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POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH SMALL PERTURBATIONS

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ABSTRACT. In this paper, we study the semilinear elliptic equation with a small perturbation. We assume the main term in the equation to have a mountain pass structure but do not suppose any condition for the perturbation term. Then we prove the existence of a positive solution. Moreover, we prove the existence of at least two positive solutions if the perturbation term is nonnegative.

1. INTRODUCTION AND MAIN RESULTS

We prove the existence of positive solutions for the semilinear elliptic equation

$$-\Delta u = f(x,u) + \lambda g(x,u) \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$ and $f(x,u)$, $g(x,u)$ are continuous on $\overline{\Omega} \times [0,\infty)$ and λ is a real parameter whose absolute value is small. We assume a condition on $f(x,u)$ such that (1.1), (1.2) with $\lambda = 0$ has a mountain pass structure, and therefore it has a positive solution when $\lambda = 0$. The most typical nonlinear term is $f(x,u) = a(x)u^p$ or $f(x,u) = a(x)u^p + b(x)u^q$, where $a, b \in C(\overline{\Omega})$ and $a(x)$ or $b(x)$ may change its sign. The purpose of this paper is to prove the existence of a positive solution for $|\lambda|$ small enough under the mountain pass assumption on $f(x,u)$ only without any conditions on $g(x,u)$. The nonlinear term $f(x,u) = a(x)u^p$ was studied by Afrouzi and Brown [1], Alama and Tarantello [2], Brown and Zhang, [4], Li and Wang [6] and the author [5]. However the assumptions in this paper are more general than those of the papers above. We emphasize that our theorem does not need any assumptions on $g(x,u)$. We assume the conditions below.

(f1) There exist positive constants p, C such that $1 < p < \infty$ if $N = 1, 2$ and $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and

$$|f(x,s)| \leq C(s^p + 1) \quad \text{for } s \geq 0,\ x \in \Omega.$$

(f2) There exist constants $\alpha > 2$, $\theta \in [0,2)$, $C > 0$ such that

$$\alpha F(x,s) - sf(x,s) \leq C|s|^\theta + C \quad \text{for } s \geq 0,\ x \in \Omega,$$

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where

$$F(x,s) := \int_0^s f(x,t)dt.$$

(f3) There exist $x_0 \in \Omega$, $\delta_0 > 0$ such that

$$\lim_{s\to\infty}\left(\min_{|x-x_0|\le\delta_0} F(x,s)/s^2\right) = \infty.$$

(f4) $\limsup_{s\to 0+}\left(\max_{x\in\overline{\Omega}} f(x,s)/s\right) < \mu_1,$

where μ_1 denotes the first eigenvalue of the Dirichlet Laplacian in Ω.

(f5) $\liminf_{s\to 0+}\left(\min_{x\in\overline{\Omega}} f(x,s)/s\right) > -\infty.$

Assumptions (f1)–(f4) guarantee that $f(x,u)$ has a mountain pass structure, and (f5) ensures that a mountain pass solution is strictly positive. For any $g(x,u)$ and $|\lambda|$ small enough, we prove the existence of a positive solution. Moreover, if $g(x,0) \ge 0$, we show the existence of another small positive solution.

Theorem 1.1. *Let $f(x,s)$ and $g(x,s)$ be continuous on $\overline{\Omega}\times[0,\infty)$. Suppose that* (f1)–(f5) *hold. Then the following assertions hold.*

(i) *There exists a $\lambda_0 > 0$ such that* (1.1), (1.2) *have a positive solution u_λ when $|\lambda| \le \lambda_0$. Furthermore, for any sequence λ_j converging to zero, along a subsequence u_{λ_j}, converges to u_0 in $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$, where u_0 is a mountain pass solution of* (1.1), (1.2) *with $\lambda = 0$ and where $W^{2,q}(\Omega)$ denotes the Sobolev space.*

