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SUMS OF PRODUCTS OF POSITIVE OPERATORS AND SPECTRA OF LÜDERS OPERATORS

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ABSTRACT. Each bounded operator T on an infinite dimensional Hilbert space $\mathcal{H}$ is a sum of three operators that are similar to positive operators; two such operators are sufficient if T is not a compact perturbation of a scalar. The spectra of Lüders operators (elementary operators on $\mathrm{B}(\mathcal{H})$ with positive coefficients) of lengths at least three are not necessarily contained in $\mathrm{B}(\mathcal{H})^+$. On the other hand, the spectra of such operators of lengths (at most) two are contained in $\mathrm{B}(\mathcal{H})^+$ if the coefficients on one side commute.

1. Introduction

Completely positive maps on $\mathrm{B}(\mathcal{H})$ (the algebra of all bounded operators on a Hilbert space $\mathcal{H}$) of the form

$$\Psi(X) = \sum_{j=1}^{n} A_j^* X A_j \tag{1.1}$$

have received renewed interest recently especially in connection with quantum information theory (see [8], [9], [13], [18] and the references there). If all the coefficients A_j in (1.1) are positive operators such a map is called a Lüders operation. If n is finite, then these are special cases of elementary operators, that is, maps of the form $X \mapsto \sum_{j=1}^{n} A_j X B_j$, whose spectra have been intensively studied in the past (see [5] and the references there), but only in the cases when both families of coefficients (A_j) and (B_j) are commutative. If $\mathcal{H}$ is finite dimensional, then $\mathrm{B}(\mathcal{H})$ is a Hilbert space for the inner product induced by the trace and it is easily verified that an elementary operator with positive coefficients A_j and B_j is a positive operator on this Hilbert space, so its spectrum is contained in $\mathbb{R}^+ := [0, \infty)$.

At the end of the paper [11] it was asked if the spectrum of a Lüders operator $X \mapsto \sum_{j=1}^{n} A_j X A_j$ with positive coefficients on $\mathrm{B}(\mathcal{H})$ is necessarily contained in $\mathbb{R}^+$ if $\mathcal{H}$ is infinite dimensional. We will show that, contrary to what one might expect, the answer to this question is negative. This will be a consequence of the fact that the operator $T = -1$ can be expressed as

$$T = \sum_{j=1}^{n} A_j B_j \quad \text{with positive } A_j, B_j \in \mathrm{B}(\mathcal{H}). \tag{1.2}$$

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At first the author did not know how to prove that every operator $T \in \mathrm{B}(\mathcal{H})$ is of the form (1.2), but then Professor Heydar Radjavi told him that by [16] and [12] T is a sum of finitely many idempotents and, since every idempotent is similar to a projection, T is a sum of products of positive operators. To see this, note that an operator Q which is similar to a positive operator, say $Q = SPS^{-1}$, is a product of two positive operators: $Q = (SS^*)((S^{-1})^*PS^{-1})$. By Pearcy and Topping [12] five idempotents are sufficient to express any T in this way, and according to [19, Proposition 5.9] this is the minimal number since scalars are in general not sums of less than five idempotents. However, since idempotents are very special elements, we cannot expect that 5 is the minimal n in (1.2).

One of the goals of this paper is to find the minimal n above. The result will imply that even the spectrum of a Lüders operator of small length is not necessarily contained in $\mathbb{R}^+$. More precisely, in the next section we will show that every $T \in \mathrm{B}(\mathcal{H})$ is a sum of three operators T_j each of which is similar to a positive operator. Moreover, if T is not a compact perturbation of a scalar, two operators T_j are sufficient. This result is optimal since compact perturbations of nonzero scalars cannot be expressed in the form (1.2) with $n \leq 2$. We will also show that the trace class operators with trace not in $\mathbb{R}^+$ cannot be expressed as $T_1 + T_2$ with both T_1 and T_2 similar to positive operators in $\mathrm{B}(\mathcal{H})$. As a preliminary step in the proof of the main result we will first show that T is a sum of four operators T_j similar to positive ones, with some additional properties needed.

In the last section we will first apply this result to answer the above-mentioned question from [11]. Then we will prove that the spectra of operators of the form $X \mapsto \sum_{j=1}^{2} A_j X B_j$ with positive A_j and B_j are contained in $\mathbb{R}^+$ if $A_1A_2 = A_2A_1$ (or if $B_1B_2 = B_2B_1$).

Throughout the paper $\mathcal{H}$ denotes an infinite dimensional separable Hilbert space and $\mathrm{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. (The results hold also for nonseparable $\mathcal{H}$, but in their formulations the ideal of compact operators must be replaced by the unique proper maximal ideal of $\mathrm{B}(\mathcal{H})$.) An operator $T \in \mathrm{B}(\mathcal{H})$ is called positive if $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$ (thus T is not necessarily definite), and the set of all positive operators is denoted by $\mathrm{B}(\mathcal{H})^+$.

