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DECOMPOSITIONS OF LOOPED CO-H-SPACES

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Abstract. We prove two homotopy decomposition theorems for the loops on simply-connected co-H-spaces, including a generalization of the Hilton-Milnor Theorem. Several examples are given.

1. Introduction

A central theme in mathematics is to decompose objects into products of simpler ones. The smaller pieces should then be simpler to analyze, and by understanding how the pieces are put back together information is obtained about the original object. In homotopy theory this takes the form of decomposing H-spaces as products of factors or decomposing co-H-spaces as wedges of summands. Powerful decomposition techniques have been developed. Some, such as those in [MNT, CMN1], are concerned with decomposing specific spaces as finely as possible, while others, such as those in [SW1, STW2], are concerned with functorial decompositions that are valid for all loop suspensions or looped co-H-spaces.

In this paper we establish two new decomposition theorems that apply to the loops on simply-connected co-H-spaces. One is a strong refinement of a result in [STW2], and the other is a generalized Hilton-Milnor Theorem, which depends on methods introduced in [GTW]. We give several examples of how the two decompositions can be used, and combine the two to produce a complete decomposition of the loops on a simply-connected co-H-space into a product of factors which are functorially indecomposable (a term to be defined momentarily).

We work in the context of functorial decompositions. As such, it may be useful to define some terms. Let **CoH** be the category of simply-connected co-H-spaces and co-H-maps and let **Top** be the category of topological spaces and continuous maps. A *functorial homotopy decomposition* is a pair of functors $A, B\colon \mathbf{CoH} \longrightarrow \mathbf{Top}$ with the property that for every $Y \in \mathbf{CoH}$ there is a natural homotopy decomposition $\Omega Y \simeq A(Y)\times B(Y)$. In effect, this means that the construction of the decomposition can be applied to any simply-connected co-H-space Y, and the output is natural for

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co-H-maps between co-H-spaces. Similarly, one can define a *functorial homotopy retract*. The space $A(Y)$ is *functorially indecomposable* if there is no other functorial homotopy decomposition of ΩY which factors $A(Y)$ into a product of nontrivial factors. The decompositions of spaces we produce are predicted by corresponding coalgebra decompositions of tensor algebras. Let $\mathcal{V}$ be the category of graded $\mathbb{Z}/p\mathbb{Z}$-modules and morphisms, and let **Coalg** be the category of coalgebras and coalgebra morphisms. A *functorial coalgebra decomposition* is a pair of functors $A, B\colon \mathcal{V} \longrightarrow \mathbf{Coalg}$ with the property that for every $V \in \mathcal{V}$ there is a natural coalgebra decomposition $T(V) \cong A(V) \otimes B(V)$, where $T(V)$ is the free tensor algebra on V. Similarly, we can define *functorial coalgebra retracts* and, as above, we can also define *functorially indecomposable* coalgebras.

We now motivate and state our results. Let p be an odd prime, and localize all spaces and maps at p. Take homology with mod-p coefficients. Let V be a graded module over $\mathbb{Z}/p\mathbb{Z}$ and let $T(V)$ be the tensor algebra on V. This tensor algebra is given a Hopf algebra structure by declaring that the generators are primitive and extending multiplicatively. In [SW1] it was shown that there is a functorial coalgebra decomposition $T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$, where $A^{\min}(V)$ is the minimal functorial coalgebra retract of $T(V)$ that contains V. The minimal statement means that $A^{\min}(V)$ is functorially indecomposable. One important property of this decomposition is that the primitive elements of $T(V)$ of tensor length not a power of p are all contained in the complement $B^{\max}(V)$. A programme of work ensued to geometrically realize these tensor algebra decompositions, which we now outline.

By the Bott-Samelson theorem, there is an algebra isomorphism $H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$. This was generalized in [Be2] to the case of a simply-connected co-H-space Y: there is an algebra isomorphism $H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y))$, where $\Sigma^{-1}\widetilde{H}_*(Y)$ is the desuspension by one degree of the graded module $\widetilde{H}_*(Y)$. Let $V = \Sigma^{-1}\widetilde{H}_*(Y)$ so $H_*(\Omega Y) \cong T(V)$. The coalgebra decomposition of $T(V)$ suggests that there are spaces $A^{\min}(Y)$ and $B^{\max}(Y)$ such that $\widetilde{H}_*(A^{\min}(Y)) \cong A^{\min}(V)$, $\widetilde{H}_*(B^{\max}(Y)) \cong B^{\max}(V)$, and there is a homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times B^{\max}(Y)$. Such a decomposition was realized in a succession of papers [SW1, SW2, STW1, STW2] which began with Y being a p-torsion double suspension and ended with the general case of Y being a simply-connected co-H-space.

However, the story does not end there, as the module $B^{\max}(V)$ has a much richer structure. There is a coalgebra decomposition $B^{\max}(V) \cong T(\bigoplus_{n=2}^{\infty} Q_nB(V))$, where $Q_nB(V)$ is a functorial retract of $V^{\otimes n}$. Ideally, this should be geometrically realized as well. This was proved in [STW1] when Y is a simply-connected, homotopy coassociative co-H-space. More precisely, there are spaces $Q_nB(Y)$ for $n \geq 2$ such that $\widetilde{H}_*(Q_nB(Y)) \cong \Sigma Q_nB(V)$, a homotopy fibration sequence $\Omega Y \xrightarrow{*} A^{\min}(Y) \longrightarrow \bigvee_{n=2}^{\infty} Q_nB(Y) \longrightarrow Y$, and a homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times \Omega(\bigvee_{n=2}^{\infty} Q_nB(Y))$.

In the more general case of a simply-connected co-H-space Y, the geometric realization of $A^{\min}(V)$ in [STW2] produced a homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times B^{\max}(Y)$, but it did not identify $B^{\max}(Y)$ as a loop space. The first goal of this paper is to do exactly that.

