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ZERO DISTRIBUTION AND FACTORIZATION OF ANALYTIC FUNCTIONS OF SLOW GROWTH IN THE UNIT DISC

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ABSTRACT. For a meromorphic function f in the unit disc, let the ρ_∞-order of the growth be the limit of the orders of L_p-norms of $\log|f(re^{i\theta})|$ over the circle. In the case when the order of the maximum modulus function is smaller than 1, we describe zero distribution of canonical products and derive a new factorization theorem and logarithmic derivative estimates.

1. Introduction and the main result

Let $D(z,t)=\{\zeta\in\mathbb{C}:|\zeta-z|<t\}$, $z\in\mathbb{C}$, $t>0$, and $\mathbb{D}=D(0,1)$. For an analytic function f in $\mathbb{D}$, we define the maximum modulus $M(r,f)=\max\{|f(z)|: |z|=r\}$, $0\le r<1$. In the sequel, the symbol C with indices stands for positive constants which depend on the parameters indicated. We write $a(r)\sim b(r)$ if $\lim_{r\uparrow 1} a(r)/b(r)=1$.

Usually, the orders of growth of an analytic function f in $\mathbb{D}$ are defined as

$$\rho_M[f]=\limsup_{r\uparrow 1}\frac{\log^+\log^+ M(r,f)}{-\log(1-r)},\ \rho_T[f]=\limsup_{r\uparrow 1}\frac{\log^+ T(r,f)}{-\log(1-r)},$$

where $T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$. It is well known that

$$\rho_T[f]\le\rho_M[f]\le\rho_T[f]+1, \tag{1.1}$$

and all admissible values of the orders are possible ([1], [2], [10]).

Many theorems on analytic functions in $\mathbb{D}$ fail to hold when the ρ_M order is smaller than 1 (see e.g. [2], [9], [11]). To be more precise we start with canonical products. Let $A=(a_n)$ be a sequence of complex numbers in $\mathbb{D}$ without accumulation points in $\mathbb{D}$. We define the exponent of convergence of A by ($\inf\emptyset=+\infty$)

$$\mu[A]=\inf\Big\{\mu\ge 0:\sum_{a_n\in A}(1-|a_n|)^{\mu+1}<\infty\Big\}.$$

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It is well known [4, 13, 15] that the Džrbašjan-Naftalevich-Tsuji canonical product

$$P(z, A, q) = \prod_{n=1}^{\infty} E\Big(\frac{1-|a_n|^2}{1-\bar a_n z}, q\Big), \tag{1.2}$$

where $E(w, 0) = (1 - w)$,

$$E(w, q) = (1 - w) \exp\{w + w^2/2 + \cdots + w^q/q\}, \quad q \in \mathbb{N},$$

is an analytic function with the zero sequence A provided that $\sum_{a_n\in A}(1-|a_n|)^{q+1} < \infty$.

C. N. Linden [9] established a connection between $\rho_M[P]$ and the zero distribution of P, where P is of the form (1.2). To clarify this connection we need some definitions.

Let

$$\square(re^{i\varphi}) = \Big\{\zeta : r \le |\zeta| \le \frac{1+r}{2}, |\arg\zeta - \varphi| \le \frac{1-r}{4}\Big\},$$

and $\nu(re^{i\varphi})$ be the number of zeros of P in $\square(re^{i\varphi})$. We define

$$\nu_1(r, P) = \max_{\varphi} \nu(re^{i\varphi}), \quad \nu[P] = \limsup_{r\uparrow 1} \frac{\log^+ \nu_1(r, P)}{-\log(1-r)}, \tag{1.3}$$

and

$$\rho_n[P] = \limsup_{r\uparrow 1} \frac{\log^+ n(r, P)}{-\log(1-r)}, \tag{1.4}$$

where $n(r, P)$ is the number of zeros in $\overline{D}(0, r)$.

Theorem A ([9, Theorem V]). *With the notation above we have*

$$\rho_T[P] = (\rho_n[P] - 1)^+, \tag{1.5}$$

$$\rho_M[P] \begin{cases} = \nu[P], & \rho_M[P] \ge 1, \\ \le \nu[P] \le 1, & \rho_M[P] < 1. \end{cases} \tag{1.6}$$

Remark 1.1. We note that relation (1.6) follows essentially from [13, 15]. Moreover, $(\rho_n[P] - 1)^+$ is equal to the convergence exponent of the zero sequence of P.

A function ρ: $[0, 1) \to \mathbb{R}_+$ is called a *proximate order* ([6, p. 55]; cf. [8]) if it satisfies the following conditions:

(i) ρ is differentiable on $[0, 1)$;
(ii) $\lim_{r\uparrow 1} \rho(r) = \rho_0 \in [0, \infty)$;
(iii) $\lim_{r\uparrow 1} \rho'(r)(1 - r) \log(1 - r) = 0$.

An advantage of this definition is that for every analytic function f of finite positive order $\rho_M[f]$ there exists a proximate order $\rho(r)$ such that

$$\limsup_{r\uparrow 1}(1 - r)^{\rho(r)} \log M(r, f) = 1.$$

Linden's proof of Theorem A is based on the following lemma [9, Lemma I], which we formulate in a suitable form for our purposes.

Lemma B. *Let* ρ: $[0, 1) \to \mathbb{R}_+$ *be a proximate order,* $\rho(r) \to \rho > 0$ $(r \uparrow 1)$. *Let* $P_s(z) = P(z, A, s)$ *be a canonical product. Suppose that for some* $C_1 > 0$, *we have*

$$\nu(re^{i\varphi}) \le \frac{C_1}{(1-r)^{\rho(r)}}, \quad 0 \le r < 1,\ 0 \le \varphi < 2\pi,$$

and $s > \rho$. Then

$$\log |P_s(z)| \leq 2^{s+2} \sum_{n=1}^{\infty} \left| \frac{1-|a_n|^2}{1-z\bar{a}_n} \right|^{s+1} \leq \frac{C_2}{(1-|z|)^{\rho(r)}}, \quad z \in \mathbb{D},$$

for some constant $C_2 > 0$.

Remark 1.2. Linden proved the lemma for the case $\rho(r) \equiv \rho$, but the same proof works in the general case as well (cf. [6, Chaps. 2, 3]).