2. Sums of operators similar to positive operators

We begin with a simple and well-known observation. Let $S \in \mathrm{B}(\mathcal{K} \oplus \mathcal{K})$ be a 2×2 operator matrix

$$S = \begin{bmatrix} u & x \\ y & z \end{bmatrix}, \tag{2.1}$$

where u is invertible. Then S is invertible if and only if $z - yu^{-1}x$ is invertible and in this case

$$S^{-1} = \begin{bmatrix} u^{-1}(1 + xdyu^{-1}) & -u^{-1}xd \\ -dyu^{-1} & d \end{bmatrix}, \quad \text{where } d = (z - yu^{-1}x)^{-1}. \tag{2.2}$$

To prove this, multiply S from the left by the invertible matrix

$$\begin{bmatrix} u^{-1} & 0 \\ -yu^{-1} & 1 \end{bmatrix}$$

to obtain an upper-triangular matrix with 1 and $z - yu^{-1}x$ along the diagonal.

The main assertion of the following lemma can be deduced from the proof of Theorem 1 in [12], but later we will need some additional information from its proof in the form presented below.

Lemma 2.1. *Every operator $T \in \mathrm{B}(\mathcal{H})$ is a sum of the form*

$$T = \sum_{j=1}^{4} S_j T_j S_j^{-1},$$

where $S_j \in \mathrm{B}(\mathcal{H})$ and the operators $T_j \in \mathrm{B}(\mathcal{H})$ are positive with disjoint spectra $\sigma(T_j)$, and each $\sigma(T_j)$ consists of at most two points, $\sigma(T_1) \subset [0,1]$ and $\sigma(T_j) \subset (1,\infty)$ for $j \neq 1$. Moreover, the range of T_1 is closed and has infinite dimension and codimension.

In particular, T can be written as $T = \sum_{j=1}^{4} A_j B_j$, where $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$.

Proof. Decompose $\mathcal{H}$ into an orthogonal sum of two isomorphic closed subspaces, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$; then T is represented by an operator matrix of the form

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.3}$$

First we will try to find diagonal positive operators $T_j = a_j \oplus b_j$ ($a_j, b_j \in \mathrm{B}(\mathcal{K})$) and invertible operators S_j ($j = 1, \ldots, 4$) of the form (2.1) such that $T = \sum_{j=1}^{4} S_j T_j S_j^{-1}$. It turns out that we can even take S_j of the form

$$S_j = \begin{bmatrix} 1 & x_j \\ y_j & 1 + y_j x_j \end{bmatrix}.$$

Then

$$S_j T_j S_j^{-1} = \begin{bmatrix} a_j + s_j y_j & -s_j \\ y_j a_j - b_j y_j + y_j s_j y_j & b_j - y_j s_j \end{bmatrix}, \quad \text{where } s_j := a_j x_j - x_j b_j.$$

There are many appropriate choices for x_j, y_j, z_j, a_j, b_j in order to make the sum $\sum_{j=1}^{4} S_j T_j S_j^{-1}$ equal to T. For example, if we let $y_1 = 0 = x_2$, $y_3 = 1$, $b_1 = 0$ and for $j \geq 2$ choose all a_j and b_j to be positive scalars with $a_j - b_j = 1$, and denote $\beta = \sum_{j=2}^{4} b_j$ (so that $\sum_{j=2}^{4} a_j = \beta + 3$), then

$$\sum_{j=1}^{4} S_j T_j S_j^{-1} = \begin{bmatrix} a_1 + \beta + 3 + x_3 + x_4 y_4 & -a_1 x_1 - x_3 - x_4 \\ y_2 + x_3 + 1 + y_4 + y_4 x_4 y_4 & \beta - x_3 - y_4 x_4 \end{bmatrix}. \tag{2.4}$$

To achieve that the matrix in (2.4) will be equal to T, we only need to choose x_3, x_4, y_4 in $\mathrm{B}(\mathcal{K})$ and invertible $a_1 \in \mathrm{B}(\mathcal{K})^+$ so that

$$a_1 + \beta + 3 + x_3 + x_4 y_4 = A \quad \text{and} \quad \beta - x_3 - y_4 x_4 = D, \tag{2.5}$$

for then the off-diagonal terms of the matrix (2.4) can be made equal to B and C by a suitable choice of y_2 and x_1. Adding the two equations (2.5), we see that we only need to choose x_4, y_4 and a_1 so that

$$x_4 y_4 - y_4 x_4 = A + D - a_1 - 2\beta - 3 =: T_0, \tag{2.6}$$

for then x_3 can be computed from either of the equations (2.5). So (for a fixed β), we first choose an invertible positive $a_1 \in \mathrm{B}(\mathcal{K})$ of the form $\lambda + \mu p$, where $\lambda, \mu \in \mathbb{R}^+$ and p is a projection of infinite rank and nullity, such that $\sigma(a_1) \subset (0,1]$ and T_0 is not a compact perturbation of a scalar. Then T_0 is a commutator by [2] (a simplified proof is in [1]), which means that there exist x_4 and y_4 satisfying

(2.6). By suitably choosing scalars a_j and b_j ($j \geq 2$) we can make the spectra of T_j disjoint for all j. $\square$

Remark 2.2. For a later use observe that in the above proof the spectra of a_j and b_j are disjoint for all j; in fact, all a_j and b_j chosen above are scalars, except possibly a_1. Also note that the operator $S_1T_1S_1^{-1}$ has the form

$$\begin{bmatrix} a_1 & * \\ 0 & 0 \end{bmatrix},$$

where $a_1 \in \mathrm{B}(\mathcal{K})^+$.