Theorem 1.1. *Let Y be a simply-connected co-H-space and let $V = \Sigma^{-1}\widetilde{H}_*(Y)$. There is a natural homotopy fibration sequence*

$$\Omega Y \longrightarrow A^{\min}(Y) \longrightarrow \bigvee_{n=2}^{\infty} Q_nB(Y) \longrightarrow Y$$

such that:

1) $\Omega Y \simeq A^{\min}(Y) \times \Omega(\bigvee_{n=2}^{\infty} Q_nB(Y))$;
2) $\widetilde{H}_*(A^{\min}(Y)) \cong A^{\min}(V)$;
3) *for each* $n \geq 2$, $\widetilde{H}_*(Q_nB(Y)) \cong \Sigma Q_nB(V)$.

In fact, Theorem 1.1 is a special case of a more general theorem proved in Section 2 which geometrically realizes any natural coalgebra-split sub-Hopf algebra $B(V)$ of $T(V)$ as a loop space.

The construction of the space $Q_nB(Y)$ exists by a suspension splitting result from [GTW]. To describe this, recall that James [J] proved that there is a homotopy decomposition $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$, where $X^{(n)}$ is the n-fold smash of X with itself. Note that $\widetilde{H}_*(\Sigma X^{(n)}) \cong \Sigma \widetilde{H}_*(X)^{\otimes n}$. James' decomposition was generalized in [GTW]. If Y is a simply-connected co-H-space, then there is a homotopy decomposition $\Sigma\Omega Y \simeq \bigvee_{n=1}^{\infty}[\Sigma\Omega Y]_n$, where each space $[\Sigma\Omega Y]_n$ is a co-H-space and there is an isomorphism $\widetilde{H}_*([\Sigma\Omega Y]_n) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$. Succinctly, $[\Omega\Sigma Y]_n$ is an $(n-1)$-fold desuspension of $Y^{(n)}$. A key point is that the space $Q_nB(Y)$ is a retract of the co-H-space $[\Sigma\Omega Y]_n$, so it too is a co-H-space.

Our second result is a generalization of the Hilton-Milnor Theorem, touched upon in [GTW]. Recall that the Hilton-Milnor Theorem states that if $X_1, \ldots, X_m$ are path-connected spaces, then there is a homotopy decomposition

$$\Omega(\Sigma X_1 \vee \cdots \vee \Sigma X_m) \simeq \prod_{\alpha \in \mathcal{I}} \Omega(\Sigma X_1^{(\alpha_1)} \wedge \cdots \wedge X_m^{(\alpha_m)}),$$

where $\mathcal{I}$ runs over a Hall space basis of the (ungraded) free Lie algebra $L\langle x_1, \ldots, x_m\rangle$, and if w_α is the basis element corresponding to α, then α_i counts the number of occurances of x_i in w_α. Note that if $\alpha_i = 0$, then, for example, we regard $X_i^{(\alpha_i)} \wedge X_j^{(\alpha_j)}$ as $X_j^{(\alpha_j)}$ rather than as $* \wedge X^{(\alpha_j)} \simeq *$. We generalize the Hilton-Milnor Theorem by replacing each ΣX_i by a simply-connected co-H-space.

Theorem 1.2. *Let $Y_1, \ldots, Y_m$ be simply-connected co-H-spaces. There is a natural homotopy decomposition*

$$\Omega(Y_1 \vee \cdots \vee Y_m) \simeq \prod_{\alpha \in \mathcal{I}} \Omega M((Y_i, \alpha_i)_{i=1}^m),$$

where $\mathcal{I}$ runs over a vector space basis of the free Lie algebra $L\langle y_1, \ldots, y_m\rangle$ and:

1) *each space $M((Y_i, \alpha_i)_{i=1}^m)$ is a simply-connected co-H-space;*
2) $\widetilde{H}_*(M((Y_i, \alpha_i)_{i=1}^m)) \cong \Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes \alpha_1} \otimes \cdots \otimes (\Sigma^{-1}\widetilde{H}_*(Y_m))^{\otimes \alpha_m}\right)$;
3) *if $Y_i = \Sigma X_i$ for $1 \leq i \leq m$, then $M((Y_i, \alpha_i)_{i=1}^m) \simeq \Sigma X^{(\alpha_1)} \wedge \cdots \wedge X^{(\alpha_m)}$.*

Again, if $\alpha_i = 0$ we interpret $(\Sigma^{-1}\widetilde{H}_*(Y_i))^{\otimes \alpha_i} \otimes (\Sigma^{-1}\widetilde{H}_*(Y_j))^{\otimes \alpha_j})$ as $\Sigma^{-1}\widetilde{H}_*(Y_j))^{\alpha_j}$ rather than 0. Note that Theorem 1.2 (3) is the usual Hilton-Milnor Theorem.

