace W of $\langle (d-n+2)O \rangle$ such that $W \cap \langle 2O \rangle = \emptyset$. Any length d zero-dimensional subscheme of Y is linearly independent. Hence $\dim(\langle \{P_1, P_2\} \cup (d-n+2)O \rangle) = d-n+3$ for all $P_1, P_2 \in Y \setminus \{O\}$ such that $P_1 \neq P_2$ and $\dim(\langle \{P_1\} \cup (d-n+2)O \rangle) = d-n+2$ for every $P_1 \in Y \setminus \{O\}$. Since $W \subset \langle (d-n+2)O \rangle$, we get that W intersects no secant line of Y. Since $\dim(\langle \{2P_1\} \cup (d-n+2)O \rangle) = d-n+3$ for every $P_1 \in Y \setminus \{O\}$ and $W \cap \langle 2O \rangle = \emptyset$, we get that W intersects no tangent line of W. Hence the linear projection $\ell_W : \mathbb{P}^d \setminus W \to \mathbb{P}^n$ induces an isomorphism of Y onto its image. Take $X := \ell_W(Y)$ and $Q := \ell_W(O)$. Since $\langle W \cup T_OY \rangle = \langle (d+2-n)O \rangle$, T_QX has order of contact at least $d+2-n$ with X at Q.

Lemma 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve. Set $d := \deg(X)$. Assume the existence of $Q \in X_{reg}$ such that the tangent line T_QX has order of contact $d+2-n$ with X at Q. Then $(T_QX \cap X)_{red} = \{Q\}$, $X \cong \mathbb{P}^1$ and X is obtained by the construction given in Example* 1.

Proof. Fix a general $A \subset X$ such that $\sharp(A) = n-3$ and set $E := \langle A \cup T_QX \rangle$. The generality of A gives $\dim(E) = n-2$. Notice that the scheme $E \cap X$ has length at least $(d+2-n)+(n-3) = d-1$. Since X is non-degenerate, Bezout's theorem gives $X \cap E = (T_QX \cap X) \cup A$. Hence $(X \cap E)_{red} = \{Q\} \cup A \subset X_{reg}$. Hence $(T_QX \cap X)_{red} = \{Q\}$ and the linear projection from E induces a degree 1 finite morphism $u : X \to \mathbb{P}^1$. Since $\mathbb{P}^1$ is smooth, u is an isomorphism by the Zariski Main Theorem.

Any degree d embedding of $\mathbb{P}^1$ is an isomorphic linear projection of a rational normal curve of $\mathbb{P}^1$. Example 1 gives all pairs (Y, O, W) such that Y is a rational normal curve of $\mathbb{P}^d$, $O \in Y$, W is a $(d-n-1)$-dimensional linear subspace of $\mathbb{P}^d$, $W \cap Y = \emptyset$ and the linear projection $\ell_W : \mathbb{P}^d \setminus W \to \mathbb{P}^n$ induces an isomorphism of Y onto a degree d smooth curve such that $T_{\ell_W(O)}(\ell_W(Y))$ has order of contact $d+2-n$ with $\ell_W(Y)$ at $\ell_W(O)$. Since $X \cong \mathbb{P}^1$, we get the last assertion of the lemma. $\square$

Easy examples (smooth plane curves with a tangent line with order of contact d) show that Lemma 1 is wrong in $\mathbb{P}^2$.

Proof of Theorem 1. First we prove the "if" part. Fix (X, Q) such that $(T_Q \cap X)_{red} = \{Q\}$ and the scheme $T_QX \cap X$ has length $d+2-n$. Lemma 1 gives $X \cong \mathbb{P}^1$ and $(T_QX \cap X)_{red} = \{Q\}$. Fix $P \in T_QX \setminus \{Q\}$ and take $S \subset X$ computing $r_X(P)$. Hence $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Set $M := \langle S \cup T_QX \rangle$. First assume $Q \in S$. Since $T_QX = \langle \{P, Q\} \rangle$, we get $T_QX \subseteq \langle S \rangle$, i.e. $M = \langle S \rangle$. By assumption the scheme $X \cap M$ has a connected component of length at least $d+2-n$ and at least $\dim(M)-1$ further points. Bezout's theorem gives $\mathrm{length}(X \cap V) \leq d + \dim(V) + 1 - n$ for any linear subspace $V \subsetneq \mathbb{P}^n$. We get $M = \mathbb{P}^n$, i.e. $r_X(P) = n+1$, contradicting [4], 5.1. Now assume $Q \notin S$. We take $S' := S \cup \{Q\}$ in the previous reasoning and get $r_X(P) + 1 \geq n+1$. Hence $r_X(P) = n$ ([4], 5.1).