Theorem 2.3. *Every $T \in \mathrm{B}(\mathcal{H})$ is of the form $T = \sum_{j=1}^3 S_jT_jS_j^{-1}$, where $S_j \in \mathrm{B}(\mathcal{H})$ and the operators $T_j \in \mathrm{B}(\mathcal{H})$ are positive (and invertible for $j \leq 2$) with finite spectra $\sigma(T_j)$, amd each $\sigma(T_j)$ consists of at most four points. Moreover, 0 is an isolated point of $\sigma(T_3)$, and the range of T_3 is closed and has infinite dimension and codimension.*

Proof. As in the proof of Lemma 2.1 we represent T by the operator matrix (2.3). Now we try to find positive block-diagonal operators $T_j = a_j \oplus b_j$ and invertible operators $S_j \in \mathrm{B}(\mathcal{H})$ of the form (2.1) (with $z - yu^{-1}x = 1$) such that $\sum_{j=1}^3 S_jT_jS_j^{-1} = T$. Denoting

$$S_j = \begin{bmatrix} u_j & x_j \\ y_j & z_j \end{bmatrix}, \text{ where } u_j \text{ is invertible and } z_j - y_ju_j^{-1}x_j = 1,$$

we compute (using (2.2)) that

$$S_jT_jS_j^{-1} = \begin{bmatrix} c_j + s_jv_j & -s_j \\ v_jc_j - b_jv_j + v_js_jv_j & b_j - v_js_j \end{bmatrix},$$

where

$$c_j := u_ja_ju_j^{-1}, \quad v_j := y_ju_j^{-1}, \quad \text{and} \quad s_j := c_jx_j - x_jb_j. \tag{2.7}$$

Note that if the spectra of b_j and c_j are disjoint, then from (2.7) a_j, y_j, b_j and x_j can all be computed from c_j, u_j, v_j, b_j, and s_j. (That the equation $c_jx_j - x_jb_j = s_j$ can be solved for x_j is Rosenblum's theorem [14, p. 8].) Further, we assume that the matrix S_3 is diagonal (that is, $x_3 = 0 = y_3$, so we will only need that the spectra of c_j and b_j are disjoint for $j = 1, 2$). Then the condition $\sum S_jT_jS_j^{-1} = T$ is equivalent to the following four equations:

$$s_1v_1 + s_2v_2 = A - c_1 - c_2 - c_3, \quad s_1 + s_2 = -B, \tag{2.8}$$

$$v_1c_1 - b_1v_1 + v_2c_2 - b_2v_2 + v_1s_1v_1 + v_2s_2v_2 = C, \tag{2.9}$$

$$v_1s_1 + v_2s_2 = -D + b_1 + b_2 + b_3.$$

Set $c := c_1 + c_2 + c_3$, $b := b_1 + b_2 + b_3$ and

$$s := s_1, \quad v := v_2, \quad w := v_2 - v_1.$$

Then from the second equation in (2.8) we get $s_2 = -(B + s)$; using this, the other three equations (2.8), (2.9) can be rewritten as

$$Bv + sw = c - A, \qquad vB + ws = D - b, \tag{2.10}$$

$$v(c_1 + c_2 - sw) - (b_1 + b_2 + ws)v - wc_1 + b_1w + wsw - vBv = C. \tag{2.11}$$

From (2.10) we have that $c_1+c_2-sw = A-c_3+Bv$ and $b_1+b_2+ws = D-b_3-vB$; hence (2.11) can be rewritten as

$$wsw - wc_1 + b_1w = C - v(A - c_3) + (D - b_3)v - vBv. \tag{2.12}$$

We are going to show that the system of equations (2.10), (2.12) has a solution.