As we can see from Theorem A, the value $\nu[P]$ coincides with $\rho_M[P]$ when the order is greater than 1. The question arises:

Question 1.3. What kind of growth characteristic can describe zero distribution in the case when $\rho_M[f] \leq 1$?

For a meromorphic function $f(z)$, $z \in \mathbb{D}$, and $p \geq 1$ we define

$$m_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |\log |f(re^{i\theta})||^p \, d\theta \right)^{\frac{1}{p}}, \quad 0 < r < 1.$$

We write

$$\rho_p[f] = \limsup_{r \uparrow 1} \frac{\log m_p(r, f)}{-\log(1-r)}.$$

A characterization of ρ_p-orders can be found in [12].

We define the ρ_∞*-order* of f as

$$\rho_\infty[f] = \lim_{p \to \infty} \rho_p[f]$$

(existence of the limit follows from the fact that L_p-norms are monotone in p). It follows from the First Fundamental Theorem of Nevanlinna that $\rho_1[f] = \rho_T[f]$. Besides, it is known (e.g. [11]) that $\rho_M[f] \leq \rho_p[f] + \frac{1}{p}$ $(p > 0)$, which generalizes (1.1). Consequently, $\rho_M[f] \leq \rho_\infty[f]$. Moreover, Linden [11] proved that $\rho_\infty[f] = \rho_M[f]$ provided that $\rho_M[f] \geq 1$. Thus, the values $\rho_\infty[f]$ and $\nu[f]$ have similar behavior with respect to the maximum modulus order when f is a canonical product. The main purpose of this paper is to prove that $\rho_\infty[f]$ coincides with $\nu[f]$ for the canonical products.

For a sequence A in $\mathbb{D}$ with the finite convergence exponent we define $\nu[A] = \nu[P(z, A, q)]$ for an appropriate choice of q. It is clear that the definition does not depend on q.

Theorem 1.4. *Given a sequence $A = (a_k)$ in $\mathbb{D}$ such that $\nu = \nu[A] < \infty$ and an integer $s \geq [\nu] + 1$, we define the canonical product $P_s(z) = P(z, A, s)$. Then $\rho_\infty[P_s] = \nu$.*

Theorem 1.5. *Let f be an analytic function in $\mathbb{D}$. Then $\nu[f] \leq \rho_\infty[f]$.*

Example 1.6. Let $a_k = 1 - 1/(k \log^2 k)$, $k \in \{3, \dots\}$. We consider the canonical product $B(z) = P(z, A, 0)$, which is a Blaschke product up to a constant factor. Since $|B|$ is bounded in $\mathbb{D}$, we have $\rho_M[B] = \rho_T[B] = 0$, and consequently $\rho_\infty[B] \leq 1$.

On the other hand, it is easy to check that

$$n(r, B) \sim \frac{1}{(1-r)\log^2(1-r)}, \quad r \uparrow 1,$$

and

$$\frac{d_1}{(1-r)\log^2(1-r)} \le \nu(r) \le \frac{d_2}{(1-r)\log^2(1-r)}, \quad r\uparrow 1,$$

for some positive constants d_1, d_2. Hence, $\nu[B]=1$, and by Theorem 1.5, $\rho_\infty[B]=1$.

In Section 2 we prove Theorem 1.4, which is the main result of the paper. In Section 3 we apply the theorem to problems of factorization of analytic and meromorphic functions in $\mathbb{D}$ and derive some new logarithmic derivative estimates.

2. Proofs of Theorems 1.4 and 1.5

We write $\rho = \rho_\infty[P_s]$.

First, we show that $\nu \le \rho$. The proof of this inequality is based on ideas from [8, Chap. 2, Lemmas 10, 11].

Lemma 2.1. *Under the assumptions of Theorem* 1.4 *we have*

$$\limsup_{r\uparrow 1} \frac{\sup_\varphi \log\log \inf_{\zeta\in\Box(re^{i\varphi})} \frac{1}{|P_s(\zeta)|}}{\log\frac{1}{1-r}} \le \rho. \tag{2.1}$$

Note that inequality (2.1) is a counterpart of *Levin's condition* (see [8, Chap. 2]), which plays an important role in the theory of subharmonic functions in the half-plane ([5, Theorem 18]) and the theory of functions of completely regular growth in an angle [8].

Proof of Lemma 2.1. Suppose that (2.1) does not hold; i.e., there exist $\varepsilon > 0$, a sequence (r_n) tending to 1, and a sequence (φ_n) such that

$$\log\frac{1}{|P_s(\zeta)|} \ge \frac{1}{(1-|\zeta|)^{\rho+\varepsilon}}$$

for all $\zeta\in\Box(r_ne^{i\varphi_n})$. Then

$$\int_{\varphi_n-\frac{1-r_n}{4}}^{\varphi_n+\frac{1-r_n}{4}} |\log|P_s(r_ne^{i\theta})||^p\,d\theta \ge \frac{1-r_n}{2(1-r_n)^{(\rho+\varepsilon)p}}.$$

Hence, $m_p(r_n,P_s) \ge \left(\frac{1}{4\pi}\right)^{\frac1p}(1-r_n)^{-\rho-\varepsilon+\frac1p}$. Thus, $\rho_p[P_s]\ge\rho+\varepsilon-\frac1p$. Passing to the limit as $p\to\infty$, we obtain $\rho_\infty[P_s]\ge\rho+\varepsilon$. This contradiction proves (2.1). $\Box$

If $\rho>0$, then the inequality $\nu\le\rho$ follows from $\rho_M[P_s]\le\rho$ and the next lemma.

Lemma 2.2. *Let f be an analytic function in $\mathbb{D}$, $\rho\colon[0,1)\to\mathbb{R}_+$ be a proximate order, $\rho(r)\to\rho>0$ $(r\uparrow 1)$. Let*

$$\log|f(re^{i\varphi})| \le C_3(1-r)^{-\rho(r)}, \quad r\in[0,1)$$

for some $C_3>0$.

