PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 4, April 2013, Pages 1323–1333
S 0002-9939(2012)11441-8
Article electronically published on August 24, 2012

A NOTE ON THE CONVERGENCE TO INITIAL DATA OF HEAT AND POISSON EQUATIONS

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(Communicated by Richard Rochberg)

Abstract. We characterize the weighted Lebesgue spaces, $L^p(\mathbb{R}^n, v(x)dx)$, for which the solutions of the Heat and Poisson problems have limits a.e. when the time t tends to zero.

1. Introduction

Consider the following classical problems in the upper half-plane,

$$(A)\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta_x u(x,t) \\ \\ u(x,0) = f(x), \end{cases} \qquad (B)\begin{cases} \frac{\partial^2 w}{\partial t^2}(x,t) = -\Delta_x w(x,t) \\ \\ w(x,0) = g(x), \end{cases}$$

$x \in \mathbb{R}^n$, $t > 0$.

It is well known that under mild size conditions of the initial data f and g, for example $f, g \in L^p(\mathbb{R}^n, dx)$, $1 \le p < \infty$, the following limits hold:

$$\lim_{t\to 0} u(x,t) = f(x), \qquad \lim_{t\to 0} w(x,t) = g(x), \qquad \text{for almost every } x. \tag{1.1}$$

The aim of this paper is to obtain optimal weighted Lebesgue spaces $L^p(\mathbb{R}^n, v(x)dx)$, $1 \le p < \infty$, for which the limits in (1.1) still hold.

We find two classes D_p^W and D_p^P (see Definition 2.2) of weights v (strictly positive and finite functions for almost all x) such that

$$\begin{aligned} &u(x,t) \text{ is a solution of (A) for } t \in (0,T] \text{ and } \lim_{t\to 0} u(x,t) = f(x) \text{ a.e. } x \\ &\text{for all } f \in L^p(\mathbb{R}^n, v(x)dx) \text{ if and only if } v \in D_p^W, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &w(x,t) \text{ is a solution of (B) for } t \in (0,T] \text{ and } \lim_{t\to 0} w(x,t) = g(x) \text{ a.e. } x \\ &\text{for all } g \in L^p(\mathbb{R}^n, v(x)dx) \text{ if and only if } v \in D_p^P. \end{aligned} \tag{1.3}$$

Received by the editors September 10, 2010 and, in revised form, August 11, 2011.
2010 *Mathematics Subject Classification.* Primary 42B99; Secondary 35G10.
The authors were partially supported by MTM2008-06621-C02-01. The first and third authors were partially supported by grants from CONICET and Universidad Nacional del Litoral, Argentina.
This research was begun during a stay of the first and third authors at the Departamento de Matemáticas of UAM, to which they are very grateful.

These two statements are included in Theorem 2.3, which states the existence of optimal spaces $L^p(\mathbb{R}^n, v(x)dx)$ adapted to either statement (1.2) or statement (1.3).

Throughout this note the wording "weighted inequality" for an operator T means to find conditions on a given weight v in order to assure the existence of a weight u for which T maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$.

Theorem 2.3 involves some weighted inequalities for local maximal operators associated to Problems (A) and (B), namely, $\sup_{t<R} |u(x,t)|$ and $\sup_{t<R} |w(x,t)|$, respectively. Even more, the finitude almost everywhere of each of these maximal operators is equivalent to the almost everywhere convergence stated either in (1.2) or (1.3).

These weighted inequalities are proved in this work by using some ideas due to E. Harboure, R.A. Macías and C. Segovia (see [7]) and also some ideas due to J.L. Rubio de Francia (see [9]). In proving them we shall need some weighted inequalities for the local Hardy-Littlewood maximal operator that we believe are of independent interest (see Lemma 3.4). For the (global) Hardy-Littlewood maximal function, some classes of weights for the weighted inequalities were obtained by L. Carleson and P. Jones, [1], Rubio de Francia, [9] and A. Gatto and C. Gutiérrez, [5], independently. These results are shown in Theorem 3.2.

Finally in Theorem 2.6 we compare all the classes of weights that appear in this note.

It is worth mentioning that the characterization of the weights v such that the Hardy-Littlewood maximal function maps $L^p(\mathbb{R}^n, v(x)dx)$, $1 < p < \infty$, and the weak $(1,1)$ boundedness, was done by B. Muckenhoupt in the celebrated paper [8]. The problem of characterization of the pairs (u,v) for which the Hardy-Littlewood function maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$ was solved by E. Sawyer in [11]. Finally we mention that the problem was solved in [7] for the case of fractional integrals. The case of Poisson integrals in light-cones was considered in [2].