Theorems 1.1 and 1.2 are very useful for producing homotopy decompositions of interesting spaces. To obtain new examples, we consider co-H-spaces which are not suspensions. The simplest example of such a space is the two-cell complex A defined as the homotopy cofibre of the map $S^{2p} \xrightarrow{\alpha_1} S^3$, where α_1 represents the generator of $\pi_{2p}(S^3)$. By [Be1], A is not even homotopy coassociative. By Theorem 1.1, $\Omega A \simeq A^{\min}(A) \times \Omega(\bigvee_{i=2}^{\infty} Q_nB(Y))$, a much more structured decomposition than the decomposition $\Omega A \simeq A^{\min}(A) \times B^{\max}(A)$ of [STW2]. The generalization of the Hilton-Milnor theorem has a wealth of applications. As an example of wide interest, by [MNT] there is a homotopy decomposition $\Sigma\mathbb{C}P^n \simeq \bigvee_{i=1}^{p-1} A_i$, where $H_*(A_i)$ consists of those elements in $H_*(\Sigma\mathbb{C}P^n)$ in degrees of the form $2i+1+2j(p-1)$ for some $j \geq 0$. The spaces A_i are co-H-spaces since they retract off a suspension, but they are usually not suspensions themselves. Applying Theorem 1.2, we obtain a decomposition of $\Omega\Sigma\mathbb{C}P^n \simeq \Omega(\bigvee_{i=1}^{p-1} A_i)$, which gives a great deal of new information about the homotopy theory of $\Sigma\mathbb{C}P^n$. Another example of how the generalized Hilton-Milnor theorem can be used is as follows. The symmetric group Σ_k on k letters acts on $A^{(k)}$ by permuting the smash factors. Suspending so one can add, we obtain a map from the group ring $\mathbb{Z}_{(p)}[\Sigma_k]$ to the set of homotopy classes of maps $[A^{(k)}, A^{(k)}]$. In particular, suppose that $e_1, \ldots, e_l \in \mathbb{Z}_{(p)}[\Sigma_k]$ is a family of mutually orthogonal idempotents such that $e_1 + \cdots + e_l = 1$. Then the corresponding maps e_i induce idempotents in homology. Let $T(e_i)$ be the mapping telescope of e_i. Then the sum of the telescope maps $\Sigma A^{(k)} \longrightarrow \bigvee_{i=1}^{l} T(e_i)$ induces an isomorphism in homology and so is a homotopy equivalence. Each $T(e_i)$ is therefore a retract of a suspension, implying that it is a co-H-space. Applying Theorem 1.2, we then have a decomposition of $\Omega\Sigma A^{(k)} \simeq \Omega(\bigvee_{i=1}^{l} T(e_i))$ which gives a great deal of information about the homotopy theory of $\Sigma A^{(k)}$. A useful special case is $\Sigma A^{(2)}$ when the idempotents are $e_1 = \frac{1}{2}(1 - \Sigma T)$ and $e_2 = 1 - e_1$, where T is the self-map of $A^{(2)}$ that interchanges the factors.

Combining Theorems 1.1 and 1.2 allows for a complete decomposition of the loops on a co-H-space into a product of functorially indecomposable spaces. That is, Theorem 1.1 states that for a simply-connected co-H-space Y there is a homotopy decomposition

$$\Omega Y \simeq A^{\min}(Y) \times \Omega(\bigvee_{n=2}^{\infty} Q_nB(Y)),$$

where $A^{\min}(Y)$ is functorially indecomposable. Since each space $Q_nB(Y)$ is a co-H-space, Theorem 1.2 implies that there is a homotopy decomposition

$$\Omega(\bigvee_{n=2}^{\infty} Q_nB(Y)) \simeq \prod_{\alpha\in\mathcal{I}} \Omega M((Q_nB(Y), \alpha_n)_{n=2}^{\infty}),$$

where each space $M((Q_nB(Y), \alpha_n)_{n=2}^{\infty})$ is a simply-connected co-H-space. The homotopy decomposition in Theorem 1.1 can now be applied to each of the factors $\Omega M((Q_nB(Y), \alpha_n)_{n=2}^{\infty})$ to produce an $A^{\min}$ that is functorially indecomposable and a complementary factor which is the loop on a wedge of simply-connected co-H-spaces. Iterating, we obtain a decomposition of ΩY as a product of $A^{\min}$'s.

Theorem 1.3. *Let Y be a simply-connected co-H-space. Then there is a functorial homotopy decomposition*

$$\Omega Y \simeq \prod_{\gamma\in\mathcal{J}} A^{\min}(Y_\gamma)$$

for some index set $\mathcal{J}$, where each Y_γ is a simply-connected co-H-space and each factor $A^{\min}(Y_\gamma)$ is functorially indecomposable.

Theorem 1.3 can be thought of as a functorial analogue of Cohen, Moore and Neisendorfer's [CMN2] complete and explicit decomposition of the loops on a Moore space as a product of indecomposable factors.

2. Geometric realization of natural coalgebra-split sub-Hopf algebras

In this section we prove Theorem 1.1 as a special case of the more general Theorem 2.3. This gives conditions for when a sub-Hopf algebra of a tensor algebra has a geometric realization as a loop space. Before proving Theorem 2.3 it will be useful to state two preliminary results.

The first preliminary result is Theorem 2.1, a geometric realization statement from [STW2]. Recall that p is an odd prime, the ground ring for all algebraic statements is $\mathbb{Z}/p\mathbb{Z}$, and all spaces and maps have been localized at p. In general, if W is a (graded or ungraded) $\mathbb{Z}/p\mathbb{Z}$-module, the tensor algebra $T(W)$ is given the structure of a Hopf algebra by saying that W is primitive. Thus T is a functor from modules to coalgebras. A *coalgebra retract* of the functor T is a functor A from ungraded modules to coalgebras such that A is a retract of T as functors from ungraded modules to coalgebras. This means that there exist natural transformations $s\colon A \longrightarrow T$ and $r\colon T \longrightarrow A$ such that:

1) $s_W\colon A(W) \longrightarrow T(W)$ and $r_W\colon T(W) \longrightarrow A(W)$ must be natural coalgebra maps for every ungraded module W;
2) the composite $r_W \circ s_W\colon A(W) \to A(W)$ is the identity.

According to [SW1], the functor admits a canonical extension as a functor from graded modules to graded coalgebras. Thus any coalgebra retract A of the functor T on ungraded modules extends canonically to a coalgebra retract of T on graded modules.

Now given a particular vector space, say $E(\bar{V})$, we say that $E(\bar{V})$ is a *functorial coalgebra retract* of $T(V)$ if:

a) there exists a coalgebra retract A of the functor T, so $A(W)$ must a natural coalgebra retract of $T(W)$ for any ungraded module W;
b) A extends canonically to a functor on graded modules (where the same name is used to denote both A and its extension);
c) evaluating on the particular module $\bar{V}$, we obtain a coalgebra isomorphism $A(\bar{V}) = E(\bar{V})$.