If there exist $N>0$ and $r_0\in(0,1)$ such that for all $r\in[r_0,1)$ and $\varphi\in[0,2\pi)$ and some $z^*\in\Box(re^{i\varphi})$ we have

$$\log|f(z^*)| > -\frac{N}{(1-r)^{\rho(r)}},$$

then

$$\nu(re^{i\varphi}) \le C_4\frac{C_3+N}{(1-r)^{\rho(r)}}, \quad r\uparrow 1,$$

where C_4 is an absolute constant.

Proof of Lemma 2.2. Without loss of generality we may assume that $\varphi = 0$. Let $r \in [0,1)$. We write $R = r + (1-r)/4$, $\varphi_r = (1-r)/4$. Then $1 - R = \frac{3}{4}(1-r)$. Since

$$\varphi_r^2 + \Big(\frac{1+r}{2} - R\Big)^2 = \frac{1}{8}(1-r)^2,$$

and $\frac{1}{2\sqrt{2}} < \frac{2}{5}$, we have

$$\square(r) \subset D\Big(R, \frac{2}{5}(1-r)\Big), \quad r \in [r_1, 1) \tag{2.2}$$

for some $r_1 \in [0,1)$.

By the assumptions of Lemma 2.2 there exists $z_r \in \square(r) \subset D(R, \frac{2}{5}(1-r))$ such that

$$\log|f(z_r)| > -\frac{N}{(1-|z_r|)^{\rho(|z_r|)}} \geq -\frac{N}{\big(\frac{1-r}{4}\big)^{\rho(1-\frac{1-r}{4})}}, \tag{2.3}$$

because $|z_r| \leq 1 - \frac{1-r}{4}$. On the other hand, we have

$$\log|f(z)| \leq \frac{C_3}{\Big(\frac{1-r}{4}\Big)^{\rho(1-\frac{1-r}{4})}}, \quad z \in D\Big(R, \frac{1-r}{2}\Big). \tag{2.4}$$

Therefore the function $\psi_r(z) := \frac{f(z+R)}{f(z_r)}$ is analytic in $D(0, \frac{1-r}{2})$. We also have $\psi_r(z_r - R) = 1$, and

$$\log|\psi_r(z)| \leq (C_3 + N)\Big(\frac{1-r}{4}\Big)^{-\rho(1-\frac{1-r}{4})}, \quad |z| \leq \frac{1-r}{2}$$

by (2.3) and (2.4).

We need the following lemma [8, Chap. 2, Lemma 9].

Lemma C. *Let Φ be an analytic function in $D(0, R_0)$, and let $z_0 \in \overline{D}(0, lR_0)$, where $l < 1$, be such that $|\Phi(z_0)| = 1$. Then*

$$n(tR_0, \Phi) \leq C_5(l,t)\log M(R_0, \Phi), \quad l < t < 1,$$

where $C_5(l,t) = \Big(\log\frac{1+lt}{l+t}\Big)^{-1}$.

Applying Lemma C with $R_0 = (1-R)/2$, $l = \frac{2}{5} : \frac{1}{2} = 0.8$, we get

$$\nu(r) \leq n\Big(D\Big(R, \frac{1-r}{2}\Big)\Big) \leq C_5(0,8;t)(C_3+N)\Big(\frac{1-r}{4}\Big)^{-\rho(1-\frac{1-r}{4})}, \quad 0.8 < t < 1.$$

Fixing $t \in (0.8; 1)$ and using the property $\big(\frac{1-r}{4}\big)^{-\rho(1-\frac{1-r}{4})} \sim 4^{-\rho}(1-r)^{-\rho(r)}$ $(r \uparrow 1)$, we obtain the required assertion. Lemma 2.2 is proved. $\square$

If $\rho = 0$, then we can apply previous arguments with arbitrary positive ε_0 instead of $\rho(r)$ to obtain $\nu \leq \varepsilon_0$. Thus $\nu = 0$. The inequality $\rho \leq \nu$ follows from the next lemma. Note that our proof of Lemma 2.3 essentially repeats the arguments from [11, Lemma 1].

Lemma 2.3. *Let $A = (a_n)$ be a sequence of complex numbers in $\mathbb{D}$, ρ be a proximate order on $[0,1)$, $\rho(r) \to \rho_0$ $(r \uparrow 1)$. Suppose that*

$$\nu(re^{i\varphi}) \leq \frac{C_6}{(1-r)^{\rho(r)}}, \quad r \in [0,1), \varphi \in [0, 2\pi) \tag{2.5}$$

for some $C_6>0$. If $s>\rho_0$, then there exists a constant C_7, independent of r and p, such that

$$m_p(r,P_s)\le C_7\frac{\log\frac{1}{1-r}}{(1-r)^{\rho(r)}},\quad p\ge 1,\ r\in[0,1).$$

Proof of Lemma 2.3. We have to prove that

$$\int_0^{2\pi}|\log|P_s(re^{i\theta},A)||^p\,d\theta\le C_7^p\frac{\log^p\frac{1}{1-r}}{(1-r)^{p\rho(r)}}. \tag{2.6}$$

We deal with the integral in (2.6) by covering the range of integration by $[\pi/(1-r)]+1$ intervals of the form $[\tau+r-1,\tau+1-r]$ for $\tau=2k(1-r)$ and $k\in\{0,\dots,[\pi/(1-r)]\}$, showing that

$$\int_{\tau+r-1}^{\tau+1-r}|\log|P_s(re^{i\theta},A)||^p\,d\theta\le C_8^p(1-r)^{-p\rho(r)+1}\log^p\frac{1}{1-r} \tag{2.7}$$

for each τ, where the constant C_8 is independent of τ. For convenience and without loss of generality, we may suppose that $\tau=0$.

We shall need some known results.