(ii) *If $g(x,0) \ge 0$, $\not\equiv 0$ in Ω, then* (1.1), (1.2) *have another nonnegative solution v_λ for $\lambda > 0$ small enough such that $0 \le v_\lambda(x) < u_\lambda(x)$ and $v_\lambda \to 0$ in $W^{2,q}(\Omega)$ as $\lambda \to 0$ for all $q \in [1,\infty)$. Moreover, if*

$$\liminf_{s\to 0+}\left(\min_{x\in\overline{\Omega}}(g(x,s)-g(x,0))/s\right) > -\infty, \tag{1.3}$$

then each v_λ is strictly positive.

We give sufficient conditions for (f3)–(f5). Assumptions (f4) and (f5) are fulfilled if

$$\lim_{s\to 0+} f(x,s)/s = 0 \quad \text{uniformly on } \Omega. \tag{1.4}$$

Assumption (f3) holds if $f(x,s)$ is superlinear at $s=\infty$ in a small neighborhood of x_0, i.e.,

$$\lim_{s\to\infty}\left(\min_{|x-x_0|\le\delta_0} f(x,s)/s\right) = \infty. \tag{1.5}$$

There are many examples of $f(x,s)$ satisfying our assumptions. An easy example of the sign-changing nonlinear term is $f(x,s) = a(x)s^p + b(x)s^q$, where $a,b \in C(\overline{\Omega})$, $1 < q < p$ if $N = 1,2$ and $1 < q < p < (N+2)/(N-2)$ if $N \ge 3$. The function $f(x,s)$ satisfies (f1)–(f5) if either (i) or (ii) below holds:

(i) $a(x)$ may change its sign, but $a(x_0) > 0$ at some $x_0 \in \Omega$ and $b(x) \le 0$ in Ω.

(ii) $a(x) \ge 0$, $\not\equiv 0$ in Ω and $b(x)$ is any function.

Indeed, it is easy to verify (f1), (1.4) and (1.5). Let us check (f2). In Case (i), we choose $\alpha = p+1$ so that

$$(p+1)F(x,s) - sf(x,s) = \frac{p-q}{q+1} b(x) s^{q+1} \le 0.$$

In Case (ii), we choose $\alpha = q+1$ so that

$$(q+1)F(x,s) - sf(x,s) = \frac{q-p}{p+1} a(x) s^{p+1} \le 0.$$

Thus (f2) holds.

2. Proof of the Theorem

We shall prove Theorem 1.1. Our approach is based on the mountain pass lemma and the maximum principle. We always assume (f1)–(f5). Assumptions (f4) and (f5) imply $f(x,0) = 0$. Throughout the paper, we put $f(x,s) = 0$ for $s < 0$, and hence $f(x,s)$ is defined on $\overline{\Omega} \times \mathbb{R}$ and continuous. Moreover, (f4) and (f5) are still valid as $s \to 0$ instead of $s \to 0+$ and (f2) holds for all $s \in \mathbb{R}$.

We call u a solution of (1.1), (1.2) if it belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and satisfies (1.1) in the distribution sense. By the bootstrap argument with the elliptic regularity theorem, u belongs to $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$ and satisfies (1.1) a.e. in Ω. Especially, u lies in $C^1(\overline{\Omega})$.

Lemma 2.1. *Any nontrivial solution u of* (1.1), (1.2) *with $\lambda = 0$ is strictly positive and $\partial u/\partial\nu < 0$ on $\partial\Omega$. Here $\partial/\partial\nu$ denotes the outward normal derivative.*