2. Preliminaries and main results

The solutions to problems (A) and (B) can be described via the Heat and Poisson semigroups. In fact, if the functions f and g belong to the Lebesgue space $L^p(\mathbb{R}^n, dx)$, it is well known that the solutions of those problems are

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy = W_t * f(x), \qquad t > 0, \tag{2.4}$$

and

$$w(x,t) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}} g(y)dy = P_t * g(x), \qquad t > 0, \tag{2.5}$$

where $W(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$, $W_t(x) = t^{-\frac{n}{2}} W(t^{-\frac{1}{2}}x)$, $P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+|x|^2)^{-\frac{n+1}{2}}$ and $P_t(x) = t^{-n}P(t^{-1}x)$. Moreover, the maximal operators

$$f \to \sup_{t>0} u(\cdot, t) \qquad \text{and} \qquad g \to \sup_{t>0} w(\cdot, t)$$

are bounded on $L^p(\mathbb{R}^n, dx)$, $p > 1$ and from $L^1(\mathbb{R}^n, dx)$ into weak-$L^1(\mathbb{R}^n, dx)$.

It is well known that the convolution maximal operator controls pointwise (a.e.) convergence and it is implicit in the "standard" argument that the maximal operator can be replaced by a local version; see [3]. This reflection shows that any weight v for which the maximal operators $\sup_{t>0} |u(x,t)|$ or $\sup_{t>0} |w(x,t)|$ have good boundedness properties would be a good weight for our problem. Even more, in order to have the limits in (1.1) it is not necessary to consider the global maximal operators $\sup_{t>0} |u(x,t)|$ or $\sup_{t>0} |w(x,t)|$ but only local versions of them. Namely

$$W_R^* f(x) := \sup_{t<R} |u(x,t)| = \sup_{t<R} |W_t * f(x)|$$

and

$$P_R^* g(x) := \sup_{t<R} |w(x,t)| = \sup_{t<R} |P_t * g(x)|,$$

for some $R > 0$.

The first question we consider is about boundedness properties of the operators $W_t * f(x)$ and $P_t * f(x)$. The following proposition gives the answer.

Proposition 2.1. *Let v be a weight in $\mathbb{R}^n$, $1 \le p < \infty$ and let $\{\phi_t\}_t$ be either the Heat, $\{W_t\}_t$, or the Poisson, $\{P_t\}_t$, semigroup (see* (2.4) *and* (2.5)*).*

The following statements are equivalent:

(a) *There exists $t_0 > 0$ and a weight u such that the operator $f \to \phi_{t_0} * f$ maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$, for $p > 1$.*
 In the case $p = 1$, it maps $L^1(\mathbb{R}^n, v(x)dx)$ into weak-$L^1(\mathbb{R}^n, u(x)dx)$.

(b) *There exists $t_0 > 0$ and a weight u such that the operator $f \to \phi_{t_0} * f$ maps $L^p(\mathbb{R}^n, v(x)dx)$ into weak-$L^p(\mathbb{R}^n, u(x)dx)$.*

(c) *There exists $t_0 > 0$ such that $\phi_{t_0} * f(x) < \infty$ a.e. x (with respect to the Lebesgue measure) for all $f \in L^p(\mathbb{R}^n, v(x)dx)$.*

(d) *There exists $t_0 > 0$ such that*

$$\|\phi_{t_0} v^{-\frac{1}{p}}\|_{L^{p'}(\mathbb{R}^n, dx)} < \infty.$$

Motivated by the above proposition we give the following definition.

Definition 2.2. Let $1 \le p < \infty$ and $\{\phi_t\}_{t>0}$ be the Heat, $\{W_t\}_{t>0}$ (respectively Poisson, $\{P_t\}_{t>0}$) semigroup.

We say that the weight v belongs to the class D_p^W (respectively D_p^P) if there exists $t_0 > 0$ such that

$$\|\phi_{t_0} v^{-\frac{1}{p}}\|_{L^{p'}(\mathbb{R}^n, dx)} < \infty.$$

The main result in this note is the following.

Theorem 2.3. *Let v be a weight in $\mathbb{R}^n$, $1 \le p < \infty$, and $\{\phi_t\}_t$ be either the Heat, $\{W_t\}_t$, or the Poisson, $\{P_t\}_t$, semigroup. Define*

$$\Phi_R^* f(x) = \sup_{t<R} |\phi_t * f(x)|,$$

for some R, $0 < R < \infty$.