Theorem D (Tsuji, [14, Theorem V.25, p. 224])**.** *For the canonical product $P_s(z)$ and positive ε, if D_m denotes the disc $D\Big(a_m,(1-|a_m|^2)^{s+4}\Big)$, then*

$$\log|P_s(z)|\ge K\log(1-|z|)\sum_m\Big|\frac{1-|a_m|^2}{1-z\bar a_m}\Big|^{s+1+\varepsilon},\quad \frac12\le|z|<1,\ z\notin\bigcup_m D_m. \tag{2.8}$$

Without loss of generality, we assume that $\frac34\le|a_m|<1$. For given r, let $\gamma_r=\{z=re^{i\theta}:r-1\le\theta\le 1-r\}$, and $F(r)=\{m:D_m\cap\gamma_r\ne\emptyset\}$, where D_m are the exceptional discs of Theorem D. By (2.5), we have

$$|F(r)|\le C_9(1-r)^{-\rho(r)}, \tag{2.9}$$

where $|F(r)|$ denotes the number of elements in the set $F(r)$. We consider the factorization $P_s=B_1B_2B_3$, where

$$B_1(z)=\prod_{m\notin F(r)}E\Big(\frac{1-|a_m|^2}{1-\bar a_m z},s\Big),$$

$$B_2(z)=\prod_{m\in F(r)}\exp\sum_{j=1}^s\frac1j\Big(\frac{1-|a_m|^2}{1-z\bar a_m}\Big)^j,$$

$$B_3(z)=\prod_{m\in F(r)}\Big(1-\frac{1-|a_m|^2}{1-z\bar a_m}\Big)=\prod_{m\in F(r)}\Big(\frac{\bar a_m(a_m-z)}{1-z\bar a_m}\Big).$$

First we note that for any positive number ε, Theorem D and Lemma B give

$$\int_{r-1}^{1-r}|\log|B_1(re^{i\theta})||^p d\theta\le\int_{r-1}^{1-r}C_{10}^p\log^p\frac{1}{1-r}\Big(\sum_m\Big|\frac{1-|a_m|^2}{1-re^{i\theta}\bar a_m}\Big|^{s+1}\Big)^p\,d\theta$$

$$\le C_{10}^p\log^p\frac{1}{1-r}\frac{1}{(1-r)^{p\rho(r)}}2(1-r)=\frac{C_{11}(\rho)\log^p\frac{1}{1-r}}{(1-r)^{p\rho(r)-1}}. \tag{2.10}$$

Next, the inequality $|1 - z\bar{a}_m| > \frac{1}{2}(1 - |a_m|^2)$ yields

$$|\log |B_2(z)|| < \sum_{m \in F(r)} \sum_{j=1}^{s} \frac{1}{j} \Big| \frac{1 - |a_m|^2}{1 - z\bar{a}_m} \Big|^j \le C_{12} |F(r)|.$$

Hence (2.9) implies that

$$\int_{r-1}^{1-r} |\log |B_2(re^{i\theta})||^p d\theta \le C_{13}(1-r)^{1-p\rho(r)}. \tag{2.11}$$

Finally, in [11, p. 124] it is proved that

$$\int_{r-1}^{1-r} |\log |B_3(re^{i\theta})||^p d\theta \le C_{14} |F(r)|^p (1-r). \tag{2.12}$$

Inequality (2.7) now follows from (2.10)–(2.12), so Lemma 2.3 is proved. □

Proof of Theorem 1.5. We note that the proof of Lemma 2.1 is valid for any analytic function f in $\mathbb{D}$. Let $A = (a_k)$ be the zero sequence of f, and $\rho > 0$. Since $\rho_M[f] \le \rho_\infty[f]$, using Lemmas 2.1 and 2.2 we deduce that $\nu[A] \le \rho_\infty[f]$, i.e. $\nu[f] \le \rho_\infty[f]$. □

3. Applications

3.1. **Factorization.** In [9, Theorem I] Linden proved the following result.

Theorem E. *Let f be analytic in $\mathbb{D}$ and of order $\rho_M[f] \ge 1$. Then*

$$f(z) = z^p P(z) g(z),$$

where P is a canonical product displaying the zeros of f, p is a nonnegative integer, g is nonzero and both P and g are analytic and of ρ_M-order at most $\rho_M[f]$.

Further, in Theorem IV [9], Linden showed that actually

$$\max\{\rho_M[P], \rho_M[g]\} \le \max\{\rho_M[f], \nu[f]\}.$$

Taking into account Theorem 1.5 we deduce that $\max\{\rho_M[P], \rho_M[g]\} \le \rho_\infty[f]$ in this case.

Theorem 3.1. *Let f be analytic in $\mathbb{D}$, and of finite order $\rho_\infty[f]$. Then*

$$f(z) = z^p P(z) g(z),$$

where P is a canonical product displaying the zeros of f, p is a nonnegative integer, g is nonzero and both P and g are analytic and of ρ_∞-order at most $\rho_\infty[f]$.

Proof of Theorem 3.1. Let $s = [\nu[f]] + 1$. Consider the canonical product $P(z) = P(z, A, s)$. This leads to the factorization $f(z) = z^p P_s(z) g(z)$, where p is the multiplicity of the zero at the origin and g is analytic and nonzero in $\mathbb{D}$. By Theorems 1.4 and 1.5, we have

$$\rho_\infty[P_s] = \nu[P_s] = \nu[f] \le \rho_\infty[f].$$

Since the multiplication by the factor z^p does not change the order ρ_p and

$$\rho_p[g] \le \max\{\rho_p[f], \rho_p[P_s]\} \le \max\{\rho_\infty[f], \rho_\infty[P_s]\},$$

passing to the limit in the latter inequality we get

$$\rho_\infty[g] \le \max\{\rho_\infty[f], \rho_\infty[P_s]\} = \rho_\infty[f].$$

□

The ρ_∞-order of nonconstant meromorphic functions has the following properties:

i) $\rho_\infty[1/f] = \rho_\infty[f]$;
ii) $\rho_\infty[fg] \le \max\{\rho_\infty[f], \rho_\infty[g]\}$.

These properties yield $\rho_\infty[f/g] \le \max\{\rho_\infty[f], \rho_\infty[g]\}$.

In view of the last inequality and Theorem 3.1 the next question arises naturally:

Question 3.2. Given a meromorphic function f in $\mathbb{D}$ of finite order, is it possible to represent it in the form

$$f(z) = z^p \frac{P(z)}{Q(z)} g(z), \tag{3.1}$$

where $p \in \mathbb{Z}$, P, Q are canonical products displaying zeros and poles of f respectively, g is a nonzero analytic function, and all P, Q, g are of ρ_∞-order at most $\rho_\infty[f]$?

It turns out that the answer is in the negative.