First suppose that T is not a compact perturbation of a scalar. Then we may assume that in the matrix representation of T we have that $D = 0$ and that B is an isometry with the range of B isomorphic to its orthogonal complement in $\mathcal{K}$ since by [2, Corollary 3.4] T is similar to such an operator. In this case we shall see that we can even afford to choose $s = 0$, so that the above system of equations simplifies to

$$Bv = c - A, \tag{2.13}$$

$$vB = -b, \tag{2.14}$$

$$b_1w - wc_1 = C - v(A - c_3) + (-b_3)v - vBv. \tag{2.15}$$

Since $B^*B = 1$, the equation (2.13) is equivalent to the following two:

$$v = B^*(c - A) \quad \text{and} \quad P^\perp(c - A) = 0, \ \text{where } P := BB^* \text{ and } P^\perp := 1 - P. \tag{2.16}$$

Using this expression for v, (2.14) can be rewritten as

$$b_1 + b_2 + b_3 = b = B^*(A - c)B. \tag{2.17}$$

If there exist v, c_j and b_j $(j = 1, 2, 3)$ such that the equations (2.16) and (2.17) are satisfied and the spectra of c_1 and b_1 are disjoint, then the equation (2.15) can be solved for w by Rosenblum's theorem.

To show that the system (2.16), (2.17) has a solution, represent A by a 2×2 operator matrix with respect to the decomposition $\mathcal{K} = P\mathcal{K} \oplus P^\perp\mathcal{K}$. By Lemma 2.1, $A = \sum_{j=1}^4 A_j$, where each A_j is similar to a positive operator; moreover, by Remark 2.2 we may assume that (with respect to the decomposition $\mathcal{K} = P\mathcal{K} \oplus P^\perp\mathcal{K}$) A_4 is of the form

$$A_4 = \begin{bmatrix} a & r \\ 0 & 0 \end{bmatrix}, \ \text{where} \ \ a \geq 0, \tag{2.18}$$

which means that $P^\perp A_4 = 0$. Thus, if we put $c_j = A_j$ for $j = 1, 2, 3$ (and $c = c_1 + c_2 + c_3$), then we have $P^\perp(A - c) = P^\perp A_4 = 0$, which is just the condition in (2.16). Further

$$B^*(A - c)B = B^*A_4B = B^*A_4PB = B^*GB, \tag{2.19}$$

where

$$G := A_4P = a \oplus 0.$$

Thus the operator $B^*(A - c)B$ is positive and hence it can be written (in many ways) as a sum of three positive operators b_j, which is just what the condition (2.17) requires. We may choose $b_3 = 0$. To see that it is possible to choose b_j and c_j $(j = 1, 2)$ so that their spectra are disjoint, note that PB is a unitary operator from $\mathcal{K}$ onto $P\mathcal{K}$ which intertwines b and a by (2.19) and (2.17); hence b and a have the same spectrum. By Lemma 2.1 we may choose a and $c_j = A_j$ so that each of their spectra consists of at most two points, $\sigma(a) \subseteq (0, 1]$ and $\sigma(A_j) \subset (1, \infty)$ $(j = 1, 2, 3)$. Since $b_j \geq 0$ and $b_1 + b_2 = b$, the spectra of b_j are contained in $[0, 1]$; hence $\sigma(b_j) \cap \sigma(c_j) = \emptyset$. Since $\sigma(b)$ consists of at most two points in $(0, 1]$, we may choose b_1, b_2 to have the same property. (We may choose for b_1 a sufficiently small positive scalar, for example.)

Since T_j is similar to $a_j \oplus b_j$ and a_j is similar to $c_j = A_j$ $(j = 1, 2, 3)$, $\sigma(T_j) = \sigma(A_j) \cup \sigma(b_j)$ consists of at most four points. Other properties of the operators T_j stated in the theorem also follow easily from those of c_j and a_j chosen above.

Now we consider the case when T is a compact perturbation of a scalar. In this case let $E = 1 \oplus 0$, the projection onto the first summand in the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. Then $\tilde{T} := T - E$ is not a compact perturbation of a scalar, so by the already proved case $\tilde{T}$ can be expressed as $\tilde{T} = \sum_{j=1}^{3} S_j(a_j \oplus b_j)S_j^{-1}$, where $a_j \geq 0$ and $b_j \geq 0$ and S_3 is block-diagonal. Since S_3 commutes with E, we have

$$T = \tilde{T} + E = \sum_{j=1}^{2} S_j(a_j \oplus b_j)S_j^{-1} + S_3((a_3 \oplus b_3) + E)S_3^{-1},$$

which is a sum of three operators similar to positive ones with (at most) four-point spectra. $\square$

Remark 2.4. Observe that in the proof of Theorem 2.3 the operator T_3 is of the form $e \oplus 0$ (since $b_3 = 0$ and S_3 is a diagonal 2×2 operator matrix), where e is similar to a positive invertible operator with at most two-point spectrum.