Proof. Let u be a nontrivial solution of (1.1), (1.2) with $\lambda = 0$. Put

$$D := \{x \in \Omega : \ u(x) < 0\}.$$

Assume that $D \ne \emptyset$. By the extension of $f(x,s)$ on $s \le 0$,

$$-\Delta u = f(x,u) = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D.$$

Thus $u \equiv 0$ in D, a contradiction. Therefore D must be empty; i.e., $u \ge 0$ in Ω. Put $A := \|u\|_\infty$. By (f5), there exists a $C > 0$ such that $f(x,s) \ge -Cs$ for $0 \le s \le A$ and $x \in \Omega$. This inequality gives us

$$(C - \Delta)u = Cu + f(x,u) \ge 0 \quad \text{in } \Omega.$$

By the Hopf maximum principle, u is strictly positive and $\partial u/\partial\nu < 0$ on $\partial\Omega$. □

For (1.1) with $\lambda = 0$, we define the Lagrangian functional $I_0(u)$ by

$$I_0(u) := \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - F(x,u) \right) dx,$$

where $F(x,u)$ is defined in (f2). In what follows, $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm. $H_0^1(\Omega)$ stands for the usual Sobolev space equipped with the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_2$. Because of (f1), I_0 is well defined in $H_0^1(\Omega)$ and becomes a C^1 functional.

Lemma 2.2. *I_0 satisfies the Palais-Smale condition.*

Proof. Let u_n be any sequence in $H_0^1(\Omega)$ such that $I_0(u_n)$ is bounded and $\|I_0'(u_n)\|$ converges to zero. From an easy calculation, we see that

$$I_0'(u)u = \int_\Omega \left(|\nabla u|^2 - uf(x,u) \right) dx,$$

which shows that

$$\begin{aligned}&\alpha I_0(u_n) - I_0'(u_n)u_n \\ &\quad = \frac{\alpha-2}{2}\|\nabla u_n\|_2^2 - \int_\Omega (\alpha F(x,u_n) - u_n f(x,u_n))\,dx.\end{aligned} \tag{2.1}$$

Hereafter we assume $\theta \geq 1$ in (f2) because in case $\theta < 1$ we replace θ by 1 and C by a larger constant. Then the norm $\|\cdot\|_\theta$ makes sense. Since $|I_0(u_n)|$ and $\|I_0'(u_n)\|$ are bounded, we use (f2) to get a constant $C > 0$ such that

$$\begin{aligned}\frac{\alpha-2}{2}\|\nabla u_n\|_2^2 &= \alpha I_0(u_n) - I_0'(u_n)u_n + \int_\Omega (\alpha F(x,u_n) - u_n f(x,u_n))\,dx \\ &\leq C + C\|\nabla u_n\|_2 + C\|u_n\|_\theta^\theta \\ &\leq C + C\|\nabla u_n\|_2 + C'\|\nabla u_n\|_2^\theta,\end{aligned}$$

where we have used the Sobolev embedding. Since $\theta < 2$, $\|\nabla u_n\|_2$ is bounded. Then a subsequence of u_n weakly converges in $H_0^1(\Omega)$. This convergence becomes a strong one, which can be proved in the standard method. See [3, 7, 8, 9] for the details. The proof is complete. $\square$

Lemma 2.3. *I_0 has a mountain pass geometry; i.e., there exist $u_1 \in H_0^1(\Omega)$ and constants $r, \rho > 0$ such that $I_0(u_1) < 0$, $\|\nabla u_1\|_2 > r$ and*

$$I_0(u) \geq \rho \quad \text{when } \|\nabla u\|_2 = r. \tag{2.2}$$

Proof. Recall that (f4) is still valid as $s \to 0$ instead of $s \to 0+$. Then we have $s_0 > 0$ and $\mu \in (0, \mu_1)$ such that

$$f(x,s)/s < \mu \quad \text{for } |s| < s_0,$$

which implies that

$$F(x,s) \leq (\mu/2)s^2 \quad \text{for } |s| \leq s_0.$$

This inequality with (f1) shows that

$$F(x,s) \leq (\mu/2)s^2 + C|s|^{p+1} \quad \text{for } s \in \mathbb{R},$$

with some $C > 0$. Since μ_1 is the first eigenvalue of $-\Delta$, it follows that $\|\nabla u\|_2^2 \geq \mu_1\|u\|_2^2$ for $u \in H_0^1(\Omega)$. Then I_0 is estimated as

$$I_0(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{2}\|u\|_2^2 - C\|u\|_{p+1}^{p+1} \geq \frac{\mu_1 - \mu}{2\mu_1}\|\nabla u\|_2^2 - C'\|\nabla u\|_2^{p+1}.$$