The following statements are equivalent:

(1) *There exists $0 < R < \infty$ and a weight u such that the operator*

$$f \to \Phi_R^* f$$

maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$ for $p > 1$. In the case $p = 1$, it maps $L^1(\mathbb{R}^n, v(x)dx)$ into weak-$L^1(\mathbb{R}^n, u(x)dx)$.

(2) *There exists $0 < R < \infty$ and a weight u such that the operator*

$$f \to \Phi_R^* f$$

maps $L^p(\mathbb{R}^n, v(x)dx)$ into weak-$L^p(\mathbb{R}^n, u(x)dx)$.

(3) *There exists* $0 < R < \infty$ *such that* $\phi_R * f(x) < \infty$ *a.e.* x *and the limit*

$$\lim_{t\to 0} \phi_t * f(x)$$

exists a.e. x *for all* $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(4) *There exists* $0 < R < \infty$ *such that*

$$\Phi_R^* f(x) < \infty,$$

a.e. x, *for all* $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(5) *The weight*

$$v \in D_p^{\phi}$$

(see Definition 2.2*).*

Remark 2.4. The motivation of the last theorem comes from Problems (A) and (B) in the introduction. An obvious academic question would be to determine a general class of functions ϕ such that Theorem 2.3 is valid for $\{\phi_t\}_t$. We could write here a list of conditions for that validity. But we think it would be just a list of conditions, that the interested reader can easily find following our proofs.

Throughout this paper more classes of weights will appear, motivating the following definitions.

Definition 2.5. Let $1 < p < \infty$. We say that the weight

- v belongs to the class D_p^* if v satisfies (II) in Theorem 3.2;
- v belongs to the class D_p^{loc} if $v^{-\frac{p'}{p}} \in L^1_{loc}(\mathbb{R}^n, dx)$ (that is, $v^{-\frac{1}{p}} \in L^{p'}_{loc}(\mathbb{R}^n, dx)$).

In the case $p = 1$ we say that the weight

- v belongs to the class D_1^* if v satisfies $\sup_{R>1} \| \frac{1}{R^n} v^{-1}(\cdot)\|_{L^\infty(B(0,R))} \le C$;
- v belongs to the class D_1^{loc} if $v^{-1} \in L^\infty_{loc}(\mathbb{R}^n, dx)$ (that is, v is locally bounded from below).

The relationship among the classes of weights in Definitions 2.2 and 2.5 is given by the next theorem.

Theorem 2.6. *The chain of inclusions*

$$D_p^* \subsetneq D_p^P \subsetneq D_p^W \subsetneq D_p^{loc}$$

holds for $1 \le p < \infty$.

3. Proofs

We need the following lemma to prove Proposition 2.1.

Lemma 3.1. *Let* v *be a weight in* $\mathbb{R}^n$, $1 \le p < \infty$, *and* $\{\phi_t\}_t$ *be either the Heat or the Poisson semigroup.*

The following statements are equivalent:

(i) *The weight*

$$v \in D_p^{\phi}.$$

(ii) *There exists* $t_1 > 0$ *such that*

$$\|\phi_{t_1}(x - \cdot) v^{-\frac{1}{p}}(\cdot)\|_{L^{p'}(\mathbb{R}^n)} < \infty$$

for all $x \in \mathbb{R}^n$.

Proof of Lemma 3.1. Assume that $\{\phi_t\}$ is the Heat semigroup and $v \in D_p^{\phi}$. There exist $t_0 > 0$ and a positive constant C_0 such that $C_0 = \|\phi_{t_0}\, v^{-\frac{1}{p}}\|_{L^{p'}(\mathbb{R}^n,dx)} < \infty$; in particular, $v^{-\frac{1}{p}} \in L^{p'}_{loc}$.

Given x we consider the ball $B_x = \{y : |x-y| \leq |x|\}$; hence for $t > 0$ we have

$$\|\phi_t(x-\cdot)v^{-\frac{1}{p}}(\cdot)\|_{L^{p'}(\mathbb{R}^n,dy)} \leq \|\,\chi_{B_x}(\cdot)\phi_t(x-\cdot)v^{-\frac{1}{p}}(\cdot)\|_{L^{p'}(\mathbb{R}^n,dy)} + \|\,\chi_{B_x^c}(\cdot)\phi_t(x-\cdot)v^{-\frac{1}{p}}(\cdot)\|_{L^{p'}(\mathbb{R}^n,dy)}.$$

If $|x-y| \leq |x|$, then $|y| \leq |x-y| + |x| \leq 2|x|$; hence

$$e^{-\frac{1}{4t}|x-y|^2} \leq 1 \leq e^{\frac{1}{4t}|x|^2}e^{-\frac{1}{4t}|x|^2} \leq e^{\frac{1}{4t}|x|^2}e^{-\frac{1}{4t}(\frac{|y|}{2})^2}.$$

If, on the other hand, $|x-y| > |x|$, then $|y| \leq |x-y| + |x| < 2|x-y|$; thus

$$e^{-\frac{1}{4t}|x-y|^2} \leq e^{-\frac{1}{4t}(\frac{|y|}{2})^2}.$$

Choosing $t_1 = t_0/4$, we get the result.

