# RIESZ BASES CONSISTING OF ROOT FUNCTIONS OF 1D DIRAC OPERATORS 

PLAMEN DJAKOV AND BORIS MITYAGIN<br>(Communicated by James E. Colliander)

Abstract. For one-dimensional Dirac operators

$$
L y=i\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \frac{d y}{d x}+v y, \quad v=\left(\begin{array}{cc}
0 & P \\
Q & 0
\end{array}\right), \quad y=\binom{y_{1}}{y_{2}}
$$

subject to periodic or antiperiodic boundary conditions, we give necessary and sufficient conditions which guarantee that the system of root functions contains Riesz bases in $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$.

In particular, if the potential matrix $v$ is skew-symmetric (i.e., $\bar{Q}=-P$ ), or more generally if $\bar{Q}=t P$ for some real $t \neq 0$, then there exists a Riesz basis that consists of root functions of the operator $L$.

## 1. Introduction

We consider one-dimensional Dirac operators of the form

$$
L_{b c}(v) y=i\left(\begin{array}{cc}
1 & 0  \tag{1.1}\\
0 & -1
\end{array}\right) \frac{d y}{d x}+v(x) y, \quad v=\left(\begin{array}{cc}
0 & P \\
Q & 0
\end{array}\right), \quad y=\binom{y_{1}}{y_{2}}
$$

with periodic matrix potentials $v$ such that $P, Q \in L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$, subject to periodic ( $\mathrm{Per}^{+}$) or antiperiodic ( $\mathrm{Per}^{-}$) boundary conditions (bc):

$$
\begin{equation*}
\text { Per }^{+}: y(\pi)=y(0) ; \quad \text { Per }^{-}: y(\pi)=-y(0) \tag{1.2}
\end{equation*}
$$

Our goal is to give necessary and sufficient conditions on potentials $v$ which guarantee that the system of periodic (or antiperiodic) root functions of $L_{P e r^{ \pm}}(v)$ contains Riesz bases.

The free operators $L_{P e r^{ \pm}}^{0}=L_{P e r^{ \pm}}(0)$ have the discrete spectra:

$$
S p\left(L_{P e r^{ \pm}}^{0}\right)=\Gamma^{ \pm}, \quad \text { where } \quad \Gamma^{ \pm}= \begin{cases}2 \mathbb{Z} & \text { if } b c=\text { Per }^{+} \\ 2 \mathbb{Z}+1 & \text { if } b c=\text { Per }^{-}\end{cases}
$$

and each eigenvalue is of multiplicity 2 . The spectra of perturbed operators $L_{P e r^{ \pm}}(v)=L_{P e r^{ \pm}}^{0}+v$ is also discrete; for $n \in \Gamma^{ \pm}$with large enough $|n|$ the perturbed operator has "twin" eigenvalues $\lambda_{n}^{ \pm}$close to $n$. In the case where $\lambda_{n}^{-} \neq \lambda_{n}^{+}$for large enough $|n|$, could the corresponding normalized "twin eigenfunctions" form a Riesz basis?

[^0]Recently, in the case of Hill operators, many authors focused on this problem (see [1, 2, 5, 6, 8, 10, 11, 12, 15, 16, and the bibliographies therein). It may happen that $\lambda_{n}^{-} \neq \lambda_{n}^{+}$for $|n|>N_{*}$ but the system of normalized eigenfunctions fails to give a convergent eigenfunction expansion (see [2, Theorem 71]).

In the present paper we consider such a problem in the case of 1D periodic Dirac operators. In [7, we have singled out a class of potentials $v$ for which smoothness could be determined only by the rate of decay of related spectral gaps $\gamma_{n}=\lambda_{n}^{+}-\lambda_{n}^{-}$, where $\lambda_{n}^{ \pm}$are the eigenvalues of $L=L(v)$ considered on $[0, \pi]$ with periodic (for even $n$ ) or antiperiodic (for odd $n$ ) boundary conditions. This class $X$ is determined by the properties of the functionals $\beta_{n}^{-}(v ; z)$ and $\beta_{n}^{+}(v, z)$ (see (2.8) below) to be equivalent in the following sense: there are $c, N>0$ such that

$$
c^{-1}\left|\beta_{n}^{+}\left(v ; z_{n}^{*}\right)\right| \leq\left|\beta_{n}^{-}\left(v ; z_{n}^{*}\right)\right| \leq c\left|\beta_{n}^{+}\left(v ; z_{n}^{*}\right)\right|, \quad|n|>N, \quad z_{n}^{*}=\left(\lambda_{n}^{+}+\lambda_{n}^{-}\right) / 2-n .
$$

Section 3 contains the main results of this paper. We prove that if $v \in X$, then the system of root functions of the operator $L_{P e r} \pm(v)$ contains Riesz bases in $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$. Theorem 3.1] which is analogous to Theorem 1 in [6] (or Theorem 2 in [5]), gives necessary and sufficient conditions for the existence of such Riesz bases. Theorem 3.2 is a modification of Theorem 3.1 that is more suitable for application to concrete classes of potentials.

Applications of Theorems 3.1 and 3.2 are given in Section 4. In particular, we prove that if the potential matrix $v$ is skew-symmetric (i.e., $\bar{Q}=-P$ ), then the system of root functions of $L_{P e r} \pm(v)$ contains Riesz bases in $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$.

## 2. Preliminaries

1. Let $H$ be a separable Hilbert space, and let $\left(e_{\alpha}, \alpha \in \mathcal{I}\right)$ be an orthonormal basis in $H$. If $A: H \rightarrow H$ is an automorphism, then the system

$$
\begin{equation*}
f_{\alpha}=A e_{\alpha}, \quad \alpha \in \mathcal{I} \tag{2.1}
\end{equation*}
$$

is an unconditional basis in $H$. Indeed, for each $x \in H$ we have

$$
x=A\left(A^{-1} x\right)=A\left(\sum_{\alpha}\left\langle A^{-1} x, e_{\alpha}\right\rangle e_{\alpha}\right)=\sum_{\alpha}\left\langle x,\left(A^{-1}\right)^{*} e_{\alpha}\right\rangle f_{\alpha}=\sum_{\alpha}\left\langle x, \tilde{f}_{\alpha}\right\rangle f_{\alpha}
$$

i.e., $\left(f_{\alpha}\right)$ is a basis, its biorthogonal system is $\left\{\tilde{f}_{\alpha}=\left(A^{-1}\right)^{*} e_{\alpha}, \alpha \in \mathcal{I}\right\}$, and the series converge unconditionally. Moreover, it follows that

$$
\begin{equation*}
0<c \leq\left\|f_{\alpha}\right\| \leq C, \quad m^{2}\|x\|^{2} \leq \sum_{\alpha}\left|\left\langle x, \tilde{f}_{\alpha}\right\rangle\right|^{2}\left\|f_{\alpha}\right\|^{2} \leq M^{2}\|x\|^{2} \tag{2.2}
\end{equation*}
$$

with $c=1 /\left\|A^{-1}\right\|, C=\|A\|, M=\|A\| \cdot\left\|A^{-1}\right\|$ and $m=1 / M$.
A basis of the form (2.1) is called a Riesz basis. One can easily see that the property (2.2) characterizes Riesz bases; i.e., a basis $\left(f_{\alpha}\right)$ is a Riesz bases if and only if (2.2) holds with some constants $C \geq c>0$ and $M \geq m>0$. Another characterization of Riesz bases is given by the following assertion (see 9, Chapter 6 , Section 5.3, Theorem 5.2]): If $\left(f_{\alpha}\right)$ is a normalized basis (i.e., $\left\|f_{\alpha}\right\|=1 \forall \alpha$ ), then it is a Riesz basis if and only if it is unconditional.