Theorem 3.3. *There exists a meromorphic function f in $\mathbb{D}$ such that $\rho_\infty[f] = 0$, and $\nu[A] = \nu[A^*] = 1$, where A and A^* are sequences of zeros and poles of f, respectively. Therefore, for any factorization of f of the form* (3.1)*, $\rho_\infty[P] \ge 1$, $\rho_\infty[Q] \ge 1$.*

Proof of Theorem 3.3. Let $r_k = 1 - 2^{-k}$, $a_k = r_k$, $a_k^* = a_k + 2^{-k^2}$, $p_k = [2^k k^{-2}]$, $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} p_k(1 - a_k) < +\infty$, we can consider the Blaschke products

$$B(z) = \prod_{k=1}^{\infty} \Big(\frac{a_k - z}{1 - z\bar{a}_k} \Big)^{p_k} \equiv \prod_{k=1}^{\infty} \Big(F(z, a_k) \Big)^{p_k}, \quad B^*(z) = \prod_{k=1}^{\infty} \Big(F(z, a_k^*) \Big)^{p_k}, \tag{3.2}$$

which are analytic in $\mathbb{D}$.

Define $f(z) = B(z)/B^*(z)$. We shall show that $\rho_\infty[f] = 0$, though $\rho_\infty[B] = \rho_\infty[B^*] = 1$. The latter equalities follow from Corollary 1.5 and the equalities $\nu[B] = \nu[B^*] = 1$, which are easy to check. We are going to prove that $m_p(r, f)$ is bounded on $[0, 1)$ for all $p > 1$. This will imply that $\rho_\infty[f] = 0$.

Let $z = re^{i\varphi}$, $r \in [r_m, r_{m+1})$ for some $m \in \mathbb{N}$. In order to estimate $|\log|f(re^{i\varphi})||$ we consider three cases.

If $\operatorname{Im} z \ne 0$, we can write

$$\Big| \log|F(z, a_k)| - \log|F(z, a_k^*)| \Big| = \Big| \operatorname{Re} \int_{[a_k, a_k^*]} \frac{1 - |z^2|}{(\zeta - z)(1 - \bar{z}\zeta)} \, d\zeta \Big|. \tag{3.3}$$

If $k \le m - 2$ or $k \ge m + 2$, and ζ is such that $r_k \le |\zeta| \le r_{k+1}$, then

$$|\zeta - z| \ge r_{m+2} - r_{m+1} \ge (1 - r)/4$$

and

$$\frac{1 - |z|^2}{|\zeta - z||1 - z\zeta|} \le \frac{1 - |z|^2}{\frac{1 - |z|}{4}(1 - |\zeta|)} = \frac{8}{1 - |\zeta|}. \tag{3.4}$$

Combining (3.3) and the latter inequality, we obtain

$$\begin{aligned}\Big(\sum_{k=1}^{m-2}+\sum_{k=m+2}^{\infty}\Big)\Big|p_k\log\Big|\frac{F(z,a_k)}{F(z,a_k^*)}\Big|\Big|\\ \le\sum_{k=1}^{\infty}\frac{8p_k(a_k^*-a_k)}{1-a_k^*}\le\sum_{k=1}^{\infty}8\frac{2^{-k^2}2^k}{(2^{-k}-2^{-k^2})k^2}<C_{15}.\end{aligned}\tag{3.5}$$

We note that (3.4) holds in the case when $r_{m-1}\le|\zeta|\le r_{m+2}$ and $|\varphi|\ge 1-r$ for $m\ge m_0$, and some $m_0\in\mathbb{N}$. Hence,

$$\sum_{k=m-1}^{m+1}p_k\Big|\log\frac{|F(z,a_k)|}{|F(z,a_k^*)|}\Big|\le\sum_{k=m-1}^{m+1}\frac{8p_k(a_k^*-a_k)}{1-a_k^*}<C_{15}.\tag{3.6}$$

Then, using (3.5) and (3.6), we deduce

$$\begin{aligned}\Big(\frac{1}{2\pi}\int_{1-r\le|\varphi|\le\pi}|\log|f(re^{i\varphi})||^p\,d\varphi\Big)^{\frac1p}\\ \le\Big(\frac{1}{2\pi}\int_{1-r\le|\varphi|\le\pi}\Big(\sum_{k=1}^{\infty}p_k\Big|\log\Big|\frac{F(re^{i\varphi},a_k)}{F(re^{i\varphi},a_k^*)}\Big|\Big|\Big)^p d\varphi\Big)^{\frac1p}\le 2C_{15}.\end{aligned}\tag{3.7}$$

In the case $|\varphi|\le 1-r$ we write

$$\log|F(z,a_k)|-\log|F(z,a_k^*)|=\log\Big|\frac{z-a_k}{z-a_k^*}\Big|+\log\Big|\frac{1-za_k^*}{1-za_k}\Big|.$$

We have

$$\Big|\log\Big|\frac{1-za_k^*}{1-za_k}\Big|\Big|=\Big|\log\Big|1-\frac{z(a_k^*-a_k)}{1-za_k}\Big|\Big|\le\frac{C_{16}(a_k^*-a_k)}{|1-za_k|}\le\frac{C_{17}(a_k^*-a_k)}{1-a_k},\quad k\in\mathbb{N}.$$

As above, we get $\sum_{k=1}^{\infty}p_k\Big|\log\Big|\frac{1-za_k^*}{1-za_k}\Big|\Big|<C_{18}$.

To finish the proof we need a lemma.

Lemma 3.4. *For any $a,b\in\mathbb{C}$, and $p>1$,*

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\log\Big|\frac{a-e^{i\varphi}}{b-e^{i\varphi}}\Big|\Big|^p d\varphi\le C(p)|a-b|.$$

Proof of Lemma 3.4. Since we are going to apply the lemma in the case $|a-b|<1$, we omit the proof of the case $|a-b|\ge 1$ for simplicity.