From now on we prove the "only if" part. Fix X and Q. If $\sharp((X \cap T_Q X)_{red}) \ge 2$, then $r_X(P) \in \{1, 2\}$ for all $P \in T_Q X$. Hence we may assume $(X \cap T_Q X)_{red} = \{Q\}$. Set $m := \mathrm{length}(X \cap T_Q X)$.

(a) Here we assume $n = 3$. Since $Q \in X_{reg}$, $(X \cap T_Q X)_{red} = \{Q\}$ and $m = \mathrm{length}(X \cap T_Q X)$, the linear projection from $T_Q X$ induces a degree $d - m$ morphism $\phi : X \to \mathbb{P}^1$. Since $\mathrm{char}(\mathbb{K}) = 0$, a general fiber of ϕ is formed by $d - m$ points. If $d - m \ge 2$, then we get $r_X(P) = 2$ for a general $P \in T_Q X$, a contradiction. Hence $m = d - 1$. Lemma 1 gives $X \cong \mathbb{P}^1$.

(b) Here we assume $n > 3$. Fix a general $A \subset X$ such that $\sharp(A) = n - 3$. Set $E := \langle A \cup T_Q X \rangle$. For general A we have $\dim(E) = n - 2$.

Claim. We claim that $E \cap X = (T_Q X \cap X) \cup A$ as schemes.

Proof of the Claim. Let $\ell_{T_Q X} : \mathbb{P}^n \backslash T_Q X \to \mathbb{P}^{n-2}$ denote the linear projection from the line $T_Q X$. Since $Q \in X_{reg}$, $(X \cap T_Q X)_{red} = \{Q\}$ and $m = \mathrm{length}(X \cap T_Q X)$, the linear projection $\ell_{T_Q X}$ induces a morphism $\psi : X \to \mathbb{P}^{n-2}$ such that $\deg(\psi) \cdot \deg(\psi(X)) = d - m$. If $\deg(\psi) \ge 2$, then $r_X(P) \le 2$ for a general $P \in T_Q X$. Hence we may assume $\deg(\psi) = 1$. Fix a general $B \subset \psi(X)$ such that $\sharp(B) = n - 3$. Set $A := \psi^{-1}(B)$. Since ψ is birational onto its image and B is general, we have $\sharp(A) = n - 3$. Since $\psi(X)$ is non-degenerate, we get $\dim(\langle B \rangle) = n - 4$. Since B is general in $\psi(X)$, $\sharp(B) \le n - 2$ and $\psi(X)$ is non-degenerate, a general hyperplane M of $\mathbb{P}^{n-2}$ containing B may be considered as a general hyperplane of $\mathbb{P}^{n-2}$. Hence our characteristic zero assumption gives that $\psi(X) \cap H$ is a reduced scheme in a linearly general position. Hence $\langle B \rangle \cap X = B$ as schemes. Since ψ is birational onto its image and B is general, A is general in X. Hence we get the Claim.

The Claim implies $E \cap X \subset X_{reg}$. Hence the Claim gives that the linear projection from E induces a morphism $\beta : X \to \mathbb{P}^1$ such that $\deg(\beta) = d - m + 3 - n$. If $\deg(\beta) \ge 2$, then we get the existence of $P_1, P_2 \in X_{reg} \backslash (\{Q\} \cup A)$ such that $P_1 \ne P_2$ and $E \cap \langle \{P_1, P_2\} \rangle \ne \emptyset$, i.e. $T_Q X \cap (\langle A \cup \{P_1, P_2\} \rangle) \ne \emptyset$. Moreover, $Q \notin (\langle A \cup \{P_1, P_2\} \rangle)$ if we take $\{P_1, P_2\} = \beta^{-1}(P')$ for a general $P' \in \beta(X)$, because $\beta(Q)$ is just the point $Q' \in \beta(X)$ which is the image of the osculating plane to X at Q by the linear projection from the line $T_Q X$. Hence $\deg(\beta) = 1$, i.e. $m = d + 1 - n$. Apply Lemma 1. $\square$

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