A countable family of bounded projections $\left\{P_{\alpha}: H \rightarrow H, \alpha \in \mathcal{I}\right\}$ is called an unconditional basis of projections if $P_{\alpha} P_{\beta}=0$ for $\alpha \neq \beta$ and

$$
x=\sum_{\alpha \in \mathcal{I}} P_{\alpha}(x) \quad \forall x \in H,
$$

where the series converge unconditionally in $H$.
If $\left\{H_{\alpha}, \alpha \in \mathcal{I}\right\}$ is a maximal family of mutually orthogonal subspaces of $H$ and $Q_{\alpha}$ is the orthogonal projection on $H_{\alpha}, \alpha \in \mathcal{I}$, then $\left\{Q_{\alpha}, \alpha \in \mathcal{I}\right\}$ is an orthogonal basis of projections. A family of projections $\left\{P_{\alpha}, \alpha \in \mathcal{I}\right\}$ is called a Riesz basis of projections if there is an orthogonal basis of projections $\left\{Q_{\alpha}, \alpha \in \mathcal{I}\right\}$ and an isomorphism $A: H \rightarrow H$ such that

$$
\begin{equation*}
P_{\alpha}=A Q_{\alpha} A^{-1}, \quad \alpha \in \mathcal{I} \tag{2.3}
\end{equation*}
$$

In view of (2.3), if $\left\{P_{\alpha}\right\}$ is a Riesz basis of projections, then there are constants $a, b>0$ such that

$$
\begin{equation*}
a\|x\|^{2} \leq \sum_{\alpha}\left\|P_{\alpha} x\right\|^{2} \leq b\|x\|^{2} \quad \forall x \in H . \tag{2.4}
\end{equation*}
$$

For a family of projections $\mathcal{P}=\left\{P_{\alpha}, \alpha \in \mathcal{I}\right\}$, the following properties are equivalent (see [9, Chapter 6]):
(i) $\mathcal{P}$ is an unconditional basis of projections;
(ii) $\mathcal{P}$ is a Riesz basis of projections.

Lemma 2.1. Let $\left(P_{\alpha}, \alpha \in \mathcal{I}\right)$ be a Riesz basis of two-dimensional projections in a Hilbert space $H$, and let $f_{\alpha}, g_{\alpha} \in \operatorname{Ran} P_{\alpha}, \alpha \in \mathcal{I}$ be linearly independent unit vectors. Then the system $\left\{f_{\alpha}, g_{\alpha}, \alpha \in \mathcal{I}\right\}$ is a Riesz basis if and only if

$$
\begin{equation*}
\kappa:=\sup \left|\left\langle f_{\alpha}, g_{\alpha}\right\rangle\right|<1 . \tag{2.5}
\end{equation*}
$$

Proof. Suppose that the system $\left\{f_{\alpha}, g_{\alpha}, \alpha \in \mathcal{I}\right\}$ is a Riesz basis in $H$. Then

$$
x=\sum_{\alpha}\left(f_{\alpha}^{*}(x) f_{\alpha}+g_{\alpha}^{*}(x) g_{\alpha}\right), \quad x \in H,
$$

where $f_{\alpha}^{*}, g_{\alpha}^{*}$ are the conjugate functionals. By (2.2), the one-dimensional projections

$$
P_{\alpha}^{1}(x)=f_{\alpha}^{*}(x) f_{\alpha}, \quad P_{\alpha}^{2}(x)=g_{\alpha}^{*}(x) g_{\alpha}, \quad \alpha \in \mathcal{I},
$$

are uniformly bounded. On the other hand, it is easy to see that

$$
\left\|P_{\alpha}^{1}\right\|^{2} \geq\left(1-\left|\left\langle f_{\alpha}, g_{\alpha}\right\rangle\right|^{2}\right)^{-1}, \quad\left\|P_{\alpha}^{2}\right\|^{2} \geq\left(1-\left|\left\langle f_{\alpha}, g_{\alpha}\right\rangle\right|^{2}\right)^{-1}
$$

so (2.5) holds.
Conversely, suppose (2.5) holds. Then we have for every $\alpha \in \mathcal{I}$,

$$
(1-\kappa)\left(\left|f_{\alpha}^{*}(x)\right|^{2}+\left|g_{\alpha}^{*}(x)\right|^{2}\right) \leq\left\|P_{\alpha}(x)\right\|^{2} \leq(1+\kappa)\left(\left|f_{\alpha}^{*}(x)\right|^{2}+\left|g_{\alpha}^{*}(x)\right|^{2}\right),
$$

which implies, in view of (2.4),

$$
\frac{a}{1+\kappa}\|x\|^{2} \leq \sum_{\alpha}\left(\left|f_{\alpha}^{*}(x)\right|^{2}+\left|g_{\alpha}^{*}(x)\right|^{2}\right) \leq \frac{b}{1-\kappa}\|x\|^{2} .
$$

Therefore, (2.2) holds, which means that the system $\left\{f_{\alpha}, g_{\alpha}, \alpha \in \mathcal{I}\right\}$ is a Riesz basis in $H$.
2. We consider the Dirac operator (1.1) with $b c=P e r^{ \pm}$in the domain
$\operatorname{Dom}\left(L_{P e r} \pm(v)\right)=\left\{y=\binom{y_{1}}{y_{2}}: y_{1}, y_{2}\right.$ are absolutely continuous, $\left.y(\pi)= \pm y(0)\right\}$.
Then the operator $L_{P e r} \pm(v)$ is densely defined and closed; its adjoint operator is

$$
\left(L_{P e r^{ \pm}}(v)\right)^{*}=L_{P e r} \pm\left(v^{*}\right), \quad v^{*}=\left(\begin{array}{cc}
0 & \bar{Q}  \tag{2.6}\\
\bar{P} & 0
\end{array}\right) .
$$

Lemma 2.2. The spectra of the operators $L_{\text {Per }} \pm(v)$ are discrete. There is an $N=$ $N(v)$ such that the union $\bigcup_{|n|>N} D_{n}$ of the discs $D_{n}=\{z:|z-n|<1 / 4\}$ contains all but finitely many of the eigenvalues of $L_{P e r^{+}}$and $L_{P e r}{ }^{-}$while the remaining finitely many eigenvalues are in the rectangle $R_{N}=\{z:|\operatorname{Re} z|,|\operatorname{Im} z| \leq N+1 / 2\}$.

Moreover, for $|n|>N$, the disc $D_{n}$ contains two (counted with algebraic multiplicity) periodic (if $n$ is even) or antiperiodic (if $n$ is odd) eigenvalues $\lambda_{n}^{-}, \lambda_{n}^{+}$such that $\operatorname{Re} \lambda_{n}^{-}<\operatorname{Re} \lambda_{n}^{+}$or $\operatorname{Re} \lambda_{n}^{-}=\operatorname{Re} \lambda_{n}^{+}$and $\operatorname{Im} \lambda_{n}^{-} \leq \operatorname{Im} \lambda_{n}^{+}$.

See the details and more general results about localization of these spectra in [13, 14] and [2, Section 1.6].

Lemma 2.2 allows us to apply the Lyapunov-Schmidt projection method and reduce the eigenvalue equation $L y=\lambda y$ for $\lambda \in D_{n}$ to an eigenvalue equation in the two-dimensional space $E_{n}^{0}=\left\{L^{0} Y=n Y\right\}$ (see [2, Section 2.4]). This leads to the following (see in [2] the formulas (2.59)-(2.80) and Lemma 30).

Lemma 2.3. (a) For large enough $|n|, n \in \mathbb{Z}$, there are functionals $\alpha_{n}(v ; z)$ and $\beta_{n}^{ \pm}(v ; z),|z|<1$ such that a number $\lambda=n+z,|z|<1 / 4$, is a periodic (for even $n$ ) or antiperiodic (for odd $n$ ) eigenvalue of $L$ if and only if $z$ is an eigenvalue of the matrix

$$
\left[\begin{array}{cc}
\alpha_{n}(v ; z) & \beta_{n}^{-}(v ; z)  \tag{2.7}\\
\beta_{n}^{+}(v ; z) & \alpha_{n}(v ; z)
\end{array}\right]
$$

(b) A number $\lambda=n+z^{*},\left|z^{*}\right|<\frac{1}{4}$, is a periodic (for even $n$ ) or antiperiodic (for odd $n$ ) eigenvalue of $L$ of geometric multiplicity 2 if and only if $z^{*}$ is an eigenvalue of the matrix (2.7) of geometric multiplicity 2 .