Theorem 2.1. *Suppose that the free graded tensor algebra functor T has a coalgebra retract A. Then for any graded module V, $A(V)$ has a geometric realization. That is, if Y is a simply-connected co-H-space such that there is an algebra isomorphism $H_*(\Omega Y) \cong T(V)$, then there is a functorial homotopy retract $\bar{A}(Y)$ of ΩY with the property that $H_*(\bar{A}(Y)) \cong A(V)$.*

For the interested reader, we have included an appendix which gives an example to distinguish between a coalgebra retract and a functorial coalgebra retract of $T(V)$, and how this fits with the existence or nonexistence of certain finite H-spaces. As well, it might be noted that while we have restricted to an odd prime, this geometric realization theorem is also useful at the prime 2. For example, in [CW], a very early version of these methods was used to produce a large family of elements of order 8 in the homotopy groups of mod-2 Moore spaces, and in [GSW] an earlier version of the geometric realization theorem for loop suspensions was used to identify the "bottom" indecomposable factor of the loops on an odd-dimensional mod-2 Moore space.

Let us return to odd primes. For the second preliminary result, given a functorial coalgebra retract $A(V)$ of $T(V)$, let $A_n(V)$ be the component of $A(V)$ consisting of homogeneous elements of tensor length n. The following suspension splitting theorem was proved in [GTW].

Theorem 2.2. *Let $A(V)$ be any functorial coalgebra retract of $T(V)$ and let $\bar{A}$ be the geometric realization of A in Theorem* 2.1*. Then for any simply-connected co-H-space Y of finite type, there is a functorial homotopy decomposition*

$$\Sigma\bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} \bar{A}_n(Y)$$

such that $\bar{A}_n(Y)$ is a functorial retract of $[\Sigma\Omega Y]_n$ and there is a coalgebra isomorphism

$$\widetilde{H}_*(\bar{A}_n(Y)) \cong A_n(\Sigma^{-1}\widetilde{H}_*(Y))$$

for each $n \geq 1$.

Now suppose that $B(V)$ is a sub-Hopf algebra of $T(V)$. We say that $B(V)$ is *coalgebra-split* if the inclusion $B(V) \longrightarrow T(V)$ has a natural coalgebra retraction. Observe that the weaker property of $B(V)$ being a sub-coalgebra of $T(V)$ which splits off $T(V)$ implies by Theorem 2.1 that $B(V)$ has a geometric realization $\overline{B}$. We aim to show that the full force of $B(V)$ being a sub-Hopf algebra of $T(V)$ implies that it has a much more structured geometric realization.

If M is a Hopf algebra, let QM be the set of indecomposable elements of M, and let IM be the augmentation ideal of M. If $B(V)$ is a natural sub-Hopf algebra of $T(V)$, then there is a natural epimorphism $IB(V) \longrightarrow QB(V)$. Let $T_n(V)$ be the component of $T(V)$ consisting of the homogeneous tensor elements of length n, and let $B_n(V) = IB(V) \cap T_n(V)$. Let $Q_nB(V)$ be the quotient of $B_n(V)$ in $QB(V)$. Recall that **CoH** is the category of simply-connected co-H-spaces and co-H-maps and let $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$.

Theorem 2.3. *Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ and let $\bar{B}$ be its geometric realization. Then there exist functors $\bar{Q}_nB$ from* **CoH** *to spaces such that for any $Y \in$* **CoH***:*

1) *$\bar{Q}_nB(Y)$ is a functorial retract of $[\Sigma\Omega Y]_n$;*
2) *there is a functorial coalgebra isomorphism*

$$\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y)) \cong Q_nB(\Sigma^{-1}\bar{H}_*(Y));$$

3) *there is a natural homotopy equivalence*

$$\bar{B}(Y) \simeq \Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_nB(Y)\right).$$

Proof. The proof is to give a geometric construction for the indecomposables of $B(V)$. Let $B^{[n]}(V)$ be the sub-Hopf algebra generated by $Q_iB(V)$ for $i \leq n$. By the method of proof of [LLW, Theorem 1.1], each $B^{[n]}(V)$ is a natural coalgebra-split sub-Hopf algebra of $T(V)$, and there is a natural coalgebra decomposition

$$B^{[n]}(V) \cong B^{[n-1]}(V) \otimes A^{[n]}(V), \tag{2.1}$$

where $A^{[n]}(V) = \mathbf{k} \otimes_{B^{[n-1]}(V)} B^{[n]}(V)$. Note that

$$Q_n B(V) \cong A^{[n]}(V)_n.$$

By Theorem 2.1, the functorial coalgebra splitting in (2.1) has a geometric realization as a natural homotopy decomposition

$$\bar{B}^{[n]}(Y) \simeq \bar{B}^{[n-1]}(Y) \times \bar{A}^{[n]}(Y) \tag{2.2}$$

for some $Y \in \mathbf{CoH}$. This induces a filtered decomposition with respect to the augmentation ideal filtration of $H_*(\Omega Y)$. By Theorem 2.2,

$$\Sigma \bar{A}^{[n]}(Y) \simeq \bigvee_{k=1}^{\infty} \bar{A}^{[n]}_k(Y),$$

where $\bar{A}^{[n]}_k(Y)$ is a functorial retract of $[\Sigma\Omega Y]_n$ and $\bar{A}^{[n]}_k(Y) \simeq *$ for $k < n$ because $A^{[n]}_k(V) = 0$ for $0 < k < n$. Define

$$\bar{Q}_n B(Y) = \bar{A}^{[n]}_n(Y).$$

Let ϕ_n be the composite of inclusions

$$\begin{array}{rcl} \bar{Q}_n B(Y) = \bar{A}^{[n]}_n(Y) & \longrightarrow & \Sigma \bar{A}^{[n]}(Y) \\ & \longrightarrow & \Sigma(\bar{B}^{[n-1]}(Y) \times \bar{A}^{[n]}(Y)) \\ & \xrightarrow{\simeq} & \Sigma \bar{B}(Y) \\ & \longrightarrow & \Sigma\Omega Y. \end{array} \tag{2.3}$$