The proof in the case of the Poisson semigroup is analogous. □

Proof of Proposition 2.1. Clearly (a) implies (b) and this implies (c).

Now assume that (c) holds. Let $\{\phi_t\}$ be the Heat kernel. Hence, for any positive function $f \in L^p(\mathbb{R}^n, v(x)dx)$, $\phi_{t_0} * f(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ be such that $\phi_{t_0} * f(x_0) < \infty$. We will first show that $\phi_{\frac{t_0}{4}} * f(x) < \infty$ for all $x \in \mathbb{R}^n$. Indeed, assume first that $x \neq x_0$.

If $|x-y| \leq |x-x_0|$, then $|y-x_0| \leq 2|x-x_0|$ and we have

$$\phi_{\frac{t_0}{4}}(x-y) \leq C\frac{1}{t_0^{n/2}} \leq C\frac{1}{t_0^{n/2}}\frac{\phi_{t_0}(x_0-y)}{\phi_{t_0}(2(x-x_0))} = C_x\phi_{t_0}(x_0-y).$$

If $|x-y| \geq |x-x_0|$, then $|x_0-y| \leq 2|x-y|$. Hence,

$$\phi_{\frac{t_0}{4}}(x-y) \leq \phi_{\frac{t_0}{4}}(\frac{x_0-y}{2}) = 4^{n/2}\phi_{t_0}(x_0-y). \tag{3.6}$$

From both of the above inequalities it follows that

$$\begin{aligned}\int_{\mathbb{R}^n}\phi_{\frac{t_0}{4}}(x-y)f(y)dy \leq &(C_x + 4^{n/2})\Big(\int_{|x-y|<|x-x_0|} + \int_{|x-y|\geq|x-x_0|}\Big)\phi_{t_0}(x_0-y)f(y)dy \\ &< \infty,\end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \{x_0\}$.

Assuming now that $x = x_0$, then (3.6) still holds and

$$0 \leq \int_{\mathbb{R}^n}\phi_{\frac{t_0}{4}}(x_0-y)f(y)dy \leq 4^{n/2}\int_{\mathbb{R}^n}\phi_{t_0}(x_0-y)f(y)dy < \infty.$$

Therefore the functional

$$f \to \int_{\mathbb{R}^n}\phi_{\frac{t_0}{4}}(x-y)f(y)dy = \int_{\mathbb{R}^n}\phi_{\frac{t_0}{4}}(x-y)v^{-1/p}(y)f(y)v^{1/p}(y)dy$$

is well defined for all $f \in L^p(\mathbb{R}^n, v(x)dx)$ and for every $x \in \mathbb{R}^n$. By duality the mapping

$$y \to \phi_{\frac{t_0}{4}}(x-y)v^{-1/p}(y)$$

belongs to $L^{p'}(\mathbb{R}^n, dx)$ for almost every $x \in \mathbb{R}^n$, thus obtaining (d) for $t_1 = \frac{t_0}{4}$.

Finally if (d) holds, then by Hölder's inequality we get that

$$\int |\phi_t * f(x)|^p u(x)dx \leq \int |f(y)|^p v(y)\, dy \int \|\phi_t(x-\cdot)v^{-\frac{1}{p}}(\cdot)\|^p_{L^{p'}(\mathbb{R}^n,dy)} u(x)dx.$$

Applying Lemma 3.1 there exists $t_0 > 0$ such that

$$\psi(x) = \|\phi_{t_0}(x-\cdot)v^{-\frac{1}{p}}(\cdot)\|^p_{L^{p'}(\mathbb{R}^n,dy)}$$

is finite for all x. Then it is enough to choose $u \in L^1_{loc}$ such that $\psi\, u \in L^1$ to obtain (a). This concludes the proof of Proposition 2.1. $\square$

If ϕ is a positive, radial, decreasing and integrable function, the maximal operator $\Phi^* f(x) = \sup_t \phi_t * f(x)$ is bounded by a constant times the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|dy.$$

Since W and P are positive, radial, decreasing and integrable functions, any good weight for the operator M would be good for our purposes.

