

A CONFORMAL INTEGRAL INVARIANT ON RIEMANNIAN FOLIATIONS

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ABSTRACT. Let M be a closed manifold which admits a foliation structure \mathcal{F} of codimension $q \geq 2$ and a bundle-like metric g_0 . Let $[g_0]_B$ be the space of bundle-like metrics which differ from g_0 only along the horizontal directions by a multiple of a positive basic function. Assume Y is a transverse conformal vector field and the mean curvature of the leaves of (M, \mathcal{F}, g_0) vanishes. We show that the integral $\int_M Y(R_{g^T}^T) d\mu_g$ is independent of the choice of $g \in [g_0]_B$, where g^T is the transverse metric induced by g and R^T is the transverse scalar curvature. Moreover if $q \geq 3$, we have $\int_M Y(R_{g^T}^T) d\mu_g = 0$ for any $g \in [g_0]_B$. However there exist codimension 2 minimal Riemannian foliations (M, \mathcal{F}, g) and transverse conformal vector fields Y such that $\int_M Y(R_{g^T}^T) d\mu_g \neq 0$. Therefore, $\int_M Y(R_{g^T}^T) d\mu_g$ is a nontrivial obstruction for the transverse Yamabe problem on minimal Riemannian foliation of codimension 2.

1. INTRODUCTION

In [13] Kazdan and Warner discovered an obstruction to the existence of metrics with prescribed scalar curvature on \mathbb{S}^2 . Let $(\mathbb{S}^2, g_{\mathbb{S}^2})$ be the unit sphere in \mathbb{R}^3 with the standard metric and h be a given function on \mathbb{S}^2 . Kazdan and Warner found that if φ is a solution to the equation

$$(1.1) \quad \Delta_{g_{\mathbb{S}^2}} \varphi + 2 - h e^{-\varphi} = 0,$$

then for any first order spherical harmonic F (i.e., the restriction to \mathbb{S}^2 of a linear function in \mathbb{R}^3) it holds that

$$(1.2) \quad \int_{\mathbb{S}^2} g(\nabla F, \nabla h) e^{-\varphi} d\mu_{g_{\mathbb{S}^2}} = 0.$$

If $g = e^{-\varphi} g_{\mathbb{S}^2}$ and φ satisfies (1.1), the scalar curvature of g is equal to h . Hence, (1.2) is just

$$(1.3) \quad \int_{\mathbb{S}^2} (\nabla F)(R_g) d\mu_g = 0.$$

An integrability condition similar to (1.2) was proved for higher dimensional spheres [14]. The integrability condition (1.2) was generalized to any closed Riemannian manifold with a conformal vector field; see [5, 6]. Note that ∇F in (1.2) is a conformal vector field with respect to the standard metric on \mathbb{S}^2 . Let (M, g_0)

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be a closed Riemannian manifold and $[g]$ be the conformal class of g_0 . Bourguignon and Ezin proved that for any conformal vector field Y

$$(1.4) \quad \int_M Y(R_g) d\mu_g = 0, \quad \forall g \in [g_0].$$

By (1.4), they found new functions which cannot be realized as the scalar curvature of a Riemannian metric on \mathbb{S}^2 . We remark that the proof given in [6] is different for dimension $n \geq 3$ and for dimension $n = 2$. There is another interesting proof given in [5]. See also [9]. One first shows that

$$(1.5) \quad \int_M Y(R_g) d\mu_g$$

is a conformal invariant, and then shows that this invariant vanishes by a result of Obata. For the case that (M, g) is a compact manifold with boundary, see [16].

Before [5] and [6], Futaki [8] found an analogous invariant for the complex Monge-Ampère equation on Kähler-Einstein manifolds of positive first Chern class. This is the well-known Futaki invariant, which is one of the main obstructions to the existence of Kähler-Einstein metrics of positive first Chern class.

Very recently we have studied a Yamabe type problem on Riemannian foliations, i.e., finding a bundle-like metric g in a given basic conformal class with constant transverse scalar curvature. This is a natural generalization of the Yamabe problem to Riemannian foliation.