We divide the unit circle into three parts:

$$\begin{aligned}&I_1=\{e^{i\theta}:|e^{i\theta}-b|>2|a-b|\}\cup\{e^{i\theta}:|e^{i\theta}-a|>2|a-b|\},\\ &I_2=\{e^{i\theta}:|e^{i\theta}-b|<|a-b|/2\}\cup\{e^{i\theta}:|e^{i\theta}-a|<|a-b|/2\},\\ &I_3=\partial\mathbb{D}\setminus(I_1\cup I_2).\end{aligned}\tag{3.8}$$

Let $e^{i\theta} \in I_1$. Without loss of generality we may assume that $|e^{i\theta} - b| > 2|a-b|$, $b = |b|$. Then

$$\Big|\frac{|e^{i\theta}-a|}{|e^{i\theta}-b|} - 1\Big| \le \frac{|a-b|}{|e^{i\theta}-b|} \le \frac{1}{2}.$$

Hence,

$$\frac{1}{2\pi}\int_{I_1}\Big|\log\frac{|e^{i\theta}-a|}{|e^{i\theta}-b|}\Big|^p \le \frac{1}{\pi}\int_{|e^{i\theta}-b|>2|a-b|}\Big(\frac{2|a-b|}{|e^{i\theta}-b|}\Big)^p\,d\theta.$$

For $J_1 = \{e^{i\theta} \in I_1 : |\theta| > |a-b|\}$, we have

$$\frac{1}{\pi}\int_{J_1}\Big(\frac{2|a-b|}{|e^{i\theta}-b|}\Big)^p\,d\theta \le \frac{\mathrm{mes}(J_1)}{\pi} \le \frac{2|a-b|}{\pi}, \tag{3.9}$$

where $\mathrm{mes}(J_1)$ is the Lebesgue measure of the set J_1.

If $e^{i\theta}$ belongs to the complement of J_1 in I_1, we consider two subcases. If $|b| \ge \frac{1}{2}$, then standard estimates yield

$$\begin{aligned}
&\frac{1}{2\pi}\int_{I_1\setminus J_1}\Big|\log\Big|\frac{|e^{i\theta}-a|}{|e^{i\theta}-b|}\Big|\Big|^p \\
&\quad\le \frac{(2|a-b|)^p}{\pi}\Big(\int_{|a-b|<|\theta|\le\frac{\pi}{2}}\frac{d\theta}{(b|\sin\theta|)^p} + \int_{\frac{\pi}{2}<|\theta|\le\pi}d\theta\Big) \\
&\quad\le \frac{(2|a-b|)^p}{\pi}\Big(\pi^p\int_{|a-b|}^{\pi/2}\frac{d\theta}{\theta^p} + \pi\Big) \le \frac{2^{p+1}\pi^{p-1}}{p-1}|a-b|.
\end{aligned}\tag{3.10}$$

So, we have the required estimate for the integral over I_1.

We then consider the integral over I_2. We deduce

$$\begin{aligned}
&\int_{I_2}\Big|\log\Big|\frac{e^{i\theta}-b}{e^{i\theta}-a}\Big|\Big|^p d\theta \le 2\int_{|e^{i\theta}-b|<|a-b|/2}\Big|\log\Big|\frac{e^{i\theta}-b}{e^{i\theta}-a}\Big|\Big|^p d\theta \\
&\quad\le 2\int_{|e^{i\theta}-a|<|a-b|/2}\Big(\log\frac{3|a-b|}{2|e^{i\theta}-b|}\Big)^p d\theta \le 2\int_{|e^{i\theta}-b|<|a-b|/2}\Big(\log\frac{3|a-b|}{2|\sin\theta|}\Big)^p \\
&\quad\le 2\int_{|\theta|<|a-b|}\Big(\log\frac{3\pi|a-b|}{4|\theta|}\Big)^p dt \le 2|a-b|\int_{|\tau|<1}\Big(\log\frac{3\pi}{4\tau}\Big)^p d\tau = C_{19}(p)|a-b|.
\end{aligned}\tag{3.11}$$

Finally, if $e^{i\theta} \in \partial\mathbb{D}\setminus(I_1\cup I_2)$, then we have $\Big|\log\Big|\frac{e^{i\theta}-a}{e^{i\theta}-b}\Big|\Big| \le 4$, and $\mathrm{mes}(\partial\mathbb{D}\setminus(I_1\cup I_2)) \le C_{20}|a-b|$. Therefore,

$$\int_{\partial\mathbb{D}\setminus(I_1\cup I_2)}\Big|\log\Big|\frac{e^{i\theta}-b}{e^{i\theta}-a}\Big|\Big|^p d\theta \le C_{21}(p)|a-b|. \tag{3.12}$$

The assertion of the lemma follows from (3.9)–(3.12). $\square$

We return to proving Theorem 3.3. Using Minkowski's inequality, the estimates (3.5), (3.6), and Lemma 3.4 we obtain, for $r_m \le r \le r_{m+1}$,

$$\begin{aligned}
&\left(\frac{1}{2\pi}\int_{|\varphi|\le 1-r} |\log|f(re^{i\varphi})||^p\,d\varphi\right)^{\frac1p}\\
&\quad\le \Bigg(\frac{1}{2\pi}\int_{|\varphi|\le 1-r}\Big|\sum_{k=m-1}^{m+1} p_k\Big(\log\Big|\frac{re^{i\varphi}-a_k}{re^{i\varphi}-a_k^*}\Big| + \log\Big|\frac{1-re^{i\varphi}a_k^*}{1-re^{i\varphi}a_k}\Big|\Big)\\
&\qquad + \Big(\sum_{k=1}^{m-2}+\sum_{k=m+2}^{\infty}\Big)p_k\log\Big|\frac{F(re^{i\varphi},a_k)}{F(re^{i\varphi},a_k^*)}\Big|\Big|^p\,d\varphi\Bigg)^{\frac1p}\\
&\quad\le \sum_{k=m-1}^{m+1} p_k\Big(\frac{1}{2\pi}\int_{|\varphi|\le 1-r}\Big|\log\Big|\frac{e^{i\varphi}-\frac{a_k}{r}}{e^{i\varphi}-\frac{a_k^*}{r}}\Big|\Big|^p d\varphi\Big)^{\frac1p} + \Big(\frac{1}{2\pi}\int_{|\varphi|\le 1-r}(C_{18}+C_{15})^p\,d\varphi\Big)^{\frac1p}\\
&\quad\le C_{22}\sum_{k=m-1}^{m+1}(2^k k^{-2}(a_k^*-a_k))^{\frac1p} + C_{23}(1-r)^{\frac1p}\\
&\quad\le C_{24}((2^{-(m-1)^2+m}m^{-2})^{\frac1p} + (1-r_m)^{\frac1p}) = o(1), \quad m\to+\infty.
\end{aligned}$$

The last estimate together with (3.7) implies $m_p(r,f) = O(1)$ as $r\uparrow 1$ for any $p>1$. □

3.2. **Logarithmic derivative estimates.** Results of this section allow us to obtain sharp lower estimates for the growth of solutions of linear differential equations in the unit disc. For this purpose one has to follow the scheme of the proof of Theorem 1.4 from [3]. However, it seems that neither the approach based on Herold's comparison theorem (see [7]) nor the Wiman-Valiron method give us sharp upper estimates for the ρ_∞-order of solutions of linear differential equations in the most interesting case when $\rho_M < 1$.