This shows the existence of r and ρ satisfying (2.2). Let x_0, δ_0 be as in (f3). Let ϕ be a function such that $\phi \in C_0^1(\Omega)$, $\phi \geq 0$, $\phi \not\equiv 0$ and the support of ϕ is in $B(x_0, \delta_0)$. Here $B(x_0, \delta_0)$ is a ball centered at x_0 with radius δ_0. By (f3),

$$\min\{F(x,s)/s^2 : \ x \in \overline{B(x_0,\delta_0)}\} \to \infty \quad \text{as } s \to \infty.$$

Put $a := \|\phi\|_\infty/2$ and

$$D := \{x \in B(x_0,\delta_0) : \ \phi(x) \geq a\}.$$

For $t \geq 0$, we compute

$$\begin{aligned}I_0(t\phi) &= (t^2/2)\|\nabla\phi\|_2^2 - \int_\Omega F(x,t\phi)dx \\ &\leq (t^2/2)\|\nabla\phi\|_2^2 - t^2\int_D \frac{F(x,t\phi)}{t^2\phi^2}\phi^2 dx \to -\infty \quad \text{as } t \to \infty.\end{aligned}$$

We fix $t > 0$ so large that $I_0(t\phi) < 0$ and $t\|\nabla\phi\|_2 > r$. Then $u_1 := t\phi$ satisfies the assertion of the lemma. □

For u_1 in Lemma 2.3, we define

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \ \gamma(0) = 0, \ \gamma(1) = u_1\},$$
$$c_0 := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I_0(\gamma(t)).$$

Lemma 2.4. *c_0 is a critical value of I_0.*

Proof. This is a well-known mountain pass lemma. For the proof, we refer the reader to [3, 7, 8, 9]. □

We call u a *mountain pass solution* of I_0 if $I_0'(u) = 0$ and $I_0(u) = c_0$. In general, a mountain pass solution is not necessarily unique but we have an a priori estimate for all mountain pass solutions in the next lemma.

Lemma 2.5. *There exists a constant $C > 0$ such that $\|u\|_{C^1(\overline{\Omega})} \le C$ for any mountain pass solution u of I_0.*

Proof. Let u be any mountain pass solution of I_0. Since $I_0'(u) = 0$ and $I_0(u) = c_0$, we use (2.1) with (f2) to get

$$\frac{\alpha-2}{2}\|\nabla u\|_2^2 \le \alpha c_0 + C\|u\|_\theta^\theta + C \le \alpha c_0 + C'\|\nabla u\|_2^\theta + C.$$

This gives an a priori bound of the $H_0^1(\Omega)$ norm of u; i.e., $\|\nabla u\|_2 \le C$ with a $C > 0$ independent of u. By the bootstrap argument with (f1) and the elliptic regularity theorem, we get the upper bound of the $W^{2,q}(\Omega)$ norm of u for all $q \in [1,\infty)$. Especially, an a priori $C^1(\overline{\Omega})$ estimate of u follows. □

By Lemma 2.5, we have an $M > 0$ such that

$$\|u\|_\infty \le M \quad \text{for any mountain pass solution } u \text{ of } I_0. \tag{2.3}$$

Now, we define

$$\widetilde{g}(x,s) = \begin{cases} g(x,0) & \text{if } s \le 0,\\ g(x,s) & \text{if } 0 \le s \le 2M,\\ g(x,2M) & \text{if } s \ge 2M.\end{cases}$$