Let us first briefly recall the definitions of basic conformal class. Let (M, \mathcal{F}, g_0) be a closed Riemannian foliation of codimension $q \geq 2$ with a bundle-like metric g_0 . From now on we assume that M is oriented and \mathcal{F} is transversally oriented. Let \mathcal{L} denote the integrable subbundle given by \mathcal{F} . The bundle-like metric g_0 induces a holonomy invariant transverse metric g_0^T on the normal bundle $\nu(\mathcal{F}) = TM/\mathcal{L}$ of the foliation. Let $\Omega_B^0(M, \mathcal{F})$ denote the space of all basic functions and

$$(1.6) \quad [g_0]_B = \{g = g_0|_{\mathcal{L}} + e^u g_0|_{\mathcal{L}^\perp} : u \in \Omega_B^0(M, \mathcal{F})\},$$

where $[g_0]_B$ is called the basic conformal class of the bundle-like metric g_0 . Any transverse metric of the form $g^T = e^u g_0^T$ is holonomy invariant if and only if u is a basic function. We call a transverse metric g^T conformal to g_0^T if $g^T = e^u g_0^T$ for a basic function u . We denote the space of all conformal transverse metrics of g_0^T by $[g_0^T]_B$. There is a one-to-one correspondence between $[g_0]_B$ and $[g_0^T]_B$, and we denote by g^T the transverse metric induced by $g \in [g_0]_B$. We denote by $R_{g^T}^T$ the transverse scalar curvature of g^T . A Riemannian foliation is called minimal if the mean curvature of the leaves vanishes. For geometry of foliations, see for instance [18] or Section 2 below. For the definition of transverse vector field, see [12] or Section 2 below. First, we observe that an integral similar to (1.5) is invariant in a basic conformal class.

Theorem 1.1. *Let Y be a transverse conformal vector field on the minimal Riemannian foliation (M, \mathcal{F}, g_0) . Then the integral*

$$(1.7) \quad \int_M Y(R_{g^T}^T) d\mu_g$$

is independent of the choice of $g \in [g_0]_B$.

Without the assumption that the Riemannian foliation (M, \mathcal{F}, g_0) is minimal, $\int_M Y(R_{g^T}^T) d\mu_g$ may depend on $g \in [g_0]_B$.

It is easy to see that invariant (1.7) is an obstruction of the following transverse Yamabe problem.

Transverse Yamabe problem. Let (M, \mathcal{F}, g_0) be a Riemannian foliation with a bundle-like metric g_0 . Does there exist any $g^T \in [g_0^T]_B$ such that

$$R_{g^T}^T = \text{const.}?$$

This is a natural generalization of the ordinary Yamabe problem, which was resolved by Yamabe, Trudinger, Aubin and finally by Schoen. The resolution of the Yamabe problem is a milestone of geometric analysis. An equivariant version of the Yamabe problem has been studied by Hebey and Vaugon [11]. Very recently, a fully nonlinear Yamabe type problem has been studied by Viaclovsky [20], Chang-Gursky-Yang and many other mathematicians. See the survey of Viaclovsky [19]. For further study of the ordinary Yamabe problem, see the survey of Brendle and Marques [4].

It is clear that the invariant (1.7) is an obstruction of the transverse Yamabe problem, at least for minimal foliations, which is the most interesting case: if there is a solution, then invariant (1.7) must vanish. Unlike the ordinary Yamabe problem, now invariant (1.7) is not a trivial invariant.

Theorem 1.2. *There are examples of minimal Riemannian foliation of codimension $q = 2$ with a transverse conformal vector field Y such that invariant (1.7) does not vanish. Hence on such a Riemannian foliation there is no solution for the transverse Yamabe problem.*

The examples come from our study of 3-dimensional Sasaki-Ricci flow in [22]. For the Sasaki-Ricci flow see [17].

However, for higher codimension ($q > 2$), this invariant still vanishes, though we believe that in general there exist obstructions for the transverse Yamabe problem.