The functionals $\alpha_{n}(z ; v)$ and $\beta_{n}^{ \pm}(z ; v)$ are well defined for large enough $|n|$ by explicit expressions in terms of the Fourier coefficients $p(m), q(m), m \in 2 \mathbb{Z}$ of the potential entries $P, Q$ about the system $\left\{e^{i m x}, m \in 2 \mathbb{Z}\right\}$ (see [2, Formulas (2.59)(2.80)]). Here we provide formulas only for $\beta_{n}^{ \pm}(v ; z)$ :

$$
\begin{equation*}
\beta_{n}^{ \pm}(v ; z)=\sum_{\nu=0}^{\infty} \sigma_{\nu}^{ \pm} \quad \text { with } \quad \sigma_{0}^{+}=q(2 n), \quad \sigma_{0}^{-}=p(-2 n) \tag{2.8}
\end{equation*}
$$

$$
\begin{aligned}
& \sigma_{\nu}^{+}=\sum_{j_{1}, \ldots, j_{2 \nu} \neq n} \frac{q\left(n+j_{1}\right) p\left(-j_{1}-j_{2}\right) q\left(j_{2}+j_{3}\right) \ldots p\left(-j_{2 \nu-1}-j_{2 \nu}\right) q\left(j_{2 \nu}+n\right)}{\left(n-j_{1}+z\right)\left(n-j_{2}+z\right) \ldots\left(n-j_{2 \nu-1}+z\right)\left(n-j_{2 \nu}+z\right)}, \\
& \sigma_{\nu}^{-}=\sum_{j_{1}, \ldots, j_{2 \nu} \neq n} \frac{p\left(-n-j_{1}\right) q\left(j_{1}+j_{2}\right) p\left(-j_{2}-j_{3}\right) \ldots q\left(j_{2 \nu-1}+j_{2 \nu}\right) p\left(-j_{2 \nu}-n\right)}{\left(n-j_{1}+z\right)\left(n-j_{2}+z\right) \ldots\left(n-j_{2 \nu-1}+z\right)\left(n-j_{2 \nu}+z\right)}
\end{aligned}
$$

where $j_{1}, \ldots, j_{2 \nu} \in n+2 \mathbb{Z}$.
Next we summarize some basic properties of $\alpha_{n}(z ; v)$ and $\beta_{n}^{ \pm}(z ; v)$.

Proposition 2.4. (a) The functionals $\alpha_{n}(z ; v)$ and $\beta_{n}^{ \pm}(z ; v)$ depend analytically on $z$ for $|z| \leq 1$. For $|n| \geq n_{0}$ the following estimates hold:

$$
\begin{align*}
& \left|\alpha_{n}(v ; z)\right|,\left|\beta_{n}^{ \pm}(v ; z)\right| \leq C\left(\mathcal{E}_{|n|}(r)+1 / \sqrt{|n|}\right), \quad|z| \leq 1 / 2  \tag{2.9}\\
& \left|\frac{\partial \alpha_{n}}{\partial z}(v ; z)\right|,\left|\frac{\partial \beta_{n}^{ \pm}}{\partial z}(v ; z)\right| \leq C\left(\mathcal{E}_{|n|}(r)+1 / \sqrt{|n|}\right), \quad|z| \leq 1 / 4 \tag{2.10}
\end{align*}
$$

where $r=(r(m)), r(m)=\max \{|p( \pm m)|,|q( \pm m)|\}, C=C(\|r\|), n_{0}=n_{0}(r)$ and

$$
\left(\mathcal{E}_{m}(r)\right)^{2}=\sum_{|k| \geq m}|r(k)|^{2} .
$$

(b) For large enough $|n|$, the number $\lambda=n+z, z \in D=\{\zeta:|\zeta| \leq 1 / 4\}$, is an eigenvalue of $L_{P e r^{ \pm}}$if and only if $z \in D$ satisfies the basic equation

$$
\begin{equation*}
\left(z-\alpha_{n}(z ; v)\right)^{2}=\beta_{n}^{+}(z ; v) \beta_{n}^{-}(z ; v) . \tag{2.11}
\end{equation*}
$$

(c) For large enough $|n|$, the equation (2.11) has exactly two roots in $D$ counted with multiplicity.

Proof. The assertion (a) is proved in [2, Proposition 35]. Lemma 2.3 implies (b). By (2.9), $\sup _{D}\left|\alpha_{n}(z)\right| \rightarrow 0$ and $\sup _{D}\left|\beta_{n}^{ \pm}(z)\right| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, (c) follows from the Rouché theorem.

In view of Lemma 2.2, for large enough $|n|$ the numbers $z_{n}^{*}=\left(\lambda_{n}^{+}+\lambda_{n}^{-}\right) / 2-n$ are well defined. The following estimate of $\gamma_{n}$ from above follows from (2.9) and (2.10) (see [2, Lemma 40]).

Lemma 2.5. For large enough $|n|$,

$$
\begin{equation*}
\gamma_{n}=\left|\lambda_{n}^{+}-\lambda_{n}^{-}\right| \leq\left(1+\delta_{n}\right)\left(\left|\beta_{n}^{-}\left(z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(z_{n}^{*}\right)\right|\right) \tag{2.12}
\end{equation*}
$$

with $\delta_{n} \rightarrow 0$ as $|n| \rightarrow \infty$.
Remark. Here and sometimes hereafter, we suppress the dependence on $v$ in the notation and write $\alpha_{n}(z)$ and $\beta_{n}^{ \pm}(z)$.
3. In view of the above consideration, there is $n_{0}=n_{0}(v)$ such that $\lambda_{n}^{ \pm}, \beta_{n}^{ \pm}(z)$ and $\alpha_{n}(z)$ are well defined for $|n|>n_{0}$, and Lemmas 2.2, 2.3, 2.5 and Proposition 2.4 hold. Let us set

$$
\begin{equation*}
\mathcal{M}^{ \pm}=\left\{n \in \Gamma^{ \pm}: \quad n \in \Gamma^{ \pm}, \quad|n|>n_{0}, \quad \lambda_{n}^{-} \neq \lambda_{n}^{+}\right\} . \tag{2.13}
\end{equation*}
$$

Definition. Let $X^{ \pm}$be the class of all Dirac potentials $v$ with the following property: there are constants $c \geq 1$ and $N \geq n_{0}$ such that

$$
\begin{equation*}
\frac{1}{c}\left|\beta_{n}^{+}\left(v ; z_{n}^{*}\right)\right| \leq\left|\beta_{n}^{-}\left(v ; z_{n}^{*}\right)\right| \leq c\left|\beta_{n}^{+}\left(v ; z_{n}^{*}\right)\right| \quad \text { if } n \in \mathcal{M}^{ \pm},|n| \geq N . \tag{2.14}
\end{equation*}
$$

Lemma 2.6. If $v \in X^{ \pm}$and the set $\mathcal{M}^{ \pm}$is infinite, then for $n \in \mathcal{M}^{ \pm}$with sufficiently large $|n|$ we have

$$
\begin{equation*}
\frac{1}{2}\left|\beta_{n}^{ \pm}\left(v ; z_{n}^{*}\right)\right| \leq\left|\beta_{n}^{ \pm}(v ; z)\right| \leq 2\left|\beta_{n}^{ \pm}\left(v ; z_{n}^{*}\right)\right| \quad \forall z \in K_{n}:=\left\{z:\left|z-z_{n}^{*}\right| \leq \gamma_{n}\right\} \tag{2.15}
\end{equation*}
$$

Proof. By Lemma [2.5] if $v \in X^{ \pm}$, then for $n \in \mathcal{M}^{ \pm}$with large enough $|n|$ we have $\beta_{n}^{ \pm}\left(z_{n}^{*}\right) \neq 0$. In view of (2.10), if $z \in K_{n}$, then for large enough $|n|$,

$$
\left|\beta_{n}^{ \pm}(z)-\beta_{n}^{ \pm}\left(z_{n}^{*}\right)\right| \leq \varepsilon_{n}\left|z-z_{n}^{*}\right| \leq \varepsilon_{n} \gamma_{n},
$$

where $\varepsilon_{n}=C\left(\mathcal{E}_{|n|}(r)+1 / \sqrt{|n|}\right) \rightarrow 0$ as $|n| \rightarrow \infty$. By Lemma 2.5, for large enough $|n|$ we have $\gamma_{n} \leq 2\left(\left|\beta_{n}^{-}\left(z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(z_{n}^{*}\right)\right|\right)$. Then, for $n \in \mathcal{M}^{ \pm}$,

$$
\left|\beta_{n}^{ \pm}(z)-\beta_{n}^{ \pm}\left(z_{n}^{*}\right)\right| \leq 2 \varepsilon_{n}\left(\left|\beta_{n}^{-}\left(z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(z_{n}^{*}\right)\right|\right) \leq 2 \varepsilon_{n}(1+c)\left|\beta_{n}^{ \pm}\left(z_{n}^{*}\right)\right|
$$

which implies, for sufficiently large $|n|$,

$$
\left[1-2 \varepsilon_{n}(1+c)\right]\left|\beta_{n}^{ \pm}\left(z_{n}^{*}\right)\right| \leq\left|\beta_{n}^{ \pm}(z)\right| \leq\left[1+2 \varepsilon_{n}(1+c)\right]\left|\beta_{n}^{ \pm}\left(z_{n}^{*}\right)\right|
$$