Consider the composite

$$\Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_n B(Y)\right) \xrightarrow{\Omega(\bigvee_{n=1}^{\infty} \phi_n)} \Omega\Sigma\Omega Y \xrightarrow{\Omega\sigma} \Omega Y \xrightarrow{r} \bar{B}(Y), \tag{2.4}$$

where σ is the evaluation map and r is the retraction map. We wish to show that this composite induces an isomorphism in homology, implying that it is a homotopy equivalence. The assertions of the theorem would then follow. To show that (2.4) induces an isomorphism in homology, it suffices to filter appropriately and show that we obtain an isomorphism of associated graded objects.

Let

$$H_*(\Omega\Sigma\Omega Y) = T(\bar{H}_*(\Omega Y))$$

be filtered by

$$I^n H_*(\Omega\Sigma\Omega Y) = \sum_{t_1 r_1 + \cdots + t_s r_s \geq n} (I^{t_1} H_*(\Omega Y))^{\otimes r_1} \otimes \cdots \otimes (I^{t_s} H_*(\Omega Y))^{\otimes r_s}.$$

Filter $H_*(\Omega Y)$ by the augmentation ideal filtration. Then

$$\Omega\sigma_* \colon H_*(\Omega\Sigma\Omega Y) \longrightarrow H_*(\Omega Y)$$

is a filtered map since $\Omega\sigma_*$ is an algebra map. Let $H_*(\bar{B}(Y))$ be filtered subject to the augmentation ideal filtration of $H_*(\Omega Y)$. Then r_* is a filtered map. Note that as an algebra

$$H_*\left(\Omega\left(\bigvee_{n=1}^{\infty}\bar{Q}_nB(Y)\right)\right)=T\left(\bigoplus_{n=1}^{\infty}\Sigma^{-1}(\bar{H}_*(\bar{Q}_nB(Y)))\right),$$

which is filtered by

$$\sum_{i_1+\cdots+i_t\geq n}\Sigma^{-1}(\bar{H}_*(\bar{Q}_{i_1}B(Y)))\otimes\cdots\otimes\Sigma^{-1}(\bar{H}_*(\bar{Q}_{i_t}B(Y))).$$

Observe that ϕ_{n*} maps $\bar{H}_*(\bar{Q}_nB(Y))$ into $\Sigma I^nH_*(\Omega Y)$ and the composite

$$\bar{H}_*(\bar{Q}_nB(Y))\xrightarrow{\phi_{n*}}\Sigma I^nH_*(\Omega Y)\twoheadrightarrow\Sigma I^nH_*(\Omega Y)/\Sigma I^{n+1}H_*(\Omega Y)\tag{2.5}$$

is a monomorphism because $\bar{Q}_nB(Y)$ is obtained from the n-homogeneous component of $\Sigma\bar{A}^{[n]}(Y)$. Thus $\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*$ is a filtered map and the image of

$$E^0(\Omega\sigma_*\circ\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*)$$

is the sub-Hopf algebra of $E^0H_*(\Omega Y)=T(\Sigma^{-1}\bar{H}_*(Y))$ generated by

$$E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y)))$$

for $n\geq 1$. From (2.5),

$$\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y))\cong E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y))).$$

By the construction of ϕ_n in (2.3), the modules

$$\{E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y)))\}$$

are algebraically independent because $\bar{Q}_iB(Y)$ is mapped into $\Sigma B^{[n]}(Y)$ for $i\leq n$ and $\bar{Q}^{[n]}(Y)$ is mapped into $\Sigma\bar{A}^{[n]}(Y)$, which is the complement to $\Sigma\bar{B}^{[n-1]}(Y)$. Since each $\bar{Q}_nB(Y)$ is mapped into $\Sigma\bar{B}(Y)$,

$$\mathrm{Im}(E^0(\Omega\sigma_*\circ\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*))=T(E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y))))$$

is a sub-Hopf algebra of $E^0H_*(\bar{B}(Y))\subseteq T(\Sigma^{-1}\bar{H}_*(Y))$. By computing the Poincaré series,

$$\mathrm{Im}(E^0(\Omega\sigma_*\circ\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*))=E^0H_*(\bar{B}(Y)).$$

Since $r\colon\Omega Y\longrightarrow\bar{B}(Y)$ is a retraction map,

$$E^0r_*|_{E^0H_*(\bar{B}(Y))}=\mathrm{id}_{E^0H_*(\bar{B}(Y))}.$$

Therefore the composite

$$E^0r_*\circ E^0(\Omega\sigma_*)\circ E^0\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*$$

of associated graded objects induced by the composition in (2.4) is an isomorphism, as required. ∎

The proof of Theorem 2.3 does more. Recall the map $\bar{Q}_nB(Y) \longrightarrow \Sigma\Omega Y$ defined in (2.3). Taking the wedge sum for $n \geq 1$ and then evaluating, we obtain a composite

$$\phi\colon \bigvee_{n=1}^{\infty} \bar{Q}_nB(Y) \xrightarrow{\bigvee_{n=1}^{\infty}\phi_n} \Sigma\Omega Y \xrightarrow{\sigma} Y.$$

The thrust of the proof of Theorem 2.3 was to show that the composite in (2.4) is a homotopy equivalence. That is, the composite $r \circ \Omega\phi$ is a homotopy equivalence. In particular, this implies that $\Omega\phi$ has a functorial retraction. Consequently, if $\bar{A}(Y)$ is the homotopy fiber of ϕ we immediately obtain the following.