When the leaves of a Riemannian foliation (M, \mathcal{F}, g_0) are all compact, the leaf space with the induced metric from g_0 is a Riemannian orbifold. Moreover, its scalar curvature is exactly the transverse scalar curvature. Hence, in this case the transverse Yamabe problem is equivalent to the Yamabe problem on orbifolds, which has been studied by Akutagawa and Botvinnik in [2] and [1]. Very recently, Viaclovsky [20] gave interesting examples of a 4-dimensional orbifold, on which the orbifold Yamabe problem has no solution. It is an interesting and natural question whether one can find a similar obstruction for the orbifold Yamabe problem. Our results can only provide an obstruction for 2-dimensional orbifolds. The nonexistence of solutions to the orbifold Yamabe problem on (bad) 2-dimensional orbifolds follows from the result of Langfang Wu in [24].

The paper is organized as follows. In Section 2, we first provide preliminaries on Riemannian foliation and the transverse Yamabe problem. Then we show the conformal invariance of (1.7). In Section 3 we provide examples of minimal Riemannian foliation of codimension 2 with nonvanishing (1.7).

2. A CONFORMAL INTEGRAL INVARIANT

In this section we show that on a minimal foliation (M, \mathcal{F}, g_0) with a transverse conformal vector field Y , the integral (1.7) is independent of the choice of $g \in [g_0]_B$. If in addition the codimension of the foliation is greater than or equal to 3, we show that (1.7) is equal to zero for $g \in [g_0]_B$.

Let (M, \mathcal{F}) be a closed manifold with a foliation \mathcal{F} of codimension q . Let \mathcal{L} denote the integrable subbundle given by \mathcal{F} and $\nu(\mathcal{F}) = TM/\mathcal{L}$. We denote the quotient map by $\pi : TM \rightarrow \nu(\mathcal{F})$. Any Riemannian metric g on M provides a splitting of the exact sequence of bundles

$$0 \rightarrow \mathcal{L} \rightarrow TM \rightarrow \nu(\mathcal{F}) \rightarrow 0$$

and an isomorphism of bundles $\sigma : \nu(\mathcal{F}) \rightarrow \mathcal{L}^\perp$ satisfying $\pi \circ \sigma = id$. The transverse metric g^T corresponding to (M, \mathcal{F}, g) is defined by

$$g^T(s_1, s_2) = g(\sigma s_1, \sigma s_2), \quad s_1, s_2 \in \Gamma \nu(\mathcal{F}).$$

The Riemannian metric g is called bundle-like if the induced transverse metric g^T is holonomy invariant; i.e., $L_\xi g^T = 0$ for any $\xi \in \Gamma \mathcal{L}$. A foliation (M, \mathcal{F}) with a bundle-like metric g is called a Riemannian foliation. For the geometry of Riemannian foliations, see [18].

Let (M, \mathcal{F}, g) be a Riemannian foliation (i.e., g is a bundle-like metric) and ∇^M be the Levi-Civita connection of (M, g) . We denote the transverse Levi-Civita connection on $(\nu(\mathcal{F}), g^T)$ by ∇ . This connection is defined by

$$\nabla_X s := \begin{cases} \pi[X, Y_s], & \text{if } X \in \Gamma \mathcal{L}, \\ \pi(\nabla_X^M Y_s), & \text{if } X \in \Gamma \mathcal{L}^\perp, \end{cases}$$

where $s \in \Gamma \nu(\mathcal{F})$ and $Y_s = \sigma s$.

The transverse curvature operator is then defined by

$$R^T(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s.$$

Let $\{e_i\}_{i=1}^q$ be a local orthonormal frame on $(\mathcal{L}^\perp, g|_{\mathcal{L}^\perp})$. The transverse Ricci curvature and the transverse scalar curvature are defined respectively by

$$Ric^T(X, Y) = g^T(R^T(X, e_i)(\pi e_i), \pi Y)$$

and

$$R^T = Ric^T(e_j, e_j).$$

Both Ric^T and R^T are well-defined; i.e., they are independent of the choice of the local frame $\{e_i\}_{i=1}^q$. Note that R^T is basic; i.e., $\xi(R^T) = 0$ for any $\xi \in \Gamma \mathcal{L}$.