Since $\varepsilon_{n} \rightarrow 0$ as $|n| \rightarrow \infty$, (2.15) follows.
Proposition 2.7. Suppose that $v \in X^{ \pm}$and the corresponding set $\mathcal{M}^{ \pm}$is infinite. Then for $n \in \mathcal{M}^{ \pm}$with large enough $|n|$,

$$
\begin{equation*}
\frac{2 \sqrt{c}}{1+4 c}\left(\left|\beta_{n}^{-}\left(v ; z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(v ; z_{n}^{*}\right)\right|\right) \leq \gamma_{n} \leq 2\left(\left|\beta_{n}^{-}\left(v ; z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(v ; z_{n}^{*}\right)\right|\right) . \tag{2.16}
\end{equation*}
$$

Proof. The estimate of $\gamma_{n}$ from above follows from Lemma 2.5. By Lemma 2.5, for $n \in \mathcal{M}^{ \pm}$with large enough $|n|$ we have $\beta_{n}^{ \pm}\left(z_{n}^{*}\right) \neq 0$. Set

$$
t_{n}=\left|\beta_{n}^{+}\left(z_{n}^{+}\right)\right| /\left|\beta_{n}^{-}\left(z_{n}^{+}\right)\right|, \quad z_{n}^{+}=\lambda_{n}^{+}-n, \quad n \in \mathcal{M}^{ \pm} .
$$

By Lemma 2.6, $t_{n}$ is well defined for large enough $|n|$. By Lemma 49 in [2], there exists a sequence $\left(\delta_{n}\right)_{n \in \mathbb{Z}}$ with $\delta_{n} \rightarrow 0$ as $|n| \rightarrow \infty$ such that, for $n \in \mathcal{M}^{ \pm}$with large enough $|n|$,

$$
\begin{equation*}
\left|\gamma_{n}\right| \geq\left(\frac{2 \sqrt{t_{n}}}{1+t_{n}}-\delta_{n}\right)\left(\left|\beta_{n}^{-}\left(z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(z_{n}^{*}\right)\right|\right) \tag{2.17}
\end{equation*}
$$

In view of (2.15) in Lemma 2.6, for large enough $|n|$ we have $1 /(4 c) \leq t_{n} \leq 4 c$. Therefore, by (2.17) it follows that

$$
\gamma_{n} \geq\left(\frac{2 \sqrt{4 c}}{1+4 c}-\delta_{n}\right)\left(\left|\beta_{n}^{-}\left(z_{n}^{*}\right)\right|+\left|\beta_{n}^{+}\left(z_{n}^{*}\right)\right|\right)
$$

which implies (since $\delta_{n} \rightarrow 0$ as $|n| \rightarrow \infty$ ) the left inequality in (2.16). This completes the proof.

## 3. Riesz bases of root functions

In view of Lemma 2.2, the Dirac operators $L_{P e r}{ }^{ \pm}(v)$ have discrete spectra; for $N$ large enough and $n \in \Gamma^{ \pm}$with $|n|>N$ the Riesz projections

$$
\begin{equation*}
S_{N}^{ \pm}=\frac{1}{2 \pi i} \int_{\partial R_{N}}\left(z-L_{P e r^{ \pm}}\right)^{-1} d z, \quad P_{n}^{ \pm}=\frac{1}{2 \pi i} \int_{|z-n|=\frac{1}{4}}\left(z-L_{P e r} \pm\right)^{-1} d z \tag{3.1}
\end{equation*}
$$

are well defined and $\operatorname{dim} S_{N}^{ \pm}<\infty, \operatorname{dim} P_{n}^{ \pm}=2$. Further we suppress in the notation the dependence on the boundary conditions Per $^{ \pm}$and write $S_{N}, P_{n}$ only. By [4, Theorem 3],

$$
\begin{equation*}
\sum_{n \in \Gamma^{ \pm},|n|>N}\left\|P_{n}-P_{n}^{0}\right\|^{2}<\infty \tag{3.2}
\end{equation*}
$$

where $P_{n}^{0}$ are the Riesz projections of the free operator. Moreover, the Bari-Markus criterion implies (see Theorem 9 in [4]) that the spectral Riesz decompositions

$$
\begin{equation*}
f=S_{N} f+\sum_{n \in \Gamma^{ \pm},|n|>N} P_{n} f \quad \forall f \in L^{2}\left([0, \pi], \mathbb{C}^{2}\right) \tag{3.3}
\end{equation*}
$$

converge unconditionally. In other words, $\left\{S_{N}, P_{n}, n \in \Gamma^{ \pm},|n|>N\right\}$ is a Riesz basis of projections in the space $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$.
Theorem 3.1. (A) If $v \in X^{ \pm}$, then there exists a Riesz basis in $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$ which consists of root functions of the operator $L_{P e r} \pm(v)$.
(B) If $v \notin X^{ \pm}$, then the system of root functions of the operator $L_{P e r} \pm(v)$ does not contain Riesz bases.

Remark. To avoid any confusion, let us emphasize that in Theorem 3.1 two independent theorems are stacked together: one for the case of periodic boundary conditions $\mathrm{Per}^{+}$, and another one for the case of antiperiodic boundary conditions Per ${ }^{-}$.

Proof. We consider only the case of periodic boundary conditions $b c=\mathrm{Per}^{+}$since the proof is the same in the case of antiperiodic boundary conditions $b c=\mathrm{Per}^{-}$.
(A) Fix $v \in X^{+}$, and let $N=N(v)>n_{0}(v)$ be chosen so large that Lemma 2.6, Proposition 2.7 and (3.1)-(3.3) hold for $|n|>N$.

If $n \notin \mathcal{M}^{+}$, then $\lambda_{n}^{*}=n+z_{n}^{*}$ is a double eigenvalue. In this case we choose $f(n), g(n) \in \operatorname{Ran}\left(P_{n}\right)$ so that

$$
\begin{equation*}
\|f(n)\|=\|g(n)\|=1, \quad L_{P e r^{+}}(v) f(n)=\lambda_{n}^{*} f(n), \quad\langle f(n), g(n)\rangle=0 \tag{3.4}
\end{equation*}
$$

If $n \in \mathcal{M}^{+}$, then $\lambda_{n}^{-}$and $\lambda_{n}^{+}$are simple eigenvalues. Now we choose corresponding eigenvectors $f(n), g(n) \in \operatorname{Ran}\left(P_{n}\right)$ so that

$$
\begin{equation*}
\|f(n)\|=\|g(n)\|=1, \quad L_{P e r^{+}}(v) f(n)=\lambda_{n}^{+} f(n), \quad L_{P e r^{+}}(v) g(n)=\lambda_{n}^{-} g(n) . \tag{3.5}
\end{equation*}
$$

Let $H$ be the closed linear span of the system

$$
\Phi=\left\{f(n), g(n): n \in \Gamma^{+},|n|>N\right\} .
$$

By (3.3), $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)=H \oplus \operatorname{Ran}\left(S_{N}\right)$. Since $\operatorname{dim} S_{N}<\infty$, the theorem will be proved if we show that the system $\Phi$ is a Riesz basis in the space $H$.

By (3.3), the system of two-dimensional projections $\left\{P_{n}: n \in \Gamma^{+},|n|>N\right\}$ is a Riesz basis of projections in $H$. By Lemma [2.1] the system $\Phi$ is a Riesz basis in $H$ if and only if

$$
\sup _{n \in \Gamma^{+},|n|>N}|\langle f(n), g(n)\rangle|<1
$$

By (3.4), we need to consider only indices $n \in \mathcal{M}^{+}$. Next we show that

$$
\begin{equation*}
\sup _{\mathcal{M}^{+}}|\langle f(n), g(n)\rangle|<1 . \tag{3.6}
\end{equation*}
$$

By Lemma [2.6 the quotient $\eta_{n}(z)=\beta_{n}^{-}(z) / \beta_{n}^{+}(z)$ is a well-defined analytic function on a neighborhood of the disc $K_{n}=\left\{z:\left|z-z_{n}^{*}\right| \leq \gamma_{n}\right\}$. Moreover, in view of (2.14) and (2.15), we have

$$
\begin{equation*}
\frac{1}{4 c} \leq\left|\eta_{n}(z)\right| \leq 4 c \quad \text { for } \quad n \in \mathcal{M}^{+}, z \in K_{n} \tag{3.7}
\end{equation*}
$$