Then $\widetilde{g}(x,s)$ is continuous and bounded on $\overline{\Omega}\times\mathbb{R}$. We choose a function $h \in C_0^\infty(\mathbb{R})$ such that $0 \le h \le 1$ in $\mathbb{R}$, $h(s) = 1$ for $|s| \le 2M$ and $h(s) = 0$ for $|s| \ge 4M$. We define

$$I_\lambda(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x,u) - \lambda h(u)\widetilde{G}(x,u)\right) dx,$$
$$\widetilde{G}(x,u) := \int_0^u \widetilde{g}(x,s)ds.$$

A critical point of I_λ is a solution of

$$-\Delta u = f(x,u) + \lambda h(u)\widetilde{g}(x,u) + \lambda h'(u)\widetilde{G}(x,u) \quad \text{in } \Omega, \tag{2.4}$$

with $u = 0$ on $\partial\Omega$. Our plan to prove Theorem 1.1 is as follows. First, we find a mountain pass solution u_λ of I_λ. Next, we prove that $0 < u_\lambda(x) \le 2M$ for $|\lambda|$ small enough. Then $h'(u_\lambda) = 0$, $h(u_\lambda) = 1$, $\widetilde{g}(x,u_\lambda) = g(x,u_\lambda)$ and therefore u_λ becomes a solution of (1.1), (1.2).

Using the same argument as in Lemma 2.2 with the fact that $h(s)\widetilde{G}(x,s)$ and its partial derivative on s are bounded, we get the next lemma.

Lemma 2.6. *For each $\lambda \in \mathbb{R}$, I_λ satisfies the Palais-Smale condition.*

Lemma 2.7. *There exists a $\lambda_0 > 0$ such that I_λ has a mountain pass geometry when $|\lambda| \le \lambda_0$.*

Proof. Since $h(s)\widetilde{G}(x,s)$ is bounded on $\overline{\Omega} \times \mathbb{R}$, we have

$$I_0(u) - |\lambda|C \le I_\lambda(u) \le I_0(u) + |\lambda|C \quad \text{for } u \in H_0^1(\Omega), \tag{2.5}$$

where $C > 0$ is independent of λ and u. Let r, ρ and u_1 be as in Lemma 2.3. For $|\lambda|$ small enough, it follows that

$$I_\lambda(u_1) \le I_0(u_1) + |\lambda|C < 0,$$

$$I_\lambda(u) \ge \rho - |\lambda|C \ge \rho/2 \quad \text{when } \|\nabla u\|_2 = r. \tag{2.6}$$

The proof is complete. □

We define the mountain pass value c_λ of I_λ by

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I_\lambda(\gamma(t)).$$

Then $c_\lambda \to c_0$ as $\lambda \to 0$ by (2.5).

Lemma 2.8. *Let $\lambda_n \in \mathbb{R}$ be a sequence converging to zero and u_n a mountain pass solution of I_{λ_n}. Then a subsequence of u_n converges to a limit u_0 in $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$, where u_0 is a mountain pass solution of I_0.*

Proof. By definition, $I_{\lambda_n}(u_n) = c_{\lambda_n}$, $I'_{\lambda_n}(u_n) = 0$ and hence u_n satisfies (2.4) with λ replaced by λ_n. Using the same argument as in Lemma 2.5 with the boundedness of c_{λ_n}, we can prove that the $W^{2,q}(\Omega)$ norm of u_n is bounded for any $q \in [1,\infty)$. By the compact embedding, a subsequence of u_n converges to a limit u_0 in $C^1(\overline{\Omega})$. Then u_0 satisfies that $I_0(u_0) = c_0$ and $I'_0(u_0) = 0$, i.e., that u_0 is a mountain pass solution of I_0. The right-hand side of (2.4) with $u = u_n$ and $\lambda = \lambda_n$ converges to that with $u = u_0$ and $\lambda = 0$ uniformly on $x \in \overline{\Omega}$. The elliptic regularity theorem again ensures that u_n converges to u_0 strongly in $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$. □

We shall prove the positivity and a priori estimate of mountain pass solutions for I_λ. To this end, for $\delta > 0$, we put

$$\Omega_\delta := \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) < \delta\},$$

where $\mathrm{dist}(x,\partial\Omega)$ denotes the distance from x to $\partial\Omega$.