We denote by τ the mean curvature vector field of leaves. That is

$$\tau = (\nabla_{\xi_\alpha}^M \xi_\alpha)^\perp,$$

where $\{\xi_\alpha\}$ is a local orthonormal frame of \mathcal{L} and X^\perp denotes the projection of X to \mathcal{L}^\perp . A Riemannian foliation (M, \mathcal{F}, g) is said to be minimal if $\tau = 0$.

Let $V(\mathcal{F})$ denote the space of all infinitesimal automorphisms of \mathcal{F} , i.e.,

$$V(\mathcal{F}) = \{Y \in \Gamma(TM) \mid L_Y \xi \in \Gamma \mathcal{L}, \quad \forall \xi \in \Gamma \mathcal{L}\}.$$

The space of transverse vector fields is defined by

$$\overline{V}(\mathcal{F}) = \{\overline{Y} := \pi(Y) \mid Y \in V(\mathcal{F})\}.$$

A function f is called basic if $df(\xi) = 0$ for any $\xi \in \Gamma \mathcal{L}$. If a vector field $Y \in V(\mathcal{F})$ satisfies

$$(2.1) \quad L_Y g^T = 2f_Y g^T$$

for a basic function f_Y depending on Y , we call \bar{Y} (or Y) a transverse conformal field. For a transverse conformal field \bar{Y} we have $f_Y = \frac{1}{q} \operatorname{div}^\nabla(\bar{Y})$, where

$$\operatorname{div}^\nabla(\bar{Y}) = g^T(\nabla_{e_i} \bar{Y}, \pi e_i)$$

is the transverse divergence of \bar{Y} with respect to ∇ . The following transverse divergence theorem can be found in [18]. See also [25].

Lemma 2.1. *Let (M, \mathcal{F}, g) be a Riemannian foliation and $X \in V(\mathcal{F})$. Then*

$$\int_M \operatorname{div}^\nabla(\bar{X}) d\mu_g = \int_M g^T(\bar{X}, \pi\tau) d\mu_g.$$

The basic Laplacian Δ_B acting on a basic function u is defined by

$$\Delta_B u = \operatorname{div}^\nabla(\nabla^M u) - \tau(u).$$

Lemma 2.2. *Let (M, \mathcal{F}, g) be a Riemannian foliation of codimension q . Assume that Y is a transverse conformal vector field. Then*

$$(2.2) \quad \frac{q}{2} Y(R^T) = -(q-1)(\Delta_B + \tau) \operatorname{div}^\nabla(\bar{Y}) - R^T \operatorname{div}^\nabla(\bar{Y}).$$

Proof. For the proof of formula (2.2) see for instance [12]. The classical version of (2.2) can be found in [15]. \square

Let (M, \mathcal{F}, g_0) be a Riemannian foliation with mean curvature vector field τ_0 . Recall that any $g \in [g_0]_B$, defined in (1.6), induces a conformal transverse metric $g^T \in [g_0^T]_B$. Let $g = g_0|_{\mathcal{L}} + e^u g_0|_{\mathcal{L}^\perp}$ be a metric in the basic conformal class $[g_0]_B$. Then (M, \mathcal{F}, g) is a Riemannian foliation with mean curvature vector field $\tau = e^{-u} \tau_0$. Hence the condition of minimal foliation (i.e., $\tau = 0$) is conformally invariant. A direct computation gives the relationship of $R_{g^T}^T$ and $R_{g_0^T}^T$

$$(2.3) \quad R_{g^T}^T = e^{-u} [-(q-1)(\Delta_B^0 + \tau_0)u - \frac{(q-1)(q-2)}{4} |\nabla u|_{g_0^T}^2 + R_{g_0^T}^T].$$