Since $\eta_{n}(z)$ does not vanish in $K_{n}$, there is an appropriate branch $\log$ of $\log z$ (which depends on $n$ ) defined on a neighborhood of $\eta_{n}\left(K_{n}\right)$. We set

$$
\log \left(\eta_{n}(z)\right)=\log \left|\eta_{n}(z)\right|+i \varphi_{n}(z)
$$

then

$$
\begin{equation*}
\eta_{n}(z)=\beta_{n}^{-}(z) / \beta_{n}^{+}(z)=\left|\eta_{n}(z)\right| e^{i \varphi_{n}(z)}, \tag{3.8}
\end{equation*}
$$

so the square root $\sqrt{\beta_{n}^{-}(z) / \beta_{n}^{+}(z)}$ is a well-defined analytic function on a neighborhood of $K_{n}$ by

$$
\begin{equation*}
\sqrt{\beta_{n}^{-}(z) / \beta_{n}^{+}(z)}=\sqrt{\left|\eta_{n}(z)\right|} e^{\frac{i}{2} \varphi_{n}(z)} \tag{3.9}
\end{equation*}
$$

Now the basic equation (2.11) splits into the following two equations:

$$
\begin{align*}
& z=\zeta_{n}^{+}(z):=\alpha_{n}(z)+\beta_{n}^{+}(z) \sqrt{\beta_{n}^{-}(z) / \beta_{n}^{+}(z)}  \tag{3.10}\\
& z=\zeta_{n}^{-}(z):=\alpha_{n}(z)-\beta_{n}^{+}(z) \sqrt{\beta_{n}^{-}(z) / \beta_{n}^{+}(z)} \tag{3.11}
\end{align*}
$$

For large enough $|n|$, each of the equations (3.10) and (3.11) has exactly one root in the disc $K_{n}$. Indeed, in view of (2.10),

$$
\sup _{|z| \leq 1 / 2}\left|d \zeta_{n}^{ \pm} / d z\right| \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty
$$

Therefore, for large enough $|n|$ each of the functions $\zeta_{n}^{ \pm}$is a contraction on the disc $K_{n}$, which implies that each of the equations (3.10) and (3.11) has at most one root in the disc $K_{n}$. On the other hand, Lemma 2.2 implies that for large enough $|n|$ the basic equation (2.11) has exactly two simple roots in $K_{n}$, so each of the equations (3.10) and (3.11) has exactly one root in the disc $K_{n}$.

For large enough $|n|$, let $z_{1}(n)$ (respectively $z_{2}(n)$ ) be the only root of the equation (3.10) (respectively (3.11)) in the disc $K_{n}$. Of course, we have
either $(i) z_{1}(n)=\lambda_{n}^{-}-n, z_{2}(n)=\lambda_{n}^{+}-n \quad$ or $(i i) \quad z_{1}(n)=\lambda_{n}^{+}-n, z_{2}(n)=\lambda_{n}^{-}-n$.
Further we assume that (i) takes place; the case (ii) may be treated in the same way, and in both cases we have

$$
\begin{equation*}
\left|z_{1}(n)-z_{2}(n)\right|=\gamma_{n}=\left|\lambda_{n}^{+}-\lambda_{n}^{-}\right| \tag{3.12}
\end{equation*}
$$

We set

$$
\begin{equation*}
f^{0}(n)=P_{n}^{0} f(n), \quad g^{0}(n)=P_{n}^{0} g(n) \tag{3.13}
\end{equation*}
$$

From (3.2) it follows that $\left\|P_{n}-P_{n}^{0}\right\| \rightarrow 0$. Therefore,

$$
\left\|f(n)-f^{0}(n)\right\|=\left\|\left(P_{n}-P_{n}^{0}\right) f(n)\right\| \leq\left\|P_{n}-P_{n}^{0}\right\| \rightarrow 0, \quad\left\|g(n)-g^{0}(n)\right\| \rightarrow 0
$$

so $\left|\left\langle f(n)-f^{0}(n), g(n)-g^{0}(n)\right\rangle\right| \rightarrow 0$. Since $\|f(n)\|^{2}=\left\|f^{0}(n)\right\|^{2}+\left\|f(n)-f^{0}(n)\right\|^{2}$ and $\langle f(n), g(n)\rangle=\left\langle f^{0}(n), g^{0}(n)\right\rangle+\left\langle f(n)-f^{0}(n), g(n)-g^{0}(n)\right\rangle$, we obtain

$$
\begin{equation*}
\left\|f^{0}(n)\right\|,\left\|g^{0}(n)\right\| \rightarrow 1, \quad \limsup _{n \rightarrow \infty}|\langle f(n), g(n)\rangle|=\limsup _{n \rightarrow \infty}\left|\left\langle f^{0}(n), g^{0}(n)\right\rangle\right| . \tag{3.14}
\end{equation*}
$$

By Lemma 2.3 $f^{0}(n)$ is an eigenvector of the matrix

$$
\left(\begin{array}{ll}
\alpha_{n}\left(z_{1}\right) & \beta_{n}^{-}\left(z_{1}\right) \\
\beta_{n}^{+}\left(z_{1}\right) & \alpha_{n}\left(z_{1}\right)
\end{array}\right)
$$

corresponding to its eigenvalue $z_{1}=z_{1}(n)$, i.e.,

$$
\left(\begin{array}{cc}
\alpha_{n}\left(z_{1}\right)-z_{1} & \beta_{n}^{-}\left(z_{1}\right) \\
\beta_{n}^{+}\left(z_{1}\right) & \alpha_{n}\left(z_{1}\right)-z_{1}
\end{array}\right) f^{0}(n)=0
$$

Therefore, $f^{0}(n)$ is proportional to the vector $\left(\frac{z_{1}-\alpha_{n}\left(z_{1}\right)}{\beta_{n}^{+}\left(z_{1}\right)}, 1\right)^{T}$. Taking into account (3.8), (3.9) and (3.10) we obtain

$$
\begin{equation*}
f^{0}(n)=\frac{\left\|f^{0}(n)\right\|}{\sqrt{1+\left|\eta_{n}\left(z_{1}\right)\right|}}\binom{\sqrt{\left|\eta_{n}\left(z_{1}\right)\right|} e^{\frac{i}{2} \varphi\left(z_{1}\right)}}{1} \tag{3.15}
\end{equation*}
$$

In an analogous way, from (3.8), (3.9) and (3.11) it follows that

$$
\begin{equation*}
g^{0}(n)=\frac{\left\|g^{0}(n)\right\|}{\sqrt{1+\left|\eta_{n}\left(z_{2}\right)\right|}}\binom{-\sqrt{\left|\eta_{n}\left(z_{2}\right)\right|} e^{\frac{i}{2} \varphi\left(z_{2}\right)}}{1} . \tag{3.16}
\end{equation*}
$$

Now, (3.15) and (3.16) imply that

$$
\begin{equation*}
\left\langle f^{0}(n), g^{0}(n)\right\rangle=\left\|f^{0}(n)\right\|\left\|g^{0}(n)\right\| \frac{1-\sqrt{\left|\eta_{n}\left(z_{1}\right)\right|} \sqrt{\left|\eta_{n}\left(z_{2}\right)\right|} e^{i \psi_{n}}}{\sqrt{1+\left|\eta_{n}\left(z_{1}\right)\right|} \sqrt{1+\left|\eta_{n}\left(z_{2}\right)\right|}}, \tag{3.17}
\end{equation*}
$$

where

$$
\psi_{n}=\frac{1}{2}\left[\varphi_{n}\left(z_{1}(n)\right)-\varphi_{n}\left(z_{2}(n)\right)\right] .
$$

Next we explain that

$$
\begin{equation*}
\psi_{n} \rightarrow 0 \quad \text { as } n \rightarrow \infty \tag{3.18}
\end{equation*}
$$

Since $\varphi_{n}=\operatorname{Im}\left(\log \eta_{n}\right)$ we obtain, taking into account (3.12), that

$$
\left|\varphi_{n}\left(z_{1}(n)\right)-\varphi_{n}\left(z_{2}(n)\right)\right| \leq \sup _{\left[z_{1}, z_{2}\right]}\left|\frac{d}{d z}\left(\log \eta_{n}\right)\right| \cdot \gamma_{n}
$$

where $\left[z_{1}, z_{2}\right]$ denotes the segment with end points $z_{1}=z_{1}(n)$ and $z_{2}=z_{2}(n)$.
By (2.10) in Proposition 2.4 and (2.15) in Lemma 2.6 we estimate

$$
\frac{d}{d z}\left(\log \eta_{n}\right)=\frac{1}{\beta_{n}^{-}(z)} \frac{d \beta_{n}^{-}}{d z}(z)-\frac{1}{\beta_{n}^{+}(z)} \frac{d \beta_{n}^{+}}{d z}(z), \quad z \in\left[z_{1}, z_{2}\right]
$$

as

$$
\left|\frac{d}{d z}\left(\log \eta_{n}\right)\right| \leq \frac{\varepsilon_{n}}{\left|\beta_{n}^{-}\left(z_{n}^{*}\right)\right|}+\frac{\varepsilon_{n}}{\left|\beta_{n}^{+}\left(z_{n}^{*}\right)\right|},
$$

where $\varepsilon_{n}=C\left(\mathcal{E}_{|n|}(r)+\frac{1}{\sqrt{|n|}}\right) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, (2.14) and (2.16) imply that $\mid \varphi_{n}\left(z_{1}(n)\right)-\varphi_{n}\left(z_{2}(n) \mid \leq 4(1+c) \cdot \varepsilon_{n} \rightarrow 0\right.$; i.e., (3.18) holds.