Theorem 2.4. *Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ and let the functor A be given by $A(V) = \mathbf{k} \otimes_{B(V)} T(V)$. Then there is a homotopy fibration sequence*

$$\Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_nB(Y)\right) \xrightarrow{\Omega\phi} \Omega Y \longrightarrow \bar{A}(Y) \longrightarrow \bigvee_{n=1}^{\infty} \bar{Q}_nB(Y) \xrightarrow{\phi} Y,$$

where $Y \in \mathbf{CoH}$ and a functorial decomposition

$$\Omega Y \simeq \Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_nB(Y)\right) \times \bar{A}(Y).$$

Note that $\bar{A}$ is a geometric realization of A.

Proof of Theorem 1.1. In Theorem 2.4 we can choose $B(V)$ to be $B^{\max}(V)$. The fiber $\bar{A}(Y)$ of ϕ is now, by definition, $A^{\min}(Y)$. The theorem follows immediately. □

3. The generalization of the Hilton-Milnor Theorem

In this section we prove Theorem 1.2. We begin by stating a key general result from [GTW].

Theorem 3.1. *Let Y and Z be simply-connected co-H-spaces. There is a natural homotopy decomposition*

$$Z \wedge \Omega Y \simeq \bigvee_{n=1}^{\infty} [Z \wedge \Omega Y]_n$$

such that:

1) *each space $[Z \wedge \Omega Y]_n$ is a simply-connected co-H-space;*
2) $\widetilde{H}_*([Z \wedge \Omega Y]_n) \cong \widetilde{H}_*(Z) \otimes (\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$;
3) *if $Z = S^1$ and $Y = \Sigma X$, then $[Z \wedge \Omega\Sigma X]_n \simeq \Sigma X^{(n)}$.*

In particular, if $Z = S^1$, then we obtain a homotopy decomposition of $\Sigma\Omega Y$ which generalizes James' decomposition of $\Sigma\Omega\Sigma X$, as discussed in the Introduction. The application of Theorem 3.1 that we need is the following.

Proposition 3.2. *Let $Y_1, \ldots, Y_m$ be simply-connected co-H-spaces. There is a natural homotopy decomposition*

$$\Sigma\Omega Y_1 \wedge \cdots \wedge \Omega Y_m \simeq \bigvee_{n_1,\ldots,n_m=1}^{\infty} M((Y_i, n_i)_{i=1}^m)$$

such that:

1) *each space* $M((Y_i, n_i)_{i=1}^m)$ *is a simply-connected co-*H*-space;*
2) $\widetilde{H}_*(M((Y_i, n_i)_{i=1}^m)) \cong \Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes n_1} \otimes \cdots \otimes (\Sigma^{-1}\widetilde{H}_*(Y_m))^{\otimes n_m}\right)$*;*
3) *if* $Y_i = \Sigma X_i$ *for* $1 \leq i \leq m$*, then* $M((\Sigma X_i, n_i)_{i=1}^m) \simeq \Sigma X^{(n_1)} \wedge \cdots \wedge X^{(n_m)}$*.*

Proof. First, consider the special case when $m = 1$. We wish to decompose $\Sigma\Omega Y_1$. Applying Theorem 3.1 with $Z = S^1$ and $Y = Y_1$, we obtain a homotopy decomposition

$$\Sigma\Omega Y \simeq \bigvee_{n_1=1}^{\infty} M(Y_1, n_1),$$

where $M(Y_1, n_1) = [\Sigma\Omega Y_1]_{n_1}$. In particular, $M(Y_1, n_1)$ is a simply-connected co-H-space, $\widetilde{H}_*(M(Y_1, n_1)) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes n_1}$, and if $Y_1 = \Sigma X_1$, then $M(\Sigma X_1, n_1) \simeq \Sigma X_1^{(n_1)}$.

Next, consider the special case when $m = 2$. We wish to decompose $\Sigma\Omega Y_1 \wedge \Omega Y_2$. From the $m = 1$ case we have

$$\Sigma\Omega Y_1 \wedge \Omega Y_2 \simeq \left(\bigvee_{n_1=1}^{\infty} M(Y_1, n_1)\right) \wedge \Omega Y_2 \simeq \bigvee_{n_1=1}^{\infty} M(Y_1, n_1) \wedge \Omega Y_2.$$

Since $M(Y_1, n_1)$ is a co-H-space, for each $n_1 \geq 1$ we can apply Theorem 3.1 with $Z = M(Y_1, n_1)$ and $Y = Y_2$ to further decompose $M(Y_1, n_1) \wedge \Omega Y_2$. Collecting these, we obtain a homotopy decomposition

$$\Sigma\Omega Y_1 \wedge \Omega Y_2 \simeq \bigvee_{n_1, n_2=1}^{\infty} M((Y_i, n_i)_{i=1}^2),$$

where each space $M((Y_i, n_i)_{i=1}^2)$ is a simply-connected co-H-space,

$$\widetilde{H}_*(M((Y_i, n_i)_{i=1}^m)) \cong \Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes n_1} \otimes (\Sigma^{-1}\widetilde{H}_*(Y_2))^{\otimes n_2}\right)$$

and if $Y_i = \Sigma X_i$, then $M((Y_i, n_i)_{i=1}^2) \simeq \Sigma X_1^{(n_1)} \wedge X^{(n_2)}$.

More generally, if $m > 2$, then the procedure in the previous paragraph is iterated to obtain the homotopy decomposition asserted in the statement of the proposition. □

Next, we state a homotopy decomposition proved in [P]. For a space X and an integer j, let $j \cdot X = \bigvee_{i=1}^{j} X$.