It follows that solving the transverse Yamabe problem is equivalent to solving the equation

$$(2.4) \quad -(q-1)(\Delta_B^0 + \tau_0)u - \frac{(q-1)(q-2)}{4} |\nabla u|_{g_0^T}^2 + R_{g_0^T}^T = ce^u.$$

When the foliation is minimal, i.e., $\tau_0 = 0$, the transverse Yamabe equation (2.4) is the Euler-Lagrange equation of the functional

$$(2.5) \quad J(g) = \frac{\int_M R_{g^T}^T d\mu_g}{(\operatorname{vol}(g))^{\frac{q-2}{q}}}, \quad \text{if } q > 2,$$

and

$$(2.6) \quad J_2(g) = \int_M \left(-\frac{1}{2} u \Delta_B^0 u + u R_{g_0^T}^T\right) d\mu_{g_0} - \int_M R_{g_0^T}^T d\mu_{g_0} \log \int_M e^u d\mu_{g_0}, \quad \text{if } q = 2.$$

Recall $g = g_0|_{\mathcal{L}} + e^u g_0|_{\mathcal{L}^\perp}$. Note that when $q = 2$, from (2.3) it is easy to see that $\int_M R_{g^T}^T d\mu_g = \int_M R_{g_0^T}^T d\mu_{g_0}$ for any $g \in [g_0]_B$.

If $\tau_0 \neq 0$, equation (2.4) could be not a variational problem. Therefore we have first restricted ourselves to the case of minimal Riemannian foliation in our study of the transverse Yamabe problem [23]. From now on we consider only minimal

Riemannian foliations. Assume that Y is a transverse conformal vector field. We define

$$(2.7) \quad I_{\overline{Y}} : [g_0]_B \rightarrow \mathbb{R}, \quad I_{\overline{Y}}(g) = \int_M Y(R_{g^T}^T) d\mu_g.$$

Theorem 2.3. *Let (M, \mathcal{F}, g_0) be a minimal Riemannian foliation and Y be a transverse conformal vector field. Then $I_{\overline{Y}}(g)$ is independent of the choice of $g \in [g_0]_B$.*

Proof. Let g^T be any transverse metric in $[g_0^T]_B$. Let $g^T(s) = e^{s\psi} g^T$, where ψ is a basic function. Let $g(s) \in [g_0]_B$ be the bundle-like metric inducing $g^T(s)$. We denote by R^T the transverse curvature of g^T .

It suffices to show that $\frac{d}{ds}|_{s=0} I_{\overline{Y}}(g(s)) = 0$. Note that (M, \mathcal{F}, g_0) is a minimal Riemannian foliation, as is (M, \mathcal{F}, g) . Hence from (2.3) we have

$$R_{g^T(s)}^T = e^{-s\psi} [-(q-1)s\Delta_B\psi - \frac{(q-1)(q-2)}{4}s^2|\nabla\psi|_{g^T}^2 + R^T]$$

and

$$d\mu_{g(s)} = e^{\frac{qs\psi}{2}} d\mu_g.$$

It follows that

$$\begin{aligned} \frac{d}{ds}|_{s=0} I_{\overline{Y}}(g(s)) &= \int_M (Y[-\psi R^T - (q-1)\Delta_B\psi] + \frac{q}{2}\psi Y(R^T)) d\mu_g \\ &= \int_M [(\frac{q}{2}-1)\psi Y(R^T) - R^T Y(\psi) - (q-1)Y\Delta_B\psi] d\mu_g. \end{aligned}$$

By the transverse divergence theorem, we have

$$\begin{aligned} \int_M -R^T Y(\psi) d\mu_g &= - \int_M [\operatorname{div}^\nabla(\psi R^T \overline{Y}) - \psi Y(R^T) - \psi R^T \operatorname{div}^\nabla(\overline{Y})] d\mu_g \\ &= \int_M \psi [Y(R^T) + R^T \operatorname{div}^\nabla(\overline{Y})] d\mu_g \end{aligned}$$

and

$$\begin{aligned} \int_M Y(\Delta_B\psi) d\mu_g &= \int_M [\operatorname{div}^\nabla(\Delta_B\psi \overline{Y}) - \operatorname{div}^\nabla(\overline{Y}) \Delta_B\psi] d\mu_g \\ &= - \int_M \psi \Delta_B(\operatorname{div}^\nabla \overline{Y}) d\mu_g. \end{aligned}$$