Seeking good weights for the operator M we recall some results going back to the 80's, due independently to J.L. Rubio de Francia [9] and to L. Carleson and P. Jones [1].

Theorem 3.2. *Let v be a weight in $\mathbb{R}^n$ and $1 < p < \infty$.*

The following statements are equivalent:

(I) *There exists a weight u such that the Hardy-Littlewood maximal operator M is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to $L^p(\mathbb{R}^n, u(x)dx)$.*

(II) *There exists a constant C such that*

$$\sup_{R>1} \frac{1}{R^{n\,p'}} \int_{B(0,R)} v^{-\frac{p'}{p}}(y)dy \leq C,$$

i.e. $v \in D^*_p$ (see Definition 2.5).

Remark 3.3. Statement II in Theorem 3.2 can be replaced by

(II′) For any $a > 0$, there exists a constant C_a such that

$$\sup_{R>a} \frac{1}{R^{n\,p'}} \int_{B(0,R)} v^{-\frac{p'}{p}}(y)dy \leq C_a.$$

To see this claim, we observe that if $a < 1$ and $a < S < 1$, then

$$\begin{aligned} \frac{1}{S^{n\,p'}} \int_{B(0,S)} v^{-\frac{p'}{p}}(y)dy &\leq \frac{1}{a^{n\,p'}} \int_{B(0,1)} v^{-\frac{p'}{p}}(y)dy \\ &\leq \frac{1}{a^{n\,p'}} \sup_{R>1} \frac{1}{R^{n\,p'}} \int_{B(0,R)} v^{-\frac{p'}{p}}(y)dy. \end{aligned}$$

Even more, statement II can be replaced by

(II″) For any $x \in \mathbb{R}^n$, and any $a > 0$, there exists a constant $C_{a,x}$ such that

$$\sup_{R>a} \frac{1}{R^{n\,p'}} \int_{B(x,R)} v^{-\frac{p'}{p}}(y)dy \leq C_{x,a}.$$

In order to prove this claim, we observe that

$$\begin{aligned}\frac{1}{R^{n\,p'}}\int_{B(x,R)} v^{-\frac{p'}{p}}(y)dy &\leq \frac{1}{R^{n\,p'}}\int_{B(0,|x|+R)} v^{-\frac{p'}{p}}(y)dy \\ &\leq \frac{(|x|+R)^{n\,p'}}{R^{n\,p'}}\frac{1}{(|x|+R)^{n\,p'}}\int_{B(0,|x|+R)} v^{-\frac{p'}{p}}(y)dy.\end{aligned}$$

Then, we use (II′). □

We remark that condition $v \in D_1^*$ can be replaced by the following condition: For any $x \in \mathbb{R}^n$, and for any $a > 0$, there exists a constant $C_{a,x}$ such that

$$\sup_{R>a}\left\|\frac{1}{R^n}v^{-1}(\cdot)\chi_{B(x,R)}(\cdot)\right\|_{L^\infty} \leq C_{x,a}.$$

Proof of Theorem 2.6. Given $t > 0$,

$$\begin{aligned}P_t(x-y) &= C_n\frac{1}{t^n\left(1+\frac{|x-y|^2}{t^2}\right)^{\frac{n+1}{2}}} \\ &\leq C_n\Big(\frac{1}{t^n}\chi_{\{|x-y|<t\}}(y) + \sum_{j=0}^{\infty}\frac{1}{t^n(2^j)^{n+1}}\chi_{\{2^j\,t<|x-y|<2^{j+1}\,t\}}(y)\Big) \\ &\leq C_n\Big(\frac{1}{t^n}\chi_{\{|x-y|<t\}}(y) + \sum_{j=0}^{\infty}2^{-j}\frac{1}{(2^{j+1}\,t)^n}\chi_{\{|x-y|<2^{j+1}\,t\}}(y)\Big).\end{aligned}$$

Thus,

$$\|P_t(x-\cdot)v^{-\frac{1}{p}}(\cdot)\|_{L^{p'}(\mathbb{R}^n,dx)} \leq C\sup_{R\geq t}\frac{1}{R^n}\|\chi_{B(x,R)}(\cdot)v^{-1/p}(\cdot)\|_{L^{p'}(\mathbb{R}^n,dx)}.$$

From Remark 3.3 it follows that $D_p^* \subset D_p^P$.

Since $W_{t^2}(x) \leq CP_t(x)$, then $D_p^P \subset D_p^W$.