The following theorem is proved in [3].

Theorem F. *Let f be an analytic function in $\mathbb{D}$ such that $\rho_M[f]<\infty$. If $\rho_M[f]>0$, let ρ be a proximate order of f. Let k and j be integers satisfying $k>j\ge 0$, and let $\delta,\varepsilon\in(0,1)$. Then there exist an at most countable collection of discs $D_\nu = D(z_\nu,r_\nu)$, where $r_\nu < 1-|z_\nu|$, and a constant $C>0$ such that*

$$\sum_{R<|z_\nu|<1} r_\nu \le \delta(1-R), \quad R\uparrow 1, \tag{3.13}$$

and

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \begin{cases} C\left(\frac{\log\frac{1}{1-|z|}}{\delta(1-|z|)^{\rho(|z|)+1}}\right)^{k-j}, & \text{if } \rho_M[f]>1,\\ \left(\frac{1}{\delta(1-|z|)}\right)^{2(k-j)+\varepsilon}, & \text{if } \rho_M[f]\le 1,\end{cases} \tag{3.14}$$

for all $z\in\mathbb{D}\setminus\bigcup_\nu D_\nu$.

For a measurable set $E\subset[0,1)$, the *upper linear density* is defined as

$$\mathcal{D}(E) = \limsup_{r\uparrow 1}\frac{\operatorname{mes}(E\cap[r,1))}{1-r}.$$

Corollary G. *Under the assumptions of Theorem* F *there exists an exceptional set* $E \subset [0,1)$ *with* $\mathcal{D}(E) \leq 2\delta$ *such that*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(\max\{\rho_M[f],1\}+1)(k-j)+\varepsilon}, \quad |z| \notin E. \tag{3.15}$$

Since Theorem F and Corollary G are sharp in the sense that one cannot remove ε in the exponent, it is not possible to obtain a better estimate than $O((1-|z|)^{-2(k-j)-\varepsilon})$ in the same notation. However, the concept of ρ_∞-order allows us to improve this estimate.

Theorem 3.5. *Let f be an analytic function in $\mathbb{D}$ such that $\rho = \rho_\infty[f] < \infty$. Let k and j be integers satisfying $k > j \geq 0$, and let $\delta, \varepsilon \in (0,1)$. Then there exist an at most countable collection of discs $D_\nu = D(z_\nu, r_\nu)$, where $r_\nu < 1 - |z_\nu|$, and a constant $C > 0$ such that*

$$\sum_{R<|z_\nu|<1} r_\nu \leq \delta(1-R), \quad R \uparrow 1, \tag{3.16}$$

and

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\left(\frac{1}{\delta(1-|z|)^{\rho+1+\varepsilon}}\right)^{k-j}, \tag{3.17}$$

for all $z \in \mathbb{D} \setminus \bigcup_\nu D_\nu$.

Example 3.6. Let $f(z) = \exp\{-(1-z)^{-\alpha}\}$, where $\alpha > 0$, $f(0) = 1/e$. Then f is bounded if $\alpha \leq 1$, and hence $\rho_M[f] = 0$. On the other hand, $\rho_M[f] = \alpha$ if $\alpha > 1$. Besides, $\rho_\infty[f] = \alpha$ for all positive α, and $\frac{f'(z)}{f(z)} = -\frac{\alpha}{(1-z)^{\alpha+1}}$, which shows that estimate (3.17) is sharp in the sense that ε cannot be removed.

The proof of Theorem 3.5 is similar to that of Theorem F [3]. Define $\mathcal{A}_1 = \overline{D}(0, \frac{1}{2})$ and the annuli $\mathcal{A}_\nu = \{\zeta : r_{\nu-1} < |\zeta| \leq r_\nu\}$ for $r_\nu = 1 - 2^\nu$, $\nu \geq 2$. Then clearly $\mathbb{D} = \bigcup_\nu \mathcal{A}_\nu$. We now state and sketch the proof of an estimate (cf. Lemma 3 [3]), which is crucial in proving Theorem 3.5.

Lemma 3.7. *Let f be an analytic function in $\mathbb{D}$ such that $\rho = \rho_\infty[f] < \infty$. Let $\{a_k\}$ denote the sequence of zeros of f listed according to multiplicities and ordered by increasing moduli, and let $0 < \delta < 1$. Then there exists an at most countable collection of discs $D_{\nu j} = D(z_{\nu j}, \rho_{\nu j})$, where $\rho_{\nu j} < 1 - |z_{\nu j}|$, such that*

$$\sum_{|a_k| \leq r_{\nu+1}} \frac{1}{|z-a_k|} \leq \frac{C(\rho, R_0)}{\delta(1-r)^{\rho+1+\varepsilon}}, \quad z \in \mathcal{A}_\nu \setminus \bigcup_j D_{\nu j}, \tag{3.18}$$

where

$$\sum_{R<|z_{\nu j}|<1} \rho_{\nu j} \leq \delta(1-R), \quad R \uparrow 1. \tag{3.19}$$

Proof of Lemma 3.7. Without loss of generality, we may assume that $\arg z = 0$. Define $I_s = \{-2^s+1, \dots, -1, 0, 1, \dots, 2^s\}$ for $s \in \mathbb{N}$. For $\tau \in I_s$, define the polar rectangles $\mathcal{A}_{s\tau} = \{\zeta \in \mathcal{A}_s : (\tau-1)\pi 2^{-s} \leq \arg \zeta < \tau\pi 2^{-s}\}$.