Hence we get

$$\frac{d}{ds}|_{s=0} I_{\overline{Y}}(g(s)) = \int_M \psi [\frac{q}{2}Y(R^T) + R^T \operatorname{div}^\nabla(\overline{Y}) + (q-1)\Delta_B(\operatorname{div}^\nabla \overline{Y})] d\mu_g.$$

The proof is then completed by formula (2.2). \square

The proof given in [5] (see also [9]) for the invariant of (1.5) would provide another proof of this theorem. For foliation of higher codimension $q > 2$, this invariant is still trivial.

Theorem 2.4. *Let (M, \mathcal{F}, g_0) be a minimal Riemannian foliation of codimension $q \geq 3$ and Y be a transverse conformal vector field. Then for any $g \in [g_0]_B$, we have $I_{\overline{Y}}(g) = 0$.*

Proof. Let $g \in [g_0]_B$, which induces g^T . Integrating (2.2) and using the transverse divergence theorem, we get

$$\left(\frac{q}{2} - 1\right) \int_M Y(R_{g^T}^T) d\mu_g = 0.$$

Hence, $I = 0$. □

3. EXAMPLES WITH NONVANISHING I_Y

In this section we compute the invariant, defined in (2.7), for a family of codimension 2 minimal Riemannian foliations. This family is given by the weighted Sasakian structures on the unit sphere in \mathbb{C}^2 . It is well-known in Sasakian geometry that a Sasakian manifold admits a minimal Riemannian foliation structure. For the weighted Sasakian structures on the unit sphere in \mathbb{C}^2 , one may refer to [7, 10, 22].

On the unit sphere \mathbb{S}^3 in \mathbb{C}^2 there is a canonical Sasakian structure described as below. Let

$$\eta = \sum_{i=1}^2 (x^i dy^i - y^i dx^i)$$

be the canonical contact form on \mathbb{S}^3 . It uniquely determines a vector field ξ by $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. The vector field ξ is called the Reeb vector field and

$$\xi = \sum_{i=1}^2 \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right).$$

The Reeb vector field ξ gives rise to a foliation \mathcal{F}_ξ on \mathbb{S}^3 . The distribution $\mathcal{D} := \ker \eta$ is called the contact distribution. Let ϕ be the linear map which satisfies $\phi\xi = 0$, and let $\phi|_{\mathcal{D}}$ be the restriction of the canonical complex structure of \mathbb{C}^2 . Let

$$(3.1) \quad g = d\eta \circ (Id \otimes \phi) + \eta \otimes \eta.$$

Then g is a bundle-like metric for the foliation \mathcal{F}_ξ and $\mathcal{D} = \mathcal{L}_\xi^\perp$, where \mathcal{L}_ξ is the integrable line bundle given by \mathcal{F}_ξ . In fact g is the standard metric on the unit sphere and $R^T = 8$.

For a given pair (a_1, a_2) of positive numbers, the weighted Sasakian structure \mathbb{S}_a^3 on the unit sphere \mathbb{S}^3 is defined as follows. Let

$$\eta_a = \sigma^{-1} \sum_{i=1}^2 (x^i dy^i - y^i dx^i) = \sigma^{-1} \eta,$$

where $\sigma = a_1|z_1|^2 + a_2|z_2|^2$. The contact form η_a uniquely determines the Reeb vector field ξ_a by $\eta_a(\xi_a) = 1$ and $i_{\xi_a} d\eta_a = 0$. It is trivial to see that $\ker \eta_a = \ker \eta = \mathcal{D}$ and easy to check that

$$\xi_a = \sum_{i=1}^2 a_i \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right).$$