The following chain of inequalities proves $D_p^W \subset D_p^{loc}$:

$$\chi_{\{|x-y|<R^{1/2}\}}(y)v^{-\frac{1}{p}}(y) \leq C\chi_{\{|x-y|<R^{1/2}\}}(y)e^{-\frac{|x-y|^2}{R}}v^{-\frac{1}{p}}(y) \leq Ce^{-\frac{|x-y|^2}{R}}v^{-\frac{1}{p}}(y).$$

To finish the proof of Theorem 2.6 it remains to show that each class is strictly included in the bigger class. We leave it to the reader to check the following assertions:

(a) The weight $v_1(x) = e^{-|x|^3 p}$ belongs to D_p^{loc}, but $v_1 \notin D_p^W$.

(b) The weight $v_2(x) = |x|^{-(n+1+\varepsilon)\,p}$, $\varepsilon > 0$, belongs to D_p^W, but $v_2 \notin D_p^P$.

(c) The weight $v_3(x) = |x|^{-(n+\varepsilon)p+n\frac{p}{p'}}$ with $0 < \varepsilon < 1$ belongs to D_p^P, but $v_3 \notin D_p^*$.

This concludes the proof of Theorem 2.6. □

In order to prove Theorem 2.3 we need a technical result about the local Hardy-Littlewood maximal function. Given $R > 0$ the local Hardy-Littlewood maximal function $\mathcal{M}_R f$ is defined by

$$\mathcal{M}_R f(x) = \sup_{0<s\leq R}\mathcal{A}_s f(x),$$

where $\mathcal{A}_s f(x) = \frac{1}{s^n}\int_{|x-y|<s} f(y)dy$. □

Lemma 3.4. *Let v be a weight in $\mathbb{R}^n$. Let $1 \leq p < \infty$ and $R > 0$ be fixed. The following statements are equivalent:*

(i) *There exists a weight u such that $\mathcal{M}_R$ is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to $L^p(\mathbb{R}^n, u(x)dx)$, for $p>1$, and from $L^1(\mathbb{R}^n, v(x)dx)$ to weak-$L^1(\mathbb{R}^n, u(x)dx)$, for $p = 1$.*

(ii) *There exists a weight u such that $\mathcal{M}_R$ is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to weak-$L^p(\mathbb{R}^n, u(x)dx)$.*

(iii) *There exists a weight u such that $\mathcal{A}_R$ is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to weak-$L^p(\mathbb{R}^n, u(x)dx)$.*

(iv) *$v^{-\frac{1}{p}} \in L^{p'}_{loc}$, for $p > 1$, and $v^{-1} \in L^{\infty}_{loc}$, for $p = 1$ (that is, v is locally bounded from below); i.e., $v \in D^{loc}_p$ (see Definition 2.5).*

To prove the above lemma we need the following technical lemma due to J.L. Rubio de Francia in [9]. It can be found in the form we need in [4].

Lemma 3.5. *Let (X, μ) be a measurable space, $\mathcal{B}$ a Banach space and T a sublinear operator from $T : \mathcal{B} \to L^s(X)$, for some $s < p$, satisfying*

$$\|(\sum_{j\in\mathbb{Z}} |Tf_j|^p)^{1/p}\|_{L^s(X)} \leq C_{p,s}(\sum_{j\in\mathbb{Z}} \|f_j\|^p_{\mathcal{B}})^{1/p},$$

where $C_{p,s}$ is a constant depending on p and s.

Then there exists a function u such that $u^{-1} \in L^{s/p}(X)$, $\|u^{-1}\|_{s/p} \leq 1$ and

$$\int_X |Tf(x)|^p u(x) d\mu(x) \leq C_X \|f\|_{\mathcal{B}}$$

for some constant C_X.

Proof of Lemma 3.4. We shall prove (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i); the rest of the implications are obvious.

We shall need the following Kolmogorov inequality; see [6]. Let μ, ν be two measures defined on $\mathbb{R}^n$. Let T be a sublinear operator of weak type (p, p), $1 \leq p < \infty$, with measures $d\mu$ and $d\nu$. Then given s, $0 < s < p$, there exists a finite constant C such that for every subset $A \subset \mathbb{R}^n$ with $\nu(A) < \infty$, we have

$$\Big(\int_A |Tf|^s d\nu\Big)^{1/s} \leq C\nu(A)^{1/s-1/p}\Big(\int |f|^p d\mu\Big)^{1/p}. \tag{3.7}$$

In our case μ and ν will be respectively $d\nu(x) = u(x)dx$ and $d\nu(x) = v(x)dx$.