Denote

$$I_s^* = \begin{cases} I_s, & 1 \leq s \leq \nu-2, \\ I_s \setminus \{0,1\}, & \nu-1 \leq s \leq \nu+1. \end{cases}$$

With this notation, we have the following result [3, Lemma 4].

Lemma H. *Let* $\nu \geq 2$, $2 \leq s \leq \nu+1$, $\tau \in I_s^*$, $\zeta \in \mathcal{A}_{s\tau}$, $z \in \mathcal{A}_\nu$ *and* $\arg z = 0$. *Then*

$$|\zeta - z| \geq \begin{cases} \tau 2^{-s-1}, & \tau > 0, \\ (|\tau|+1)2^{-s-1}, & \tau \leq 0. \end{cases}$$

We return to the proof of Lemma 3.7. Let $n(U)$ denote the number of the points a_k in a set $U \subset \mathbb{D}$. Recall that $\square(re^{i\varphi}) = \left\{\zeta : r \leq |\zeta| \leq \frac{1+r}{2}, |\arg\zeta - \varphi| \leq \frac{\pi}{4}(1-r)\right\}$. Then $\nu(re^{i\varphi}) \leq C(1-r)^{-\rho-\varepsilon/2}$.

If $z \in \mathcal{A}_\nu$ and $z \neq a_k$ for all k, write

$$\begin{aligned} \sum_{|a_k| \leq r_{\nu+1}} \frac{1}{|z-a_k|} &= \sum_{s=1}^{\nu+1} \sum_{\tau \in I_s} \sum_{a_k \in \mathcal{A}_{s\tau}} \frac{1}{|z-a_k|} \\ &\leq \sum_{s=1}^{\nu+1} \sum_{\tau \in I_s^*} \sum_{a_k \in \mathcal{A}_{s\tau}} \frac{1}{|z-a_k|} + \sum_{s=\nu-1}^{\nu+1} \sum_{\tau=0}^{1} \sum_{a_k \in \mathcal{A}_{s\tau}} \frac{1}{|z-a_k|} \\ &= \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned} \tag{3.20}$$

To deal with the sum $\sum_1$ in (3.20), we first observe that $r_s = \frac{1+r_{s-1}}{2}$ and $\tau\pi 2^{-s} - (\tau-1)\pi 2^{-s} = \pi 2^{-s} = 2\pi 2^{-2}(1-r_{s-1})$. Lemma H now yields

$$\begin{aligned} \sum\nolimits_1 &\leq \sum_{s=1}^{\nu+1} \sum_{\tau \in I_s^*} \frac{n(\mathcal{A}_{s\tau})}{\inf_{\zeta \in \mathcal{A}_{s\tau}} |z-\zeta|} \\ &\leq \sum_{s=1}^{\nu+1} \left(\sum_{\tau \in I_s^*,\, \tau>0} \frac{n_1(r_{s-1})}{\tau} 2^{s+1} + \sum_{\tau \in I_s^*,\, \tau \leq 0} \frac{n_1(r_{s-1})}{|\tau|+1} 2^{s+1} \right) \\ &\leq 8 \sum_{s=1}^{\nu+1} \frac{n_1(r_{s-1})}{1-r_{s-1}} \left(\sum_{\tau=1}^{2^s} \frac{1}{\tau} \right) \\ &\leq 8(1+(\nu+1)\log 2) \sum_{s=1}^{\nu+1} \frac{n_1(r_{s-1})}{1-r_{s-1}} \leq 24\nu \sum_{s=1}^{\nu+1} 2^{(s-1)(\rho+1+\varepsilon)} \\ &< 48\nu 2^{\nu(\rho+1+\varepsilon/2)} < \frac{C}{(1-r_\nu)^{\rho+1+\varepsilon}}. \end{aligned} \tag{3.21}$$

To deal with the sum $\sum_2$ in (3.20), define

$$U = \bigcup_{s=\nu-1}^{\nu+1} \bigcup_{\tau=0}^{1} \mathcal{A}_{s\tau}, \quad N_\nu = n(U), \quad \text{and} \quad \delta_\nu = \delta \cdot 2^{-\nu-6}.$$

Then, by the Cartan lemma [8, pp. 19–21], there exists a finite collection of discs $D(w_{\nu j}, h_{\nu j})$ with $\sum_j h_{\nu j} = 2\delta_\nu$ and a permutation of $\{a_k\} \subset U$, say $b_1, \ldots, b_{N_\nu}$, such that $|z - b_m| > m\delta_\nu / N_\nu$ for all $m = 1, \ldots, N_\nu$ and $z \notin \bigcup_j D(w_{\nu j}, h_{\nu j})$. Hence, by noting that

$$n(\mathcal{A}_{s\tau}) \leq n_1(r_{s-1}), \quad s = \nu-1, \ldots, \nu+1,\ \tau = 0, 1,$$

it follows that

$$\begin{aligned}\sum\nolimits_2 &= \sum_{a_k\in U}\frac{1}{|z-a_k|} = \sum_{m=1}^{N_\nu}\frac{1}{|z-b_m|}\\ &\le \frac{N_\nu}{\delta_\nu}\sum_{m=1}^{N_\nu}\frac{1}{m} \le \frac{2^5 N_\nu}{\delta(1-r_{\nu+1})}(1+\log N_\nu)\\ &\le \frac{C}{\delta}\sum_{s=\nu-2}^{\nu+1}\frac{n_1(r_s)}{1-r_s}\log n_1(r_s) \le \frac{C}{\delta(1-r_\nu)^{\rho+1+\varepsilon}}\end{aligned} \tag{3.22}$$

for all $z\in\mathcal{A}_\nu$ such that $z\notin\bigcup_j D(w_{\nu j},h_{\nu j})$.

By combining (3.20)–(3.22) we conclude that

$$\sum_{|a_k|\le r_{\nu+1}}\frac{1}{|z-a_k|} \le \frac{C(\rho,R_0)}{\delta(1-r)^{\rho+1+\varepsilon}} \tag{3.23}$$

for all $z\in\mathcal{A}_\nu$ such that $z\notin\bigcup_j D(w_{\nu j},h_{\nu j})$.

An estimate of the exceptional set repeats that given in [3]. □

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