The line bundle \mathcal{L}_{ξ_a} , spanned by the Reeb vector field ξ_a , defines the characteristic foliation structure \mathcal{F}_{ξ_a} of \mathbb{S}_a^3 . Let ϕ_a be the linear map which satisfies $\phi_a \xi_a = 0$ and $\phi_a|_{\mathcal{D}} = \phi|_{\mathcal{D}}$, and let

$$(3.2) \quad g_a = d\eta_a \circ (Id \otimes \phi_a) + \eta_a \otimes \eta_a.$$

Then $(\mathbb{S}^3, \mathcal{F}_{\xi_a}, g_a)$ is a Sasakian manifold, and hence its foliation is minimal. See [10].

If $\frac{a_1}{a_2} (\neq 1)$ is a rational number, the leaf space of \mathcal{F}_{ξ_a} is the orbifold $\mathbb{P}_{\mathbb{C}}^1(a_1, a_2)$, i.e., the weighted projective line. Any holonomy invariant transverse metric g^T on $(\mathbb{S}^3, \mathcal{F}_{\xi_a})$ induces naturally an orbifold metric on $\mathbb{P}_{\mathbb{C}}^1(a_1, a_2)$, and the scalar curvature, of the orbifold metric is exactly the transverse scalar curvature $R_{g^T}^T$. It was known that $\mathbb{P}_{\mathbb{C}}^1(a_1, a_2)$ admits no orbifold metric of constant scalar curvature; see [24]. In [3] the Futaki invariant of $\mathbb{P}_{\mathbb{C}}^1(a_1, a_2)$ was shown to be nonvanishing.

If $\frac{a_1}{a_2}$ is an irrational number, the leaf space of \mathcal{F}_{ξ_a} has no manifold structure. In the following we show that the invariant I is zero if and only if $a_1 = a_2$; i.e., the leaf space is the standard sphere. This phenomenon is very similar to the existence of \mathbb{S}^1 -equivariant harmonic maps from \mathbb{S}^3 into \mathbb{S}^2 in [21].

We have the following two (real) tangent vector fields on \mathbb{S}^3 :

$$Z_1 = \sigma^{-1}(-i|z_2|^2 z_1, i|z_1|^2 z_2), \quad Z_2 = \sigma^{-1}(|z_2|^2 z_1, -|z_1|^2 z_2),$$

where z_1 denotes the vector field $x^1 \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial y^1}$ and iz_1 denotes the vector field $-y^1 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial y^1}$. A direct computation gives

$$Z_i \in \ker \eta_a, \quad Z_2 = \phi_a Z_1, \quad [\xi_a, Z_i] = 0, \quad [Z_1, Z_2] = -2\sigma^{-3}|z_1|^2|z_2|^2 \xi_a.$$

Proposition 3.1. *The vector field Z_2 is a transverse conformal vector field on $(\mathbb{S}^3, \mathcal{F}_{\xi_a}, g_a)$.*

Proof. By the fact that $[\xi_a, Z_2] = 0$, we see that $Z_2 \in V(\mathcal{F}_{\xi_a})$. Note that g , given by (3.1), is the standard metric on \mathbb{S}^3 and $Z_1, Z_2 \in \mathcal{D}$. Then by the definition (3.2) of g_a , we have

$$g_a^T(Z_i, Z_j) = \sigma^{-1} g^T(Z_i, Z_j) = \sigma^{-1} g(Z_i, Z_j) = \sigma^{-3} |z_1|^2 |z_2|^2 \delta_{ij}.$$

One can then verify that $L_{Z_2} g_a^T = 2f_{Z_2} g_a^T$ with

$$f_{Z_2} = \frac{1}{2} Z_2 \log(\sigma^{-3} |z_1|^2 |z_2|^2). \quad \square$$

Proposition 3.2. *The transverse scalar curvature of $(\mathbb{S}^3, \mathcal{F}_{\xi_a}, g_a)$ is*

$$R^T(g_a^T) = -24(a_1 - a_2)^2 \sigma^{-1} |z_1|^2 |z_2|^2 - 16(a_1 - a_2)(|z_1|^2 - |z_2|^2) + 8\sigma.$$