Assume that (iii) holds. Let $x_0 \in \mathbb{R}^n$ and $R > 0$ be fixed. Since $B(x_0, R/2) \subset B(x, R)$ for $x \in B(x_0, R/2)$, then, for any nonnegative f, we have

$$\mathcal{A}_R f(x) = \frac{1}{R^n}\int_{B(x,R)} f(y)dy \geq \frac{1}{R^n}\int_{B(x_0,R/2)} f(y)dy,\ x \in B(x_0, R/2).$$

Therefore, by (iii) and (3.7), we have for $s < p$,

$$\begin{aligned}\Big(\frac{1}{R^n}\int_{B(x_0,R/2)} f(y)dy\Big)\Big(u(B(x_0, R/2))\Big)^{1/s} &\leq C\Big(\int_{B(x_0,R/2)} \mathcal{A}_R f(x)^s u(x)dx\Big)^{1/s}\\ &\leq Cu(B(x_0, R/2))^{1/s-1/p}\Big(\int f^p(y)v(y)dy\Big)^{1/p}.\end{aligned}$$

Given an arbitrary positive function $g \in L^p(B(x_0, R/2))$ we choose $f = gv^{-1/p}$. Then we have

$$\Big(\frac{1}{R^n}\int_{B(x_0,R/2)} g(y)v^{-1/p}(y)dy\Big) \le Cu(B(x_0,R/2))^{-1/p}\Big(\int g^p(y)dy\Big)^{1/p}.$$

By duality we conclude that $v^{-1/p}$ belongs to $L^{p'}(B(x_0,R))$; that is, we have proved (iv).

Now assume that (iv) holds. Let p be in the range $1 < p < \infty$. To prove (i) we define the sets $E_0 = B(0,R), E_k = \{x : 2^{k-1}R \le |x| < 2^k R\}$, $k \ge 1$.

For each k fixed we split $f = f' + f''$, where $f'(x) = f(x)\chi_{B(0,R2^{k+1})}(x)$.

By Kolmogorov's inequality and the weak $(1,1)$ inequality in the vector-valued setting (see [10]), for each $0 < s < 1 < p$ and for each k, we have that

(3.8)

$$\begin{aligned}
&\Big\|\Big(\sum_j |\mathcal{M}_R f'_j|^p\Big)^{1/p}\Big\|_{L^s(E_k)} \le C|E_k|^{1/s-1}\Big\|\Big(\sum_j |f'_j|^p\Big)^{1/p}\Big\|_{L^1}\\
&\le C|E_k|^{1/s-1}\int_{B(0,R2^{k+1})}\Big(\sum_j |f_j(x)|^p\Big)^{1/p}dx\\
&\le C|E_k|^{1/s-1}\Big(\int_{B(0,R2^{k+1})}\sum_j |f_j(x)|^p v(x)\,dx\Big)^{1/p}\Big(\int_{B(0,R2^{k+1})} v^{-\frac{p'}{p}}(x)dx\Big)^{1/p'}\\
&\le C_{k,v}|E_k|^{1/s-1}\Big(\int\sum_j |f_j(x)|^p v(x)dx\Big)^{1/p}.
\end{aligned}$$

On the other hand, if $x \in E_k$ and $y \notin B(0, R\,2^{k+1})$, then

$$R2^{k+1} < |y| \le |y-x| + |x| \le |y-x| + R2^k$$

and, thus, $|y-x| > R2^k$. Hence,

$$\mathcal{M}_R f_j''(x) = 0, \text{ for all } j \in \mathbb{N}, x \in E_k. \tag{3.9}$$

Pasting together (3.8) and (3.9), we see that the operator satisfies Lemma 3.5 in each set E_k. Hence a family of weights U_k, each one with support in E_k, can be found satisfying the statements in that lemma. The weight $u(x) = \sum_k \frac{1}{(2^k C_{E_k})^p} U_k(x)\chi_{E_k}(x)$ satisfies (i).

For the case $p = 1$, we use the weak $(1,1)$ continuity of the Hardy-Littlewood maximal operator and we have

$$\begin{aligned}
&|\{x \in E_k : \mathcal{M}_R f'(x) > \lambda\}| \le \frac{C}{\lambda}\|f'\|_{L^1} = \frac{C}{\lambda}\int_{B(0,R2^{k+1})}|f(x)|dx\\
&\le \frac{C}{\lambda}\Big(\int_{B(0,R2^{k+1})}|f(x)|v(x)\,dx\Big)\Big\|v^{-1}(\cdot)\chi_{B(0,R2^{k+1})}(\cdot)\Big\|_{L^\infty(\mathbb{R}^n,dx)}\\
&\le \frac{C_{k,v}}{\lambda}\int |f(x)|v(x)dx.
\end{aligned}\tag{3.10}$$

Pasting together (3.9) and (3.10), we get

$$|\{x \in E_k : \mathcal{M}_R f(x) > \lambda\}| \le \frac{C_{k,v}}{\lambda}\int |f(x)v(x)dx. \tag{3.11}$$

Hence the weight $u(x) = \sum_k \frac{1}{2^k Ck,v}\chi_{E_k}(x)$ gives (i). This concludes the proof of Lemma 3.4. $\square$

Proof of Theorem 2.3. The density of continuous functions with compact support on $L^p(\mathbb{R}^n, v(x)dx)$ gives (2) $\Rightarrow$ (3).