From (3.17) it follows that

$$
\begin{equation*}
\left|\left\langle f^{0}(n), g^{0}(n)\right\rangle\right|^{2}=\left\|f^{0}(n)\right\|^{2}\left\|g^{0}(n)\right\|^{2} \cdot \Pi_{n} \tag{3.19}
\end{equation*}
$$

with

$$
\begin{equation*}
\Pi_{n}=\frac{1+\left|\eta_{n}\left(z_{1}\right)\right|\left|\eta_{n}\left(z_{2}\right)\right|-2 \sqrt{\left|\eta_{n}\left(z_{1}\right)\right|\left|\eta_{n}\left(z_{2}\right)\right|} \cos \psi_{n}}{\left(1+\left|\eta_{n}\left(z_{1}\right)\right|\right)\left(1+\left|\eta_{n}\left(z_{2}\right)\right|\right)} . \tag{3.20}
\end{equation*}
$$

Now (3.18) implies that $\cos \psi_{n}>0$ for large enough $n$, so taking into account that $\left\|f^{0}(n)\right\|,\left\|g^{0}(n)\right\| \leq 1$, we obtain by (3.7),

$$
\left|\left\langle f^{0}(n), g^{0}(n)\right\rangle\right|^{2} \leq \Pi_{n} \leq \frac{1+\left|\eta_{n}\left(z_{1}\right)\right|\left|\eta_{n}\left(z_{2}\right)\right|}{\left(1+\left|\eta_{n}\left(z_{1}\right)\right|\right)\left(1+\left|\eta_{n}\left(z_{2}\right)\right|\right)} \leq \delta<1
$$

with

$$
\delta=\sup \left\{\frac{1+x y}{(1+x)(1+y)}: \frac{1}{4 c} \leq x, y \leq 4 c\right\} .
$$

Finally, (3.14) shows that (3.6) holds, which completes the proof of (A).

Proof of (B). For every Dirac potential $v$ we set

$$
t_{n}(z)= \begin{cases}\left|\beta_{n}^{-}(z) / \beta_{n}^{+}(z)\right| & \text { if } \quad \beta_{n}^{+}(z) \neq 0  \tag{3.21}\\ \infty & \text { if } \quad \beta_{n}^{+}(z)=0, \beta_{n}^{-}(z) \neq 0 \\ 1 & \text { if } \quad \beta_{n}^{+}(z)=0, \beta_{n}^{-}(z)=0\end{cases}
$$

then $t_{n}(z),|z|<1$, is well defined for large enough $|n|$.
If $v \notin X^{+}$, then there is a subsequence of indices $\left(n_{k}\right)$ in $\mathcal{M}^{+}$such that one of the following holds:

$$
\begin{array}{r}
t_{n_{k}}\left(z_{n_{k}}^{*}\right) \rightarrow 0 \quad \text { as } k \rightarrow \infty, \\
t_{n_{k}}\left(z_{n_{k}}^{*}\right) \rightarrow \infty \quad \text { as } k \rightarrow \infty . \tag{3.23}
\end{array}
$$

Next we consider only the case (3.22) because the case (3.23) could be handled in a similar way: if $1 / t_{n_{k}}\left(z_{n_{k}}^{*}\right) \rightarrow 0$, then one may exchange the roles of $\beta_{n}^{+}$and $\beta_{n}^{-}$ and use the same argument.

In the above notation, if (3.22) holds, then there is a sequence $\left(\tau_{k}\right)$ of positive numbers such that

$$
\begin{equation*}
t_{n_{k}}(z) \leq \tau_{k} \rightarrow 0 \quad \forall z \in\left[z_{n_{k}}^{-}, z_{n_{k}}^{+}\right] \tag{3.24}
\end{equation*}
$$

where $\left[z_{n}^{-}, z_{n}^{+}\right]$denotes the segment with end points $z_{n}^{-}$and $z_{n}^{+}$.
Indeed, Lemma 2.5 and (3.22) imply that for large enough $k$,

$$
\begin{equation*}
\left|\gamma_{n_{k}}\right| \leq 2\left(\left|\beta_{n_{k}}^{-}\left(z_{n_{k}}^{*}\right)\right|+\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right|\right) \leq 4\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right| . \tag{3.25}
\end{equation*}
$$

In view of (2.10) in Proposition [2.4) for $z \in\left[z_{n}^{-}, z_{n}^{+}\right]$and $n \in \mathcal{M}^{+}$with large enough $|n|$ we have

$$
\begin{equation*}
\left|\beta_{n}^{ \pm}(z)-\beta_{n}^{ \pm}\left(z_{n}^{*}\right)\right| \leq \sup _{\left[z_{n}^{-}, z_{n}^{+}\right]}\left|\frac{\partial \beta_{n}^{ \pm}}{\partial z}(z)\right| \cdot\left|z-z_{n}^{*}\right| \leq \varepsilon_{n}\left|\gamma_{n}\right| \tag{3.26}
\end{equation*}
$$

with $\varepsilon_{n} \rightarrow 0$ as $|n| \rightarrow \infty$. Therefore, from (3.25) and (3.26) it follows that

$$
\begin{equation*}
\left|\beta_{n_{k}}^{+}(z)\right| \geq\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right|-4 \varepsilon_{n_{k}}\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right|=\left(1-4 \varepsilon_{n_{k}}\right)\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right| . \tag{3.27}
\end{equation*}
$$

On the other hand, (3.25) and (3.26) imply that

$$
\left|\beta_{n_{k}}^{-}(z)\right| \leq\left|\beta_{n_{k}}^{-}(z)-\beta_{n_{k}}^{-}\left(z_{n_{k}}^{*}\right)\right|+\left|\beta_{n_{k}}^{-}\left(z_{n_{k}}^{*}\right)\right| \leq 4 \varepsilon_{n_{k}}\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right|+\left|\beta_{n_{k}}^{-}\left(z_{n_{k}}^{*}\right)\right| .
$$

Thus, since $\varepsilon_{n_{k}} \rightarrow 0$, we obtain

$$
\frac{\left|\beta_{n_{k}}^{-}(z)\right|}{\left|\beta_{n_{k}}^{+}(z)\right|} \leq \frac{4 \varepsilon_{n_{k}}\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right|+\left|\beta_{n_{k}}^{-}\left(z_{n_{k}}^{*}\right)\right|}{\left(1-4 \varepsilon_{n_{k}}\right)\left|\beta_{n_{k}}^{+}\left(z_{n_{k}}^{*}\right)\right|}=\frac{4 \varepsilon_{n_{k}}+t_{n_{k}}\left(z_{n_{k}}^{*}\right)}{1-4 \varepsilon_{n_{k}}} \rightarrow 0 ;
$$

i.e., (3.24) holds with $\tau_{k}=\frac{4 \varepsilon_{n_{k}}+t_{n_{k}}\left(z_{n_{k}}^{*}\right)}{1-4 \varepsilon_{n_{k}}}$.

Let the vectors $f\left(n_{k}\right), g\left(n_{k}\right) \in \operatorname{Ran}\left(P_{n_{k}}\right)$ be chosen as in (3.5). Then $f\left(n_{k}\right)$ and $g\left(n_{k}\right)$ are unit eigenvectors which correspond to the simple eigenvalues $\lambda_{n_{k}}^{+}$and $\lambda_{n_{k}}^{-}$, so they are uniquely determined up to constant multipliers of absolute value one. Therefore, if the system of root functions of $L_{P e r^{+}}(v)$ contains Riesz bases, then the system $\left\{f\left(n_{k}\right), g\left(n_{k}\right): k \in \mathbb{N}\right\}$ has to be a Riesz basis in its closed linear span which coincides with the closed linear span of $\left\{\operatorname{Ran} P_{n_{k}}, k \in \mathbb{N}\right\}$. By Lemma 2.1 and (3.14), this would imply that

$$
\begin{equation*}
\sup _{k}\left\langle f\left(n_{k}\right), g\left(n_{k}\right)\right\rangle=\sup _{k}\left\langle f^{0}\left(n_{k}\right), g^{0}\left(n_{k}\right)\right\rangle<1 . \tag{3.28}
\end{equation*}
$$

Thus, the proof of (B) will be completed if we show that (3.28) fails.