Proof. Let $Z = Z_1 - iZ_2$. We have $\phi Z = iZ$ and $g_a^T(Z, \bar{Z}) = 2\sigma^{-3} |z_1|^2 |z_2|^2$, where \bar{Z} denotes the complex conjugate of Z . The transversal scalar curvature follows from the formula

$$R^T(g_a^T) = -2[g_a^T(Z, \bar{Z})]^{-1} Z \bar{Z} \log[g_a^T(Z, \bar{Z})].$$

We notice that the transverse scalar curvature of \mathbb{S}_a^3 was also given in [10] in a slightly different form. \square

Proposition 3.3. *On the minimal Riemannian foliation $(\mathbb{S}^3, \mathcal{F}_{\xi_a}, g_a)$, we have*

$$I_{Z_2}(\tilde{g}) = -8\pi^2 \frac{a_1^2 - a_2^2}{a_1^2 a_2^2}, \quad \forall \tilde{g} \in [g_a]_B.$$

Proof. By Theorem 2.3, it suffices to show that

$$\int_{\mathbb{S}^3} Z_2(R_{g_a^T}^T) d\mu_{g_a} = -8\pi^2 \frac{a_1^2 - a_2^2}{a_1^2 a_2^2}.$$

It follows from Proposition 3.2 that

$$Z_2(R_{g_a^T}^T) = -48(a_1 - a_2)a_1a_2\sigma^{-3}|z_1|^2|z_2|^2.$$

Let $t = |z_1|^2$, $s = (a_1 - a_2)t + a_2 = \sigma$. Then

$$Z_2(R_{g_a^T}^T) = 48(a_1 - a_2)^{-1}a_1a_2s^{-3}(s - a_1)(s - a_2).$$

Note that the volume element with respect to g_a is

$$d\mu_{g_a} = \eta_a \wedge d\eta_a = \sigma^{-2}\eta \wedge d\eta = \sigma^{-2}d\mu_g,$$

where $d\mu_g$ is the standard volume element of \mathbb{S}^3 . For each $t = |z_1|^2 \in (0, 1)$, we have a torus $\{(z_1, z_2) : |z_1|^2 = t, |z_2|^2 = 1 - t\}$ which has area $(2\pi\sqrt{t})(2\pi\sqrt{1-t})$ with respect to the standard metric g on \mathbb{S}^3 . Note that

$$\xi, \quad (|z_1||z_2|)^{-1}\sigma Z_1, \quad (|z_1||z_2|)^{-1}\sigma Z_2,$$

form an orthonormal frame of (\mathbb{S}^3, g) . It is easy to see that $\xi|z_1|^2 = Z_1|z_1|^2 = 0$. It implies that

$$|\nabla|z_1|^2|_g = (|z_1||z_2|)^{-1}\sigma|Z_2(|z_1|^2)| = 2|z_1||z_2|.$$

Then by the coarea formula, we have

$$\begin{aligned} \int_{\mathbb{S}^3} Z_2(R_{g_a^T}^T)\eta_a \wedge d\eta_a &= \int_0^1 Z_2(R_{g_a^T}^T)\sigma^{-2}(2\pi\sqrt{t})(2\pi\sqrt{1-t})\frac{dt}{|\nabla|z_1|^2|_g} \\ &= \int_0^1 Z_2(R_{g_a^T}^T)\sigma^{-2}2\pi^2 dt \\ &= \int_{a_2}^{a_1} 96\pi^2(a_1 - a_2)^{-2}a_1a_2s^{-5}(s - a_1)(s - a_2)ds \\ &= -8\pi^2\frac{a_1^2 - a_2^2}{a_1^2a_2^2}. \end{aligned} \quad \square$$

Therefore the invariant vanishes if and only if $a_1 = a_2$. This also gives us the proof of Theorem 1.2.

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