Corollary 2.5. *Each $T \in \mathrm{B}(\mathcal{H})$ can be expressed as $T = \sum_{j=1}^{3} A_j B_j$, where $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$.*

Theorem 2.6. *If $T \in \mathrm{B}(\mathcal{H})$ is not a compact perturbation of a scalar, then T is a sum of two operators similar to positive operators.*

Proof. We have to show that in the proof of Theorem 2.3, a_3 and b_3 can be taken to be 0. That b_3 can be taken to be 0 has been already observed in that proof. Now note that in the matrix representation (2.3) of T we may assume, in addition to $D = 0$ and B is an isometry, that A is not a compact perturbation of a scalar. For this, we simply decompose the second copy of $\mathcal{K}$ into two orthogonal isomorphic closed subspaces, $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, and decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{K}_1^{\perp} \oplus \mathcal{K}_1$. Since B maps $\mathcal{K}_1$ isometrically into $\mathcal{K}_1^{\perp}$ the matrix of T has 0 in the $(2, 2)$ position and an isometry with infinitely codimensional range in the $(1, 2)$ position. The new element in the position $(1, 1)$ is then not a compact perturbation of a scalar. So we will assume that already in the initial matrix representation of T, the element A is not a compact perturbation of a scalar. Now consider the matrix of A relative to the decomposition of the Hilbert space of A into the range of B and its orthogonal complement. Since A is not a compact perturbation of a scalar, by Theorem 2.3 and Remark 2.4, A is of the form $A = \sum_{j=1}^{3} \tilde{A}_j$, where $\tilde{A}_1$ and $\tilde{A}_2$ are similar to positive invertible operators each with at most four points in its spectrum and $\tilde{A}_3$ is of the form $e \oplus 0$ with e similar to a positive invertible operator with a two-point spectrum. By the same reasoning as in the proof of Theorem 2.3 (see the paragraph containing (2.18); the role of A_4 is now played by $\tilde{A}_3$) we see that the system of equations (2.16), (2.17) has a solution such that $c_j = \tilde{A}_j$ for $j = 1, 2$ and $c_3 = 0 = b_3 = 0$. But we have to show also that we can achieve $\sigma(c_j) \cap \sigma(b_j) = \emptyset$ $(j = 1, 2)$ in order to assure that (2.15) has a solution for w and that x_j can be computed from the last equation in (2.7). For this we now note that the operator $B^*(A - c)B = B^*\tilde{A}_3B$ is unitarily equivalent to e. Since $\sigma(c_j)$ $(j = 1, 2)$ is a finite subset of $(0, \infty)$ and $\sigma(B^*(A - c)B)$ consists of just two positive points, it follows that $B^*(A-c)B$ is similar to a sum b_1+b_2, where $b_j \geq 0$ and $\sigma(b_j)\cap\sigma(c_j)=\emptyset$ for both $j=1,2$. $\square$

An operator $T \in \mathrm{B}(\mathcal{H})$ of the form $\lambda + K$, where $\lambda \in \mathbb{C}\setminus\mathbb{R}^+$ and K is compact, is not of the form

$$PQ + RS \quad \text{for any } P,Q,R,S \in \mathrm{B}(\mathcal{H})^+. \tag{2.20}$$

To see this, just note that the spectrum of the coset $\dot{R}\dot{S}$ in the Calkin algebra is the same as the spectrum of $\dot{S}^{1/2}\dot{R}\dot{S}^{1/2}$, hence contained in $\mathbb{R}^+$, while the spectrum of $\lambda - \dot{P}\dot{Q}$ is contained in the ray $\lambda - \mathbb{R}^+$ which is disjoint with $\mathbb{R}^+$.

Each compact operator on a Hilbert space is an additive commutator of two bounded operators [1]. By an analogy one might conjecture that each compact operator is a sum of two operators similar to positive ones, but this is not true.

Proposition 2.7. *If $T \in \mathrm{C}^1(\mathcal{H})$ (the trace class) is nonzero and $Tr\,(T)$ is not positive, then T is not a sum of two operators in $\mathrm{B}(\mathcal{H})$ similar to positive ones.*

Proof. Assume the contrary, that $T = S_1AS_1^{-1} + S_2BS_2^{-1}$, where $A,B \in \mathrm{B}(\mathcal{H})^+$. Put $F := -S_1^{-1}TS_1$ and $S = S_1^{-1}S_2$. Then

$$F + A = -SBS^{-1}. \tag{2.21}$$

Considering the essential spectra, it follows from (2.21) and the positivity of A and B that A and B must be compact. We claim that A and B must be in the Hilbert-Schmidt class $\mathrm{C}^2(\mathcal{H})$. For a proof we may first replace B by a unitarily equivalent operator (and modify S accordingly) to reduce to the situation when A and B can be diagonalized in the same orthonormal basis $\mathbb{B}$ of $\mathcal{H}$. Let (α_j) and (β_j) be the lists of eigenvalues of A and B in decreasing order (each eigenvalue repeated according to its multiplicity). From (2.21) we have $AS + SB = G$, where $G := -FS$. Denoting by $\sigma_{i,j}$ and $\psi_{i,j}$ the entries of the matrices of S and G in the basis $\mathbb{B}$, we see that

