PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 4, April 2013, Pages 1313–1322
S 0002-9939(2012)11640-5
Article electronically published on August 23, 2012

A SIMPLE PROOF OF L^q-ESTIMATES FOR THE STEADY-STATE OSEEN AND STOKES EQUATIONS IN A ROTATING FRAME. PART II: WEAK SOLUTIONS

GIOVANNI P. GALDI AND MADS KYED

(Communicated by Walter Craig)

ABSTRACT. This is the second of two papers in which simple proofs of L^q-estimates of solutions to the steady-state three-dimensional Oseen and Stokes equations in a rotating frame of reference are given. In this part, estimates are established in terms of data in homogeneous Sobolev spaces of negative order.

1. INTRODUCTION

As in [GK11a], we study the system

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$ are dimensionless constants. Here, $v : \mathbb{R}^3 \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \to \mathbb{R}$ represent Eulerian velocity and pressure fields, respectively, of a Navier-Stokes liquid in a frame of reference rotating with angular velocity $\mathcal{T}\,\mathrm{e}_3$ relative to some inertial frame. The above system is the classical steady-state whole space Oseen ($\mathcal{R} > 0$) or Stokes ($\mathcal{R} = 0$) problem with the extra term $\mathcal{T}\big(\mathrm{e}_3 \wedge x\cdot\nabla v - \mathrm{e}_3 \wedge v\big)$, which stems from the rotating frame of reference. Due to the unbounded coefficient $\mathrm{e}_3 \wedge x$, this term cannot be treated as a perturbation to the Oseen or Stokes operator.

In [GK11a] we gave an elementary proof of L^q-estimates of solutions (v, p) to (1.1) in terms of data $f \in L^q(\mathbb{R}^3)^3$, $1 < q < \infty$. Such estimates had already been shown in [FHM04] and [Far06], but with very technical and non-trivial proofs based on an appropriate coupling of the Littlewood-Payley decomposition theorem and multiplier theory. In [His06], [KNP08], and [KNP10] an approach similar to the one in [FHM04] and [Far06] was used to prove L^q-estimates of weak solutions to (1.1) in terms of data f in the homogeneous Sobolev space $D_0^{-1,q}(\mathbb{R}^3)^3$ of negative order. Our aim in this paper is to extend our approach from [GK11a] and give an elementary proof of these estimates of weak solutions.

Our main theorem reads:

Theorem 1.1. *Let $1 < q < \infty$, $\mathcal{R}_0 > 0$, $0 \leq \mathcal{R} < \mathcal{R}_0$, and $\mathcal{T} > 0$. For any $f \in D_0^{-1,q}(\mathbb{R}^3)^3$ there exists a solution $(v, p) \in D^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ to (1.1) that*

Received by the editors August 9, 2011.
2010 *Mathematics Subject Classification.* Primary 35Q30, 35B45, 76D07.
The first author was partially supported by NSF grant DMS-1062381.
The second author was supported by the DFG and JSPS as a member of the International Research Training Group Darmstadt-Tokyo IRTG 1529.

satisfies

$$\|\nabla v\|_q + \|p\|_q \leq C_1 |f|_{-1,q}, \tag{1.2}$$

with C_1 independent of $\mathcal{R}_0$, $\mathcal{R}$, and $\mathcal{T}$. Moreover,

$$|\mathcal{R}\partial_3 v|_{-1,q} + |\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)|_{-1,q} \leq C_2 \bigg(1 + \frac{1}{\mathcal{T}^2}\bigg) |f|_{-1,q}, \tag{1.3}$$

with $C_2 = C_2(\mathcal{R}_0)$. Furthermore, if $(\tