

ON THE ASYMPTOTICS OF $\Gamma_q(z)$ AS q APPROACHES 1

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ABSTRACT. In this note we give a derivation of the asymptotic main term for the q -Gamma function $\Gamma_q(z)$ as q approaches 1. Our formula is valid for all fixed $z \in \mathbb{C}$ except non-positive integers.

1. INTRODUCTION

Recall that the q -Gamma function is defined by [1, 2, 3]

$$(1.1) \quad \Gamma_q(z) = \frac{(q; q)_\infty}{(1 - q)^{z-1} (q^z; q)_\infty},$$

where

$$(1.2) \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad a \in \mathbb{C}, \quad q \in (0, 1).$$

Let $\Gamma(z)$ be the Euler Gamma function. All the standard textbooks on q -series present W. Gosper's heuristic argument for

$$(1.3) \quad \lim_{q \rightarrow 1} \Gamma_q(z) = \Gamma(z)$$

without verifying the validity of the term-by-term limiting process, [1, 2, 3]. An alternative proof by T. H. Koornwinder is also given in [1] by using a convexity argument, but all these proofs failed to give an error term. In [6] we give a proof with an error term using a q -Beta integral from [1]. In this note we will give yet another proof with an error term valid on the whole complex plane except at poles of $\Gamma(z)$.

2. MAIN RESULTS

Lemma 1. *Let $|z| < 1$ and $0 < q < 1$. Then*

$$(2.1) \quad (z; q)_\infty = \exp \left\{ - \sum_{k=1}^{\infty} \frac{z^k}{k(1 - q^k)} \right\}.$$

Proof. Using

$$(2.2) \quad \log(1 - z) = - \sum_{k=1}^{\infty} \frac{z^k}{k}, \quad |z| < 1,$$

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one obtains

$$(2.3) \quad \begin{aligned} \log(z, q)_\infty &= \sum_{j=0}^{\infty} \log(1 - zq^j) = - \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{(zq^j)^k}{k} \\ &= - \sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{j=0}^{\infty} q^{jk} = - \sum_{k=1}^{\infty} \frac{z^k}{k(1 - q^k)} \end{aligned}$$

for $q \in (0, 1)$, where all the logarithms are taken as their principle branches. Equation (2.1) follows by exponentiating (2.3). \square

Lemma 2. *For each fixed w with $\Re(w) > 0$, let $q = e^{-\pi\tau}$ with $\tau > 0$. Then*

$$(2.4) \quad (q^w; q)_\infty = \frac{\sqrt{2\pi} w^{w-1/2} \exp(-\frac{\pi}{6\tau})}{\Gamma(w) (1 - e^{-\tau\pi w})^{w-1/2}} \{1 + \mathcal{O}(\tau)\}$$

as $\tau \rightarrow 0^+$.

Proof. Taking $z = qe^{-\tau\pi w}$ in (2.1) with $\Re(w) > 0$, one obtains

$$(2.5) \quad (qe^{-\tau\pi w}, q)_\infty = \exp \left\{ - \sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k(1 - q^k)} \right\}$$

and

$$(2.6) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k(1 - q^k)} &= \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k} \left\{ \frac{q^k}{1 - q^k} - \frac{1}{k\pi\tau} + \frac{1}{2} - \frac{k\pi\tau}{12} \right\} \\ &\quad + \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k} \left\{ \frac{1}{k\pi\tau} - \frac{1}{2} + \frac{k\pi\tau}{12} \right\} \\ &= S + \frac{1}{\pi\tau} \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k^2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k} + \frac{\pi\tau}{12} \sum_{k=1}^{\infty} e^{-k\tau\pi w} \\ &= S + \frac{1}{\pi\tau} \text{Li}_2(\exp(-\pi\tau w)) + \frac{1}{2} \log(1 - e^{-\tau\pi w}) + \frac{\pi\tau}{12(\exp(\tau\pi w) - 1)}, \end{aligned}$$

where

$$(2.7) \quad S = \sum_{k=1}^{\infty} \frac{e^{-k\tau\pi w}}{k} \left\{ \frac{1}{e^{k\pi\tau} - 1} - \frac{1}{k\pi\tau} + \frac{1}{2} - \frac{k\pi\tau}{12} \right\}$$

and

$$(2.8) \quad \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| \leq 1.$$

Using [1, 4],

$$(2.9) \quad \text{Li}_2(z) + \text{Li}_2(1 - z) = \frac{\pi^2}{6} - \log z \cdot \log(1 - z),$$

one gets

$$(2.10) \quad \begin{aligned} \text{Li}_2(\exp(-\pi\tau w)) &= -\text{Li}_2(1 - \exp(-\pi\tau w)) + \frac{\pi^2}{6} + \pi\tau w \log(1 - \exp(-\pi\tau w)) \\ &= -\pi\tau w + \frac{\pi^2}{6} + \pi\tau w \log(1 - \exp(-\pi\tau w)) + \mathcal{O}(\tau^2) \end{aligned}$$

as $\tau \rightarrow 0^+$. Then (2.10) transforms (2.6) into

$$(2.11) \quad \sum_{k=1}^{\infty} \frac{q^k e^{-k\pi\tau w}}{k(1-q^k)} = S - w + \frac{\pi}{6\tau} + \left(w + \frac{1}{2}\right) \log(1 - \exp(-\pi\tau w)) \\ + \frac{\pi\tau}{12(\exp(\pi\tau w) - 1)} + \mathcal{O}(\tau).$$

Using [1],

$$(2.12) \quad \log \Gamma(w) = \left(w - \frac{1}{2}\right) \log w - w + \frac{\log(2\pi)}{2} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tw}}{t} dt,$$

one obtains

$$(2.13) \quad \begin{aligned} &\int_0^\infty \left(\frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{1}{e^t - 1}\right) \frac{e^{-tw}}{t} dt \\ &= \log \Gamma(w) - \left(w - \frac{1}{2}\right) \log w + w - \frac{\log(2\pi)}{2} - \frac{1}{12w} \end{aligned}$$

for $\Re(w) > 0$.

Let

$$(2.14) \quad \begin{aligned} I &= \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{1}{e^t - 1}\right) \frac{e^{-tw}}{t} dt \\ &= \sum_{k=1}^{\infty} \int_{(k-1)\pi\tau}^{k\pi\tau} \left(\frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{1}{e^t - 1}\right) \frac{e^{-tw}}{t} dt \end{aligned}$$

and

$$(2.15) \quad f(t) = \left(\frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{1}{e^t - 1}\right) \frac{e^{-tw}}{t}.$$

Then

$$(2.16) \quad f'(t) = \mathcal{O}(t)$$

as $t \rightarrow 0^+$ and

$$(2.17) \quad f'(t) = \mathcal{O}(\exp(-t\Re(w)))$$

as $t \rightarrow +\infty$. Subtracting (2.14) from (2.7) one obtains

$$(2.18) \quad \begin{aligned} S - I &= \sum_{k=1}^{\infty} \int_{(k-1)\pi\tau}^{k\pi\tau} dt \int_t^{k\pi\tau} f'(y) dy \\ &= \sum_{k=1}^{\infty} \int_{(k-1)\pi\tau}^{k\pi\tau} f'(y) \int_{(k-1)\pi\tau}^y dt dy \\ &= \sum_{k=1}^{\infty} \int_{(k-1)\pi\tau}^{k\pi\tau} f'(y) (y - (k-1)\pi\tau) dy. \end{aligned}$$

Then,

$$(2.19) \quad |S - I| \leq \pi\tau \int_0^{\infty} |f'(y)| dy,$$

which implies

$$(2.20) \quad S - I = \mathcal{O}(\pi\tau)$$

as $\tau \rightarrow 0^+$. Combining (2.11), (2.13), (2.14) and (2.20) one gets

$$(2.21) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k(1-q^k)} &= \log \Gamma(w) - \left(w - \frac{1}{2}\right) \log w - \frac{\log(2\pi)}{2} \\ &\quad + \frac{\pi\tau}{12} \left(\frac{1}{\exp(\tau\pi w) - 1} - \frac{1}{\pi\tau w} \right) + \frac{\pi}{6\tau} \\ &\quad + \left(w + \frac{1}{2}\right) \log(1 - \exp(-\pi\tau w)) + \mathcal{O}(\tau) \\ &= \log \Gamma(w) - \left(w - \frac{1}{2}\right) \log w - \frac{\log(2\pi)}{2} \\ &\quad + \frac{\pi}{6\tau} + \left(w + \frac{1}{2}\right) \log(1 - \exp(-\pi\tau w)) + \mathcal{O}(\tau) \end{aligned}$$

as $\tau \rightarrow 0^+$. For $\Re(w) > 0$, applying Lemma 1 to (2.21) one obtains

$$(2.22) \quad (q^{w+1}; q)_{\infty} = \frac{\sqrt{2\pi} w^{w-1/2} \exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w) (1 - e^{-\tau\pi w})^{w+1/2}} \{1 + \mathcal{O}(\tau)\}$$

as $\tau \rightarrow 0^+$ and

$$(2.23) \quad \begin{aligned} (q^w; q)_{\infty} &= (1 - e^{-\tau\pi w}) (qe^{-\tau\pi w}, q)_{\infty} \\ &= \frac{\sqrt{2\pi} w^{w-1/2} \exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w) (1 - e^{-\tau\pi w})^{w-1/2}} \{1 + \mathcal{O}(\tau)\} \end{aligned}$$

as $\tau \rightarrow 0^+$. □

Theorem 3. Let $q = \exp(-\pi\tau)$ with $\tau > 0$. Then

$$(2.24) \quad \Gamma_q(w) = \Gamma(w) \{1 + \mathcal{O}(\tau)\}$$

as $\tau \rightarrow 0^+$ for $-z \notin \mathbb{N} \cup \{0\}$.

Proof. Using (2.4) one gets

$$(2.25) \quad (q; q)_\infty = \frac{\sqrt{2\pi} \exp\left(-\frac{\pi}{6\tau}\right)}{(1 - e^{-\tau\pi})^{1/2}} \{1 + \mathcal{O}(\tau)\}$$

as $\tau \rightarrow 0^+$. For $\Re(w) > 0$, applying (2.4) and (2.25) to (1.1) one obtains

$$(2.26) \quad \Gamma_q(w) = \frac{(q; q)_\infty}{(1 - q)^{w-1} (q^w; q)_\infty} = \Gamma(w) \left\{ \frac{1 - e^{-\pi\tau w}}{w(1 - e^{-\pi\tau})} \right\}^{w-\frac{1}{2}} \{1 + \mathcal{O}(\tau)\}$$

as $\tau \rightarrow 0^+$. Then (2.26) and

$$(2.27) \quad \left\{ \frac{1 - e^{-\pi\tau w}}{w(1 - e^{-\pi\tau})} \right\}^{w-\frac{1}{2}} = 1 + \mathcal{O}(\tau)$$

as $\tau \rightarrow 0^+$ prove (2.24) for $\Re(w) > 0$.

Using [5],

$$(2.28) \quad \theta_1(v|t) = 2 \sum_{k=0}^{\infty} (-1)^k p^{(k+1/2)^2} \sin(2k+1)\pi v,$$

$$(2.29) \quad \theta_1(v|t) = 2p^{1/4} \sin \pi v (p^2; p^2)_\infty (p^2 e^{2\pi i v}; p^2)_\infty (p^2 e^{-2\pi i v}; p^2)_\infty,$$

and

$$(2.30) \quad \theta_1\left(\frac{v}{t} \mid -\frac{1}{t}\right) = -i \sqrt{\frac{t}{i}} e^{\pi i v^2/t} \theta_1(v \mid t),$$

where $p = e^{\pi i t}$ with $\Im(t) > 0$, one obtains

$$(2.31) \quad (q, q^{1+w}, q^{1-w}; q)_\infty = \frac{\exp\left(\frac{\pi\tau}{8} + \frac{\pi\tau w^2}{2}\right) \theta_1(w \mid \frac{2i}{\tau})}{\sqrt{2\tau} \sinh \frac{\pi\tau w}{2}}$$

and

$$(2.32) \quad (q; q)_\infty^3 = \frac{\sqrt{2} \exp\left(\frac{\pi\tau}{8}\right) \theta'_1(0 \mid \frac{2i}{\tau})}{\pi\tau^{3/2}},$$

where $q = \exp(-\pi\tau)$ with $\tau > 0$. Applying (2.31) and (2.32) to (1.1) one gets a reflection formula for $\Gamma_q(w)$,

$$(2.33) \quad \Gamma_q(1+w) \Gamma_q(1-w) = \frac{(q; q)_\infty^3}{(q, q^{1+w}, q^{1-w}; q)_\infty} = \frac{2\theta'_1(0 \mid \frac{2i}{\tau}) \sinh \frac{\pi\tau w}{2}}{\pi\tau \exp\left(\frac{\pi\tau w^2}{2}\right) \theta_1(w \mid \frac{2i}{\tau})}$$

or

$$(2.34) \quad \Gamma_q(w) \Gamma_q(1-w) = \frac{1-q}{1-q^w} \Gamma_q(1+w) = \frac{(e^{\pi\tau} - 1) \theta'_1(0 \mid \frac{2i}{\tau})}{\pi\tau \exp\left(\frac{\pi\tau(w^2+w+2)}{2}\right) \theta_1(w \mid \frac{2i}{\tau})}$$

for $w \notin \mathbb{Z}$. Using (2.24) for the $\Re(w) > 0$ case we proved earlier and the observations

$$(2.35) \quad \frac{e^{\pi\tau} - 1}{\pi\tau} = 1 + \mathcal{O}(\tau),$$

$$(2.36) \quad \theta'_1 \left(0 \mid \frac{2i}{\tau} \right) = 2\pi \exp \left(-\frac{\pi}{2\tau} \right) \{1 + \mathcal{O}(\tau)\},$$

$$(2.37) \quad \theta_1 \left(w \mid \frac{2i}{\tau} \right) = 2 \sin \pi w \exp \left(-\frac{\pi}{2\tau} \right) \{1 + \mathcal{O}(\tau)\},$$

$$(2.38) \quad \Gamma(w) \Gamma(1-w) = \frac{\pi}{\sin \pi w}, \quad w \notin \mathbb{Z}$$

applied to (2.34), one gets

$$(2.39) \quad \Gamma_q(w) = \frac{\pi}{\sin \pi w} \frac{1}{\Gamma(1-w)} \{1 + \mathcal{O}(\tau)\} = \Gamma(w) \{1 + \mathcal{O}(\tau)\}$$

for $\Re(w) < 1$ and $w \notin \mathbb{Z}$ as $\tau \rightarrow 0^+$. The theorem follows by combining the $\Re(w) > 0$ and $\Re(w) < 1$ cases. \square

REFERENCES

- [1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999. MR1688958 (2000g:33001)
- [2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990. MR1052153 (91d:33034)
- [3] Mourad E. H. Ismail, *Continuous and Discrete Orthogonal Polynomials*, Cambridge University Press, Cambridge, 2005.
- [4] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, New York, 1981. MR618278 (83b:33019)
- [5] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, Cambridge, 1962. MR0178117 (31:2375)
- [6] R. Zhang, On asymptotics of q-gamma functions, *Journal of Mathematical Analysis and Applications* **339**, no. 2 (2008), 1313-1321. MR2377088 (2008k:33063)

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