By Lemma 2.3 $f^{0}\left(n_{k}\right)$ is an eigenvector of the matrix

$$
\left(\begin{array}{cc}
\alpha_{n_{k}}\left(z_{n_{k}}^{+}\right) & \beta_{n_{k}}^{-}\left(z_{n_{k}}^{+}\right) \\
\beta_{n_{k}}^{+}\left(z_{n_{k}}^{+}\right) & \alpha_{n_{k}}\left(z_{n_{k}}^{+}\right)
\end{array}\right)
$$

corresponding to its eigenvalue $z_{n_{k}}^{+}$, so it follows that $f^{0}(n)$ is proportional to the vector $\binom{a(k)}{1}$ with $a(k)=\frac{z_{n_{k}}^{+}-\alpha_{n_{k}}\left(z_{n_{k}}^{+}\right)}{\beta_{n_{k}}^{+}\left(z_{n_{k}}^{+}\right)}$. Moreover, from (2.11), (3.21) and (3.24) it follows that

$$
|a(k)|=\sqrt{t_{n_{k}}\left(z_{n_{k}}^{+}\right)} \leq \sqrt{\tau_{k}} \rightarrow 0 \quad \text { as } k \rightarrow \infty
$$

Therefore, we obtain

$$
\begin{equation*}
f^{0}\left(n_{k}\right)=\frac{\left\|f^{0}\left(n_{k}\right)\right\|}{\sqrt{|a(k)|^{2}+1}}\binom{a(k)}{1} \rightarrow\binom{0}{1} \quad \text { as } k \rightarrow \infty \tag{3.29}
\end{equation*}
$$

In the same way we obtain that $g^{0}\left(n_{k}\right) \rightarrow\binom{0}{1}$ as $k \rightarrow \infty$. Hence, $\left\langle f^{0}\left(n_{k}\right), g^{0}\left(n_{k}\right)\right\rangle \rightarrow$ 1 as $k \rightarrow \infty$, so (3.28) fails, which completes the proof of (B).

By Theorem 3.1 the condition (2.14) guarantees that there exists a Riesz basis in $L^{2}\left([0, \pi], \mathbb{C}^{2}\right)$ which consists of root functions of the operator $L_{P e r} \pm(v)$. Besides the case $v \in X_{t}$ (see the next section for a definition of the class of potentials $X_{t}$ ) it seems difficult to verify the condition (2.14). Moreover, since the points $z_{n}^{*}$ are not known in advance, in order to check (2.14) one has to compare the values of $\beta_{n}^{ \pm}(z)$ for all $z$ close to 0 . Next we give a modification of Theorem 3.1, which is more suitable for applications.

Consider potentials $v$ such that for $n \in \Gamma^{+}=2 \mathbb{Z}$ (or $n \in \Gamma^{-}=2 \mathbb{Z}+1$ ) with large enough $|n|$,

$$
\begin{equation*}
\beta_{n}^{-}(0) \neq 0, \quad \beta_{n}^{+}(0) \neq 0 \tag{3.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\exists d>0: \quad d^{-1}\left|\beta_{n}^{ \pm}(0)\right| \leq\left|\beta_{n}^{ \pm}(z)\right| \leq d\left|\beta_{n}^{ \pm}(0)\right| \quad \forall z \in D=\{z:|z|<1 / 4\} . \tag{3.31}
\end{equation*}
$$

Theorem 3.2. Suppose $b c=$ Per $^{+}$(or $b c=$ Per $^{-}$), and $v$ is a Dirac potential such that (3.30) and (3.31) hold for $n \in \Gamma^{+}$(respectively $n \in \Gamma^{-}$). Then
(a) the system of root functions of $L_{P e r^{+}}(v)$ (respectively $\left.L_{P e r^{-}}(v)\right)$ is complete and contains at most finitely many linearly independent associated functions;
(b) the system of root functions of $L_{P e r^{+}}(v)$ (respectively $\left.L_{P e r}{ }^{-}(v)\right)$ contains Riesz bases if and only if

$$
\begin{equation*}
0<\liminf _{n \in \Gamma^{+}} \frac{\left|\beta_{n}^{-}(0)\right|}{\left|\beta_{n}^{+}(0)\right|}, \quad \limsup _{n \in \Gamma^{+}} \frac{\left|\beta_{n}^{-}(0)\right|}{\left|\beta_{n}^{+}(0)\right|}<\infty \tag{3.32}
\end{equation*}
$$

(or, respectively, liminf and limsup are taken over $\Gamma^{-}$).
Remark. Although the conditions (3.30)-(3.32) look too technical there is, after [2, 3], a well-elaborated technique to evaluate these parameters and check these conditions. To compare with the case of Hill operators with trigonometric polynomial coefficients, see [5, 6.

Proof. By Proposition [2.4 for large enough $|n|$, the basic equation

$$
\begin{equation*}
\left(z-\alpha_{n}(z)\right)^{2}=\beta_{n}^{+}(z) \beta_{n}^{-}(z) \tag{3.33}
\end{equation*}
$$

has exactly two roots (counted with multiplicity) in the disc $D=\{z:|z|<1 / 4\}$. Therefore, a number $\lambda=n+z$ with $z \in D$ is a periodic or antiperiodic eigenvalue of algebraic multiplicity two if and only if $z \in D$ satisfies the system of two equations (3.33) and

$$
\begin{equation*}
2\left(z-\alpha_{n}(z)\right) \frac{d}{d z}\left(z-\alpha_{n}(z)\right)=\frac{d}{d z}\left(\beta_{n}^{+}(z) \beta_{n}^{-}(z)\right) . \tag{3.34}
\end{equation*}
$$

In view of [4. Theorem 9], the system of root functions of the operator $L_{P e r}{ }^{ \pm}(v)$ is complete, so Part (a) of the theorem will be proved if we show that there are at most finitely many $n$ such that the system (3.331), (3.34) has a solution $z \in D$.

Suppose that $z^{*} \in D$ satisfies (3.33) and (3.34). By (2.10), for each $z \in D$,

$$
\begin{equation*}
\left|\frac{d \alpha_{n}}{d z}(z)\right| \leq \varepsilon_{n}, \quad\left|\frac{d \beta_{n}^{ \pm}}{d z}(z)\right| \leq \varepsilon_{n} \quad \text { with } \varepsilon_{n} \rightarrow 0 \text { as }|n| \rightarrow \infty . \tag{3.35}
\end{equation*}
$$

In view of (3.35), the equation (3.34) implies that

$$
2\left|z^{*}-\alpha_{n}\left(z^{*}\right)\right|\left(1-\varepsilon_{n}\right) \leq \varepsilon_{n}\left(\left|\beta_{n}^{+}\left(z^{*}\right)\right|+\left|\beta_{n}^{-}\left(z^{*}\right)\right|\right)
$$

By (3.33),

$$
\left|z^{*}-\alpha_{n}\left(z^{*}\right)\right|=\left|\beta_{n}^{+}\left(z^{*}\right) \beta_{n}^{-}\left(z^{*}\right)\right|^{1 / 2}
$$

so it follows, in view of (3.31), that

$$
2\left(1-\varepsilon_{n}\right) \leq \varepsilon_{n}\left(\left|\frac{\beta_{n}^{+}\left(z^{*}\right)}{\beta_{n}^{-}\left(z^{*}\right)}\right|^{1 / 2}+\left|\frac{\beta_{n}^{-}\left(z^{*}\right)}{\beta_{n}^{+}\left(z^{*}\right)}\right|^{1 / 2}\right) \leq 2 d \varepsilon_{n}
$$

Since $\varepsilon_{n} \rightarrow 0$ as $|n| \rightarrow \infty$, the latter inequality holds for at most finitely many $n$, which completes the proof of (a).

In view of (a), all but finitely many of the eigenvalues of $L_{P e r}{ }^{ \pm}$are simple; i.e., $\lambda_{n}^{-} \neq \lambda_{n}^{+}$for large enough $|n|$. One can easily see that Conditions (3.30)-(3.32) imply (2.14), respectively for $n \in \Gamma^{+}$or $n \in \Gamma^{-}$, i.e., $v \in X^{+}$or $v \in X^{-}$. Hence (b) follows from Theorem 3.1.

Remark. For Hill-Schrödinger operators with $L^{2}$-potentials, an analog of Theorem 3.2 has been proven in [6, Theorem 1] (see also [5, Theorem 2]).