Lemma 2.9. *There exist constants $\lambda_0, \delta, a, b > 0$ such that any mountain pass solution u of I_λ with $|\lambda| \le \lambda_0$ satisfies* (i) *and* (ii) *below.*

(i) *$0 < u(x) \le 2M$ in Ω, where M has been defined by* (2.3).

(ii) *$\partial u/\partial\nu < -a$ in Ω_δ and $u(x) > b$ in $\Omega \setminus \Omega_\delta$. Here $\partial/\partial\nu$ is well defined at each point in Ω_δ for $\delta > 0$ small because $\partial\Omega$ is smooth.*

Proof. First, we shall prove $|u(x)| \le 2M$ for $|\lambda| > 0$ small enough. Suppose that our claim is false. Then there exist sequences $\lambda_n \in \mathbb{R}$ and u_n such that λ_n converges to zero, u_n is a mountain pass solution of I_{λ_n} and $\|u_n\|_\infty > 2M$. By Lemma 2.8, a subsequence of u_n converges to a mountain pass solution u_0 of I_0 in $C^1(\overline{\Omega})$. Since $\|u_0\|_\infty \le M$ by (2.3), it follows that $\|u_n\|_\infty < 2M$ for n large enough. A contradiction occurs. Thus we have $\|u\|_\infty \le 2M$. The positivity of u in (i) follows from (ii).

Next, we shall prove that $\partial u/\partial\nu < -a$ in Ω_δ with some $a, \delta > 0$ independent of u. Suppose on the contrary that there exist λ_n, x_n, u_n such that $\lambda_n \to 0$, $\operatorname{dist}(x_n, \partial\Omega) \to 0$, u_n is a mountain pass solution of I_{λ_n} and

$$\liminf_{n\to\infty} \partial u_n(x_n)/\partial\nu \ge 0.$$

We choose a subsequence of x_n which converges to a limit $x_0 \in \partial\Omega$. By Lemma 2.8, a subsequence of u_n converges to a mountain pass solution u_0 of I_0 in $C^1(\overline{\Omega})$. Then $\partial u_0/\partial\nu(x_0) \ge 0$, a contradiction to Lemma 2.1. Thus $\partial u/\partial\nu < -a$ in Ω_δ with some $a, \delta > 0$. Fix such a $\delta > 0$. Then by the same method as above, we can prove that $u(x) > b$ in $\Omega \setminus \Omega_\delta$ with some $b > 0$. □

In Lemma 2.3, we replace r by any positive constant smaller than r. Then (2.2) is still valid after ρ is replaced by a smaller positive constant. Hence (2.6) still holds if $|\lambda|$ is replaced by a small one. Thus the next lemma follows.

Lemma 2.10. *There exists an $r_0 > 0$ such that for any $r \in (0, r_0)$, there exist constants $\rho, \lambda' > 0$ which satisfy*

$$I_\lambda(u) \ge \rho \quad \text{when } \|\nabla u\|_2 = r,\ |\lambda| < \lambda'.$$

The lemma above will be used to find a small positive solution of (1.1), (1.2). We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Choose $\lambda_0 > 0$ which satisfies Lemmas 2.7 and 2.9. Let u_λ be a mountain pass solution of I_λ with $|\lambda| \le \lambda_0$. Then $0 < u_\lambda(x) \le 2M$ by Lemma 2.9. Thus $h'(u_\lambda) = 0$, $h(u_\lambda) = 1$, $\widetilde{g}(x, u_\lambda) = g(x, u_\lambda)$ and therefore u_λ becomes a solution of (1.1), (1.2). Let λ_j be any sequence converging to zero. By Lemma 2.8, a subsequence $u_{\lambda'_j}$ converges to a mountain pass solution u_0 of I_0 in $W^{2,q}(\Omega)$ for all $q \in [1, \infty)$.