$$(\alpha_i + \beta_j)\sigma_{i,j} = \psi_{i,j}. \tag{2.22}$$

Let $\gamma_j := (\sum_i |\psi_{i,j}|^2)^{1/2}$ and note that $\sum_j \gamma_j^2 < \infty$ since $G \in \mathrm{C}^2(\mathcal{H})$. Since S is invertible (in particular, bounded from below), there exists a scalar $\gamma > 0$ such that $\sum_i |\sigma_{i,j}|^2 \geq \gamma$ for all i; hence (2.22) implies that

$$\beta_j^{-2}\gamma_j^2 = \beta_j^{-2}\sum_i |\psi_{i,j}|^2 = \sum_i \frac{(\alpha_i+\beta_j)^2}{\beta_j^2}|\sigma_{i,j}|^2 \geq \sum_i |\sigma_{i,j}|^2 \geq \gamma,$$

whenever $\beta_j \neq 0$. Thus $\beta_j^2 \leq \gamma_j^2\gamma^{-1}$ and consequently $\sum_j \beta_j^2 < \infty$, which means that $B \in \mathrm{C}^2(\mathcal{H})$. Similarly (or from (2.21), since $F \in \mathrm{C}^2(\mathcal{H})$) we see that $A \in \mathrm{C}^2(\mathcal{H})$.

By considering the polar decomposition of S of the form $S = RU$, where R is positive and U is unitary, we may rewrite (2.21) in the form

$$F + A = -RCR^{-1}, \tag{2.23}$$

where $C := UBU^* \geq 0$. Assume for a moment that in some orthonormal basis of $\mathcal{H}$ the operator R can be represented by a diagonal matrix and let $[\alpha_{i,j}]$, $[\phi_{i,j}]$ and

$[\gamma_{i,j}]$ be the matrices of A, F and C (respectively) in this basis. Then, considering the sums of diagonal terms of matrices, (2.23) implies that

$$-\sum_{j=1}^{n}\phi_{j,j}=\sum_{j=1}^{n}\alpha_{j,j}+\sum_{j=1}^{n}\gamma_{j,j}. \tag{2.24}$$

Letting $n\to\infty$, the first sum in (2.24) tends to $-Tr\,(F)=Tr\,(T)\in\mathbb{C}\setminus(0,\infty)$, while the second and the third sums converge to elements in $[0,\infty]$. This shows that the equality (2.24) can hold for all n only if $Tr\,(T)=0$ and $\phi_{j,j}=0=\alpha_{j,j}$ for all j. Since $A\in\mathrm{B}(\mathcal{H})^+$, the condition $\alpha_{j,j}=0$ for all j implies that $A=0$. But then B is similar to T; hence $Tr\,(B)=0$, which implies (since $B\geq 0$) that $B=0$. In this case $T=0$, which was excluded by the hypothesis of the proposition. Now we will show by an approximation argument that (2.23) leads to a contradiction even if R cannot be diagonalized.

By the Weyl–von Neumann theorem [4, p. 214], given $\varepsilon>0$, there exist a diagonal Hermitian operator D and an operator $H\in\mathrm{C}^2(\mathcal{H})$ with $\|H\|_2<\varepsilon$ (where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm) such that $R=D+H$. If ε is small enough, then D is invertible (since $D=R-H=R(1-R^{-1}H)$) and

$$\|D^{-1}\|\leq\|R^{-1}\|\sum_{n=0}^{\infty}\|R^{-1}H\|^n\leq\frac{\|R^{-1}\|}{1-\varepsilon\|R^{-1}\|}.$$

Further, if ε is small enough, then $1+HD^{-1}$ is invertible and

$$RCR^{-1}=(1+HD^{-1})DCD^{-1}(1+HD^{-1})^{-1}.$$

Since $(1+HD^{-1})^{-1}=1-HD^{-1}(1+HD^{-1})^{-1}$, we may write

$$\begin{aligned}RCR^{-1}=DCD^{-1}&-DCD^{-1}HD^{-1}(1+HD^{-1})^{-1}\\&+HCD^{-1}\left[1-HD^{-1}(1+HD^{-1})^{-1}\right];\end{aligned}$$

hence (since B and therefore also C is in $\mathrm{C}^2(\mathcal{H})$ by the first paragraph of this proof)

$$\begin{aligned}\|RCR^{-1}-DCD^{-1}\|_1\leq{}&\|H\|_2\|C\|_2\|D^{-1}\|\\&\cdot\left[\|D\|\|D^{-1}\|\|(1+HD^{-1})^{-1}\|+\|1-HD^{-1}(1+HD^{-1})^{-1}\|\right].\end{aligned}$$

It follows that $\|RCR^{-1}-DCD^{-1}\|_1\to 0$ as $\varepsilon\to 0$. This allows us to conclude in essentially the same way as in the previous paragraph (by considering the sums of diagonal entries of matrices) that (2.23) leads to a contradiction. ☐

For most of the above proof it would be sufficient if we assumed that $T\in\mathrm{C}^2(\mathcal{H})$ (instead of $T\in\mathrm{C}^1(\mathcal{H})$), but the problem is that for an operator T not in $\mathrm{C}^1(\mathcal{H})$ the sum of diagonal entries of its matrix relative to a general orthogonal basis can be quite arbitrary (it need not even be defined [7]).