Theorem 3.1 gives a criterion for the existence of a Riesz basis consisting of root functions in the case of Dirac operators $L_{P e r} \pm(v)$ with $L^{2}$-potentials. Technically its proof is based on the same argument as in [6, Theorem 1]. Moreover, analogs of Theorems 3.1 and 3.2 hold for Hill-Schrödinger operators with $H^{-1}$-potentials and the proofs are essentially the same.

## 4. Applications

Consider the classes of Dirac potentials

$$
X_{t}=\left\{v=\left(\begin{array}{cc}
0 & P  \tag{4.1}\\
Q & 0
\end{array}\right), \quad Q(x)=t \overline{P(x)}, P, Q \in L^{2}([0, \pi])\right\}, t \in \mathbb{R} \backslash\{0\}
$$

If $t=1$ we get the class $X_{1}$ of symmetric Dirac potentials (which generate selfadjoint Dirac operators); $X_{-1}$ is the class of skew-symmetric Dirac potentials. Next we show that if $v \in X_{t}$, then the system of root functions of $L_{P e r}+(v)$ or $L_{P e r^{-}}(v)$ contains Riesz bases.

Proposition 4.1. Suppose $v \in X_{t}, t \in \mathbb{R} \backslash\{0\}$.
(a) If $t>0$, then there is a symmetric potential $\tilde{v}$ such that $L_{P e r} \pm(v)$ is similar to the self-adjoint operator $L_{P e r} \pm(\tilde{v})$, so its spectrum $S p\left(L_{P e r^{ \pm}}(v)\right) \subset \mathbb{R}$.
(b) If $t<0$, then there is a skew-symmetric potential $\tilde{v}$ such that $L_{P e r} \pm(v)$ is similar to $L_{P e r} \pm(\tilde{v})$. Moreover, there is an $N=N(v)$ such that for $|n|>N$ either
(i) $\lambda_{n}^{-}$and $\lambda_{n}^{+}$are simple eigenvalues and $\overline{\lambda_{n}^{+}}=\lambda_{n}^{-}, \operatorname{Im} \lambda_{n}^{ \pm} \neq 0$
or
(ii) $\lambda_{n}^{+}=\lambda_{n}^{-}$is a real eigenvalue of algebraic and geometric multiplicity 2 .
(c) For large enough $|n|$,

$$
\begin{equation*}
\overline{\beta_{n}^{+}\left(z_{n}^{*}, v\right)}=t \cdot \beta_{n}^{-}\left(z_{n}^{*}, v\right) \tag{4.2}
\end{equation*}
$$

which implies $X_{t} \subset X^{+} \cup X^{-}$.
(d) The system of root functions of $L_{\text {Per }}{ }^{+}(v)$ (or $\left.L_{P e r^{-}}(v)\right)$ contains Riesz bases.

Proof. For every $c \neq 0$, the Dirac operator $L_{P e r} \pm(v)$ is similar to the Dirac operator $L_{P e r} \pm\left(v_{c}\right)$ with $v_{c}=\left(\begin{array}{cc}0 & c P \\ \frac{1}{c} Q & 0\end{array}\right)$. Indeed, if $C=\left(\begin{array}{ll}c & 0 \\ 0 & 1\end{array}\right)$, then a simple calculation shows that $C L_{P e r \pm} \pm(v)=L_{P e r} \pm\left(v_{c}\right) C$.

If $v \in X_{t}$ we set $\tilde{v}=v_{c}$ with $c=\sqrt{|t|}$. Then $\frac{1}{c} Q=\frac{t}{c} \bar{P}=\frac{t}{\sqrt{|t|}} \bar{P}= \pm c \bar{P}$. Therefore, $\tilde{v}$ is symmetric or skew-symmetric, respectively, for $t>0$ and $t<0$.
(b) By (2.6), $\left(L_{P e r^{ \pm}}(v)\right)^{*}=L_{P e r^{ \pm}}\left(v^{*}\right)$ with

$$
v^{*}=\left(\begin{array}{cc}
0 & \bar{Q} \\
\bar{P} & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & t P \\
\frac{1}{t} Q & 0
\end{array}\right)=v_{t},
$$

so the operator $L_{P e r} \pm(v)$ is similar to its adjoint operator. Therefore, if $\lambda \in$ $S p\left(L_{P e r^{ \pm}}(v)\right)$, then $\bar{\lambda} \in S p\left(L_{P e r^{ \pm}}(v)\right)$ as well.

On the other hand, by Lemma [2.2, there is an $N=N(v)$ such that for $|n|>N$ the disc $D_{n}=\{z:|z-n|<1 / 4\}$ contains exactly two (counted with algebraic multiplicity) periodic (for even $n$ ) or antiperiodic (for odd $n$ ) eigenvalues of the operator $L_{P e r^{ \pm}}$. Therefore, if $\lambda \in D_{n}$ with $\operatorname{Im} \lambda \neq 0$ is an eigenvalue of $L_{P e r}{ }^{ \pm}$, then $\bar{\lambda} \in D_{n}$ is also an eigenvalue of $L_{P e r^{ \pm}}$and $\bar{\lambda} \neq \lambda$, so $\lambda$ and $\bar{\lambda}$ are simple; i.e., (i) holds.

Suppose $\lambda \in D_{n}$ is a real eigenvalue. If $\binom{w_{1}}{w_{2}}$ is a corresponding eigenvector, then passing to conjugates we obtain $L\binom{\overline{w_{2}}}{-\overline{w_{1}}}=\lambda L\binom{\overline{w_{2}}}{-\overline{w_{1}}}$; i.e., $\binom{\overline{w_{2}}}{-\overline{w_{1}}}$ is also an eigenvector corresponding to the eigenvalue $\lambda$. $\operatorname{But}\left\langle\binom{ w_{1}}{w_{2}},\binom{\overline{w_{2}}}{-\overline{w_{1}}}\right\rangle=0$, so these vectors are linearly independent. Hence (ii) holds.
(c) By (i) and (ii) it follows that

$$
z_{n}^{*}=\frac{1}{2}\left(\lambda_{n}^{-}+\lambda_{n}^{+}\right)-n \quad \text { is real for } \quad|n|>N .
$$

In view of (2.8), this implies that (4.2) holds.
(d) In view of (4.2), we have $v \in X$, so the claim follows from Theorem 3.1,

Example 4.2. If $a, b, A, B$ are nonzero complex numbers and

$$
v=\left(\begin{array}{ll}
0 & P  \tag{4.3}\\
Q & 0
\end{array}\right) \quad \text { with } \quad P(x)=a e^{2 i x}+b e^{-2 i x}, \quad Q(x)=A e^{2 i x}+B e^{-2 i x}
$$

then the system of root functions of $L_{\text {Per }}+(v)\left(\right.$ or $\left.L_{P e r}-(v)\right)$ contains at most finitely many linearly independent associated functions. Moreover, the system of root functions of $L_{P e r^{+}}(v)$ contains Riesz bases always, while the system of root functions of $L_{\text {Per }}(v)$ contains Riesz bases if and only if $|a A|=|b B|$.

Let us mention that if $b c=\operatorname{Per}^{+}$, then it is easy to see by (2.8) that $\beta_{n}^{ \pm}(z)=0$ whenever defined, so the claim follows from Theorem 3.1.

If $b c=\mathrm{Per}^{-}$, then the result follows from Theorem [3.2 and the asymptotics

$$
\begin{align*}
& \beta_{n}^{+}(0)=A^{\frac{n+1}{2}} a^{\frac{n-1}{2}} 4^{-n+1}\left[\left(\frac{n-1}{2}\right)!\right]^{-2}(1+O(1 / \sqrt{|n|}),  \tag{4.4}\\
& \beta_{n}^{-}(0)=b^{\frac{n+1}{2}} B^{\frac{n-1}{2}} 4^{-n+1}\left[\left(\frac{n-1}{2}\right)!\right]^{-2}(1+O(1 / \sqrt{|n|}) . \tag{4.5}
\end{align*}
$$

Proofs of (4.4), (4.5) and similar asymptotics related to other trigonometric polynomial potentials and implying Riesz basis existence or nonexistence will be given elsewhere (see similar results for the Hill-Schrödinger operator in [5, 6]).

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Faculty of Engineering and Natural Sciences, Sabanci University, Orhanli, 34956 Tuzla, Istanbul, Turkey

E-mail address: djakov@sabanciuniv.edu
Department of Mathematics, The Ohio State University, 231 West 18 th Avenue, Columbus, Ohio 43210

E-mail address: mityagin.1@osu.edu


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