We now suppose that $g(x, 0) \ge 0$, $g(x, 0) \not\equiv 0$ in Ω. By (2.6), we have

$$\inf_{\|\nabla u\|_2 = r} I_\lambda(u) \ge \rho/2 > 0 = I_\lambda(0).$$

Let B be the set of $u \in H_0^1(\Omega)$ such that $\|\nabla u\|_2 \le r$. Then the minimum of I_λ in B is achieved at an interior point v_λ. Indeed, choose a sequence u_n in B such that $I_\lambda(u_n)$ converges to the infimum of I_λ in B. A subsequence of u_n weakly converges in $H_0^1(\Omega)$ to a point v_λ in B. By the weakly lower semicontinuity of I_λ, we have

$$I_\lambda(v_\lambda) \le \liminf_{n\to\infty} I_\lambda(u_n),$$

which means that v_λ is a minimum point of I_λ in B. Since $I_\lambda(0) = 0$, we have $I_\lambda(v_\lambda) \le 0 < I_\lambda(u_\lambda)$, where u_λ is a mountain pass solution of I_λ. Therefore $v_\lambda \ne u_\lambda$. In the same way as in Lemma 2.5 with $|\lambda|$ and $r > 0$ small enough, we can prove that $\|v_\lambda\|_\infty \le M$. Hence $\widetilde{g}(x, v_\lambda) = g(x, v_\lambda)$ and v_λ is a solution of (1.1). Moreover, $v_\lambda \not\equiv 0$ because $g(x, 0) \not\equiv 0$. Thus v_λ is a nontrivial solution. We shall show that $v_\lambda(x) \ge 0$ for $\lambda > 0$. Let D be the set of $x \in \Omega$ such that $v_\lambda(x) < 0$. Since $f(x, s) = f(x, 0) = 0$ and $g(x, s) = g(x, 0) \ge 0$ for $s < 0$, we see that for $\lambda > 0$,

$$-\Delta v_\lambda = f(x, v_\lambda) + \lambda g(x, v_\lambda) \ge 0 \quad \text{in } D, \quad v_\lambda = 0 \text{ on } \partial D,$$

which shows that $v_\lambda \ge 0$ in D, a contradiction to the definition of D. Thus D must be empty, and $v_\lambda(x) \ge 0$ in Ω. By Lemma 2.10, $\|\nabla v_\lambda\|_2 \to 0$ as $\lambda \to 0$. By the bootstrap argument, the $W^{2,q}(\Omega)$ norm of v_λ converges to zero for all $q \in [1, \infty)$, and hence $v_\lambda \to 0$ in $C^1(\overline{\Omega})$. Then Lemma 2.9 (ii) shows that $v_\lambda(x) < u_\lambda(x)$ in Ω for $\lambda > 0$ small enough.

We suppose that (1.3) holds. Put $A := \|v_\lambda\|_\infty$. By (1.3), there is a $C > 0$ such that

$$g(x,s) - g(x,0) \geq -Cs \quad \text{for } 0 \leq s \leq A,\ x \in \Omega.$$

Moreover, $f(x,s) \geq -Cs$ for $0 \leq s \leq A$ in the proof of Lemma 2.1. Then we have

$$\begin{aligned}((1+\lambda)C - \Delta)v_\lambda &= f(x, v_\lambda) + Cv_\lambda \\ &\qquad + \lambda(g(x,v_\lambda) - g(x,0) + Cv_\lambda) + \lambda g(x,0) \\ &\geq 0.\end{aligned}$$

By the strong maximum principle, v_λ is strictly positive. The proof is complete. □

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