Problem. Which compact operators on an infinite dimensional Hilbert space can be written as T_1+T_2, where T_1 and T_2 are similar to positive operators?

Theorem 2.6 implies that all operators can be approximated in norm by sums of two operators similar to positive ones, but concerning such an approximation a much stronger result holds: it follows from [6, Theorem 3.10] that both summands can be taken to be similar to the same positive operator.

3. On spectra of Lüders operators

For two commutative m-tuples (A_j) and (B_j) of elements of $\mathrm{B}(\mathcal{H})$ the spectrum $\sigma(\Phi)$ of the map $\Phi(X) := \sum_{j=1}^{m} A_j X B_j$ on $\mathrm{B}(\mathcal{H})$ can be described in terms of spectra of (A_j) and (B_j) ([5], [11]); in particular, $\sigma(\Phi) \subseteq \mathbb{R}^+$ if $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$. For noncommutative (A_j) and (B_j) the situation may be completely different. One consequence of Theorem 2.3 is that for an infinite dimensional Hilbert space $\mathcal{H}$ the spectra of Lüders operators on $\mathrm{B}(\mathcal{H})$ are not necessarily contained in $\mathbb{R}^+$.

Proposition 3.1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Every complex number λ can be an eigenvalue of a Lüders operator on $\mathrm{B}(\mathcal{H})$ of length 3 (or more).*

Proof. Decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. By Corollary 2.5 there exist $A_j, B_j \in \mathrm{B}(\mathcal{K})^+$ such that $\sum_{j=1}^{3} A_j B_j = \lambda$. By a simple calculation this implies that the operator

$$X_0 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue λ of the Lüders operator Φ on $\mathrm{B}(\mathcal{H})$ defined by $\Phi(X) = \sum_{j=1}^{3} T_j X T_j$, where

$$T_j = \begin{bmatrix} A_j & 0 \\ 0 & B_j \end{bmatrix}. \qquad \square$$

Theorem 3.2. *Suppose that $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$ ($j = 1, 2$) and let Φ be the map on $\mathrm{B}(\mathcal{H})$ defined by $\Phi(X) = \sum_{j=1}^{2} A_j X B_j$. If $A_1 A_2 = A_2 A_1$ (or if $B_1 B_2 = B_2 B_1$), then the spectrum of Φ is contained in $\mathbb{R}^+$.*

Proof. Since boundary points of the spectrum of any operator are approximate eigenvalues [3], it suffices to show that each approximate eigenvalue λ of Φ is in $\mathbb{R}^+$. By considering the space $\mathcal{B} := \ell_\infty(\mathrm{B}(\mathcal{H}))/c_0(\mathrm{B}(\mathcal{H}))$, where $\ell_\infty(\mathrm{B}(\mathcal{H}))$ is the space of all bounded sequences with entries in $\mathrm{B}(\mathcal{H})$, and $c_0(\mathrm{B}(\mathcal{H}))$ is the subspace of all sequences converging (in norm) to 0, we may reduce the approximate eigenvalues of Φ to proper eigenvalues of the corresponding operator $\tilde{\Phi}$ on $\mathcal{B}$. Here of course $\tilde{\Phi}$ is defined by $\tilde{\Phi}([X_n]) = [\Phi(X_n)]$, where $[X_n]$ denotes the coset of a sequence $(X_n) \in \ell_\infty(\mathrm{B}(\mathcal{H}))$. Note that $\tilde{\Phi}$ is again an elementary operator, namely of the form

$$\tilde{\Phi}(Y) = \sum_{j=1}^{2} \tilde{A}_j Y \tilde{B}_j \quad (Y \in B), \tag{3.1}$$

where $\tilde{A}$ denotes the coset in $\mathcal{B}$ of the constant sequence $(A, A, \ldots) \in \ell_\infty(\mathrm{B}(\mathcal{H}))$ for each $A \in \mathrm{B}(\mathcal{H})$. Since $\mathcal{B}$ is a C^*-algebra, we can regard it as a subalgebra of $\mathrm{B}(\mathcal{K})$ for some (nonseparable) Hilbert space $\mathcal{K}$, and by the formula (3.1) we may regard the map $\tilde{\Phi}$ to be defined on $\mathrm{B}(\mathcal{K})$. Any approximate eigenvalue λ of Φ is then an eigenvalue of $\tilde{\Phi}$. Choose a nonzero eigenvector Y corresponding to λ. $\mathcal{K}$ is not separable, but it can be expressed as an orthogonal sum of separable subspaces $\mathcal{K}_i$ that reduce all the operators A_j, B_j and Y. If i is such that $Y|\mathcal{K}_i \neq 0$, then λ is an eigenvalue of the operator Ψ on $\mathrm{B}(\mathcal{K}_i)$ defined by $\Psi(X) = \sum_{j=1}^{2} C_j X D_j$, where $C_j = A_j|\mathcal{K}_i$ and $D_j = B_j|\mathcal{K}_i$. So it suffices to show that all eigenvalues of such operators are in $\mathbb{R}^+$. Thus (adapting the notation), we may assume that λ is an eigenvalue of Φ. Denote by X a corresponding eigenvector with $\|X\| = 1$; hence