Let us prove (3) $\Rightarrow$ (4). By taking separately the positive and the negative part of f, we can assume that $f \geq 0$. We present a proof in the case $\phi = P$. Let x be such that $\lim_{t\to 0}\phi_t * f(x)$ exists and $\phi_R * f(x) < \infty$. Then there exist a constant $0 < C(x,f) < \infty$ and $t_{x,f} = t(x,f) > 0$ such that

$$\sup_{t<t_{x,f}} \phi_t * f(x) < C(x,f). \tag{3.12}$$

We can clearly choose $t_{x,f} < R$. Let us now consider $t_{x,f} < t < R$. Since ϕ is radial and nonincreasing, then

$$\phi_t(x-y) = t^{-n}\phi\left(\frac{x-y}{t}\right) \leq t_{x,f}^{-n}\phi\left(\frac{x-y}{R}\frac{R}{t}\right) \leq t_{x,f}^{-n}\phi\left(\frac{x-y}{R}\right) \leq \left(\frac{R}{t_{x,f}}\right)^n \phi_R(x-y).$$

Therefore,

$$\sup_{t_{x,f}<t<R} \phi_t * f(x) \leq \left(\frac{R}{t_{x,f}}\right)^n \phi_R * f(x) < \infty. \tag{3.13}$$

(3.12) and (3.13) give (4).

On the other hand, Proposition 2.1 together with the arguments in its proof gives (4) $\Rightarrow$ (5).

The last implication to be proved is (5) $\Rightarrow$ (1). We shall give the proof in the case $\phi = W$. Given $R > 0$ and $0 < t < R$ we split

$$W_t = W_t^1 + W_t^2,$$

where $W_t^1 = W_t\chi_{\{|x|\leq(2nR)^{1/2}\}}$.

If $j_0 \in \mathbb{Z}$ is such that $2^{j_0}t < R < 2^{j_0+1}t$, then

$$\begin{aligned} W_t^1(x) &\leq W_t(x)\Big(\chi_{\{|x|\leq(2nt)^{1/2}\}}(x) + \sum_{j=0}^{j_0}\chi_{\{(2n2^jt)^{1/2}\leq|x|\leq(2n2^{j+1}t)^{1/2}\}}(x)\Big) \\ &\leq C\Big(\frac{1}{t^{\frac{n}{2}}}\chi_{\{|x|\leq(2nt)^{1/2}\}}(x) + \sum_{j=0}^{j_0}(2n2^j)^{\frac{n}{2}}e^{-\frac{n}{2}2^j}\frac{1}{(2n2^jt)^{\frac{n}{2}}}\chi_{\{|x|\leq(2n2^{j+1}t)^{1/2}\}}(x)\Big). \end{aligned}$$

Thus for $f \geq 0$,

$$\sup_{t<R} W_t^1 * f(x) \quad \leq \quad C_n\mathcal{M}_{(2nR)^{1/2}}f(x) \tag{3.14}$$

with $C_n = C\Big((2n)^{\frac{n}{2}} + \sum_{j=0}^{\infty}(2n2^j)^{\frac{n}{2}}e^{-\frac{n}{2}2^j}\Big) < \infty$.

On the other hand, since $W_t^2(x)$ is increasing in the time interval $(0,R)$ we also have

$$\sup_{0<t<R} W_t^2 * f(x) = W_R^2 * f(x) \leq W_R * f(x). \tag{3.15}$$

Thus, by (3.14) and (3.15),

$$W_R^* f(x) \leq C\left(\mathcal{M}_{(2nR)^{1/2}}f(x) + W_R * f(x)\right). \tag{3.16}$$

Then the result follows by using Proposition 2.1, Theorem 2.6 and Lemma 3.4.

The proof in the case $\phi = P$ follows by choosing $P_t^1 = P_t \chi_{\{|x| \leq (n)^{1/2} R\}}$ and repeating the above argument. $\square$

Acknowledgments

We are very grateful to the referee for detailed comments. The variety of substantial suggestions certainly pushed us to improve the results and presentation of the paper in an essential way.

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