Theorem 3.3. *Let* $X_1, \ldots, X_m$ *be simply-connected CW-complexes of finite type. Let* F *be the homotopy fiber of the inclusion* $\bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i$*. There is a natural homotopy equivalence*

$$F \simeq \bigvee_{j=2}^{m}\left(\bigvee_{1 \leq i_1 < \cdots < i_j \leq m} (j-1) \cdot \Sigma\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_j}\right).$$

Remark 3.4. A version of Theorem 3.3 holds for an infinite wedge $\bigvee_{i=1}^{\infty} X_i$, provided the spaces X_i can be ordered so that the connectivity of X_i is nondecreasing and tends to infinity. This guarantees that the fiber F of the inclusion $\bigvee_{i=1}^{\infty} X_i \longrightarrow \prod_{i=1}^{\infty} X_i$ is of finite type.

Proof of Theorem 1.2. The proof follows along the same lines as the proof of the usual Hilton-Milnor Theorem when $Y_i = \Sigma X_i$. So it may be useful when reading to keep in mind this special case.

First, including the wedge into the product gives a homotopy fibration

$$F_1 \longrightarrow \bigvee_{i=1}^{m} Y_i \longrightarrow \prod_{i=1}^{m} Y_i$$

that defines the space F_1. This fibration splits after looping as

$$\Omega(\bigvee_{i=1}^{m} Y_i) \simeq \prod_{i=1}^{m} \Omega Y_i \times \Omega F_1. \tag{3.1}$$

Second, by Theorem 3.3,

$$F_1 \simeq \bigvee_{j=2}^{m} \left(\bigvee_{1\leq i_1<\cdots<i_j\leq m} (j-1)\cdot \Sigma\Omega Y_{i_1} \wedge \cdots \wedge \Omega Y_{i_j} \right).$$

Applying the generalization of the James decomposition in Proposition 3.2 we can decompose each term $\Sigma\Omega Y_{i_1} \wedge \cdots \wedge \Omega Y_{i_j}$ to obtain a refined decomposition

$$F_1 \simeq \bigvee_{\alpha_1\in\mathcal{J}_1} M_{\alpha_1}$$

for some index set $\mathcal{J}_1$, where the summands M_{α_1} have the property that

$$H_*(M_{\alpha_1}) \cong \Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_{t_1})^{\otimes n_1}) \otimes \cdots \otimes (\Sigma^{-1}\widetilde{H}_*(Y_{t_l})^{\otimes n_m})\right)$$

for indices $l \geq 2$, $1 \leq t_1 < \cdots < t_l \leq m$ and $n_1, \ldots, n_l \geq 1$. Third, including the wedge into the product gives a homotopy fibration

$$F_2 \longrightarrow \bigvee_{\alpha_1\in\mathcal{J}_1} M_{\alpha_1} \longrightarrow \prod_{\alpha_1\in\mathcal{J}_1} M_{\alpha_1}$$

which defines the space F_2. This fibration splits after looping, so (3.1) refines to a decomposition

$$\Omega(\bigvee_{i=1}^{m} Y_i) \simeq \prod_{i=1}^{m} \Omega Y_i \times \prod_{\alpha_1\in\mathcal{J}_1} \Omega M_{\alpha_1} \times \Omega F_2. \tag{3.2}$$

Observe that since each Y_i is simply-connected, the spaces M_{α_1} can be ordered so that their connectivity is nondecreasing and tending to infinity. Therefore, Remark 3.4 implies that Theorem 3.3 can be applied to decompose F_2. The process can now be iterated to produce fibers F_k for $k \geq 3$ and a decomposition

$$\Omega(\bigvee_{i=1}^{m} Y_i) \simeq \prod_{i=1}^{m} \Omega Y_i \times \left(\prod_{j=1}^{k-1} \prod_{\alpha_j\in\mathcal{J}_j} \Omega M_{\alpha_j} \right) \times \Omega F_k, \tag{3.3}$$

where each M_{α_j} has the property that

$$H_*(M_{\alpha_1}) \cong \Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_{t_1})^{\otimes n_1}) \otimes \cdots \otimes (\Sigma^{-1}\widetilde{H}_*(Y_{t_l})^{\otimes n_m})\right)$$

for indices $l \geq j+1$, $1 \leq t_1 < \cdots < t_l \leq m$ and $n_1, \ldots, n_l \geq 1$. Note that the condition $l \geq j+1$ implies that the connectivity of F_k is strictly increasing with k, and so tends to infinity. Thus, the decompositions of $\Omega(\bigvee_{i=1}^m Y_i)$ stabilize, giving a homotopy decomposition

$$\Omega(\bigvee_{i=1}^m Y_i) \simeq \prod_{i=1}^m \Omega Y_i \times \prod_{\alpha\in\mathcal{J}} M_\alpha$$

for some index set $\mathcal{J}$. What remains is a bookkeeping argument that makes explicit the index set $\mathcal{J}$. Since the bookkeeping is recording the tensor algebra terms $(\Sigma^{-1}\widetilde{H}_*(Y_{t_1})^{\otimes n_1}) \otimes \cdots \otimes (\Sigma^{-1}\widetilde{H}_*(Y_{t_l})^{\otimes n_m})$, it is the same bookkeeping process as in the usual Hilton-Milnor Theorem. Thus the index set $\mathcal{J}$ can be taken to be the Hall basis of the (ungraded) free Lie algebra on m letters. $\square$

4. Appendix

In this appendix we give an example which distinguishes between a coalgebra retract of a tensor algebra and a functorial coalgebra retract. Let $p = 3$ and let $V = \mathbb{Z}/3\mathbb{Z}\{u, v\}$ be a two-dimensional $\mathbb{Z}/3\mathbb{Z}$-vector space, where $|u|$ and $|v|$ are odd. Let $E(V)$ be the exterior algebra generated by V, and let $T(V)$ be the tensor algebra generated by V. Then by [W], $E(V)$ is a coalgebra retract of $T(V)$, and this holds even if one puts any possible Steenrod algebra structure on V. However, we will show that $E(V)$ is not a functorial coalgebra retract of $T(V)$.