$$\sum_{j=1}^{2} A_j X B_j = \lambda X. \tag{3.2}$$

Suppose that A_1 and A_2 commute. Then by Voiculescu's version [17] of the Weyl-von Neumann-Berg theorem, given $\varepsilon > 0$, there exist commuting diagonal Hermitian operators $C_j \in \mathrm{B}(\mathcal{H})$ and Hilbert-Schmidt operators $H_j \in \mathrm{C}^2(\mathcal{H})$ such that $A_j = C_j + H_j$ and $\|H_j\|_2 < \varepsilon$ $(j = 1, 2)$. Let $C_j = C_j^+ - C_j^-$ be the decomposition of C into the positive and the negative part and denote by Q_j the range projection of C_j^-. Then $A_j + C_j^- = C_j^+ + H_j$; hence (since $Q_j C_j^+ = 0$ and $Q_j C_j^- = C_j^-$)

$$Q_j A_j Q_j + C_j^- = Q_j H_j Q_j \in \mathrm{C}^2(\mathcal{H}).$$

This implies that $C_j^- \in \mathrm{C}^2(\mathcal{H})$ and $\|C_j^-\|_2 \leq \|H_j\|_2 < \varepsilon$. So, replacing C_j by C_j^+ and H_j by $H_j - C_j^-$ (and the initial ε by $\varepsilon/2$), we may assume that $C_j \geq 0$. Let P be any finite rank projection that commutes with C_1 and C_2. (Note that, since C_1, C_2 are commuting diagonal operators, there exists a net of such projections P converging strongly to the identity.) From (3.2) we have that $\sum P A_j X B_j X^* P = \lambda P X X^* P$; hence applying the trace Tr we obtain

$$\sum_{j=1}^{2} \left(Tr\,(P C_j X B_j X^* P) + Tr\,(P H_j X B_j X^* P) \right) = \lambda Tr\,(P X X^* P). \tag{3.3}$$

Since P commutes with C_j,

$$Tr\,(P C_j X B_j X^* P) = Tr\,(C_j P X B_j X^* P) = Tr\,(C_j^{1/2} P X B_j X^* P C_j^{1/2}) \geq 0. \tag{3.4}$$

Further (since $\|Z\|_2 = \|Z^*\|_2$ for all $Z \in \mathrm{B}(\mathcal{H})$),

(3.5)
$$|Tr\,(P H_j X B_j X^* P| \leq \|H_j\|_2 \|X B_j X^* P\|_2 = \|H_j\|_2 \|P X B_j X^*\|_2 < \varepsilon \|P X\|_2,$$

where we have assumed (without lost of generality) that $\|B_j\| \leq 1$. If P is sufficiently close to 1 so that $PX \neq 0$, then from (3.3) and (3.5) we have that

$$\begin{aligned} \left| \lambda - \sum_{j=1}^{2} \frac{Tr\,(P C_j X B_j X^* P)}{Tr\,(P X X^* P)} \right| &\leq \varepsilon \sum_{j=1}^{2} \frac{\|PX\|_2}{Tr\,(P X X^* P)} \\ &= \frac{2\varepsilon}{\|PX\|_2}. \end{aligned}$$

Letting in this estimate $P \to 1$, $\varepsilon \to 0$ and using (3.4), we see that $\lambda \geq 0$. □

Remark 3.3. Theorem 3.2 can be extended to operators of the form

$$X \mapsto \sum_{j=1}^{n} A_j X B_j \tag{3.6}$$

if the coefficients on one side, say all the A_j, are smooth nonnegative functions $A_j = f_j(H_1, H_2)$ of a pair of commuting Hermitian operators (H_1, H_2). Namely, in this case it can be shown (using the Fourier transform) that small Hilbert-Schmidt perturbations of (H_1, H_2) result in small Hilbert-Schmidt perturbations of $f_j(H_1, H_2)$. The author does not know if the theorem can be extended to the general situation, when all the A_j commute, but the B_j do not necessarily commute.

Problems. 1. Can Theorem 3.2 be generalized to operators of length greater than 2?

2. Suppose that all A_j, B_j are positive and for each j at least one of A_j, B_j is compact. Then it can be deduced from [15, Corollary 6.6] (see [10]) that all eigenvalues of the operator (3.6) are contained in $\mathbb{R}^+$. Is the same true for the entire spectrum?

3. In Theorem 3.2 can the commutativity condition be replaced by commutativity modulo compact operators?

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