This is an important distinction. For if $E(V)$ were a functorial coalgebra retract of $T(V)$, then Theorem 2.1 would imply that there is a simply-connected H-space X such that $H_*(X;\mathbb{Z}/3\mathbb{Z}) \cong E(V)$. However, this contradicts [Z], which states that if $V = \{\bar{u}, \bar{v}\}$ with $|\bar{u}| = 2n-1$, $|\bar{v}| = 2n+3$, $\mathcal{P}^1_*(\bar{v}) = \bar{u}$, $n > 2$ and $n \not\equiv 0 \bmod 3$, then there is no finite H-space X such that $H_*(X;\mathbb{Z}/3\mathbb{Z}) \cong E(V)$.

Proposition 4.1. *At $p = 3$, $E(V)$ is not a functorial coalgebra retract of $T(V)$.*

Proof. Aiming for a contradiction, suppose that $E(V)$ is a functorial coalgebra retract of $T(V)$. By definition of functorial coalgebra retract (see Section 2), there exists a coalgebra retract A of the functor T such that $A(V)$ and $E(V)$ coincide. To be clear, any choice of coalgebra retract A of T such that $A(V) = E(V)$ would go to show that $E(V)$ is a functorial coalgebra retract. By [SW1], there is a smallest coalgebra retract $A^{\min}$ of T with the property that $W \subseteq A^{\min}(W) \subseteq T(W)$ for any graded $\mathbb{Z}/3\mathbb{Z}$-module W. In our case, we have $V \subseteq A(V) = E(V) \subseteq T(V)$, so by minimality, $A^{\min}(V) \subseteq A(V)$. Thus we may assume that $A = A^{\min}$.

For $n \geq 2$, let L_n be the Lie functor which takes a module W to the module of homogeneous Lie brackets of length n in $T(W)$. Let T_n be the tensor power functor which takes a module W to the module of length n tensor elements in $T(W)$. Let $L_n^{\max}$ be the largest summand of L_n which is also a summand of T_n. By [SW1], $L_n^{\max}$ exists and is unique up to natural equivalence. Diagrammatically, this states that there is a natural commutative diagram

$$\begin{tikzcd} T_n(W) = W^{\otimes n} \arrow[r, "\beta_n"] & L_n(W) \arrow[r, "i"] \arrow[d, "\pi"] & T_n(W) = W^{\otimes n} \\ L_n^{\max}(W) \arrow[u, "\phi"] \arrow[r, equal] & L_n^{\max}(W) & \end{tikzcd} \tag{4.1}$$

for any graded module W, where β_n is an epimorphism, i is an inclusion, and the composite $i \circ \beta_n$ sends an element $a_1 \otimes \cdots \otimes a_n$ to the iterated graded commutator $[[a_1, a_2], \ldots, a_n]$.

By [SW1], there is a coalgebra decomposition

$$T(V) \cong A^{\min}(V) \otimes B^{\max}(V), \tag{4.2}$$

where $B^{\max}(V)$ is the sub-Hopf algebra of $T(V)$ generated by $\{L_n^{\max}(V)\}_{n=2}^{\infty}$. The decomposition (4.2) induces a decomposition of Lie elements

$$L_3(V) \cong L_3(B^{\max}(V)) \otimes L_3(A^{\min}(V)).$$

But $L_3(B^{\max}(V)) = L_3^{\max}(V)$ since $B^{\max}(V)$ is generated by $L_n^{\max}(V)$ for $n \geq 2$, and $L_3(A^{\min}(V)) = 0$ since $A^{\min}(V) = E(V)$. Thus $L_3(V) \cong L_3^{\max}(V)$.

Note that the isomorphism $L_3(V) \cong L_3^{\max}(V)$ holds regardless of the Steenrod algebra structure on V. We will show that this isomorphism leads to a contradiction if we assume that $V = \{u, v\}$ with $\mathcal{P}_*^1(v) = u$. A basis for $L_3(V)$ is $\{[u,v],v],[[u,v],u]\}$. Let $A = [[u,v],v]$ and observe that

$$A = [[u,v],v] = [u,v]v - v[u,v] = (uv+vu)v - v(uv+vu) = uv^2 - v^2u.$$

In $V^{\otimes 3}$, let $a = v^2$ and $b = \mathcal{P}_*^1(a) = uv + vu$. Observe that

$$\begin{aligned} \mathcal{P}_*^1(v^3) = \mathcal{P}_*^1(av) = bv + au = (uv+vu)v + v^2u &= uv^2 + vuv + v^2u \\ &= A + v(uv - vu). \end{aligned} \tag{4.3}$$

Next, observe that the map $V^{\otimes 3} \xrightarrow{\beta_3} L_3(V)$ factors as a composite

$$V^{\otimes 3} \xrightarrow{r} V \otimes L_2(V) \xrightarrow{s} L_3(V).$$

A basis for $L_2(V)$ is $\{[u,u],[v,v],[u,v]\}$. Note that $[u,u] = 2u^2 = -u^2$; similarly $[v,v] = -v^2$, and $[u,v] = uv+vu$. In particular, $r(v^3) = v^3$ while $r(v(uv-vu)) = 0$. Thus applying r to (4.3) we obtain

$$\mathcal{P}_*^1(v^3) = A \tag{4.4}$$

in $V \otimes L_2(V)$. On the other hand, since $\beta^3 = s \circ r$, (4.1) implies that $L_3(V) \cong L_3^{\max}(V)$ retracts off $V \otimes L_2(V)$. But no such retraction can exist by (4.4) and the fact that A is the highest dimensional element in $L_3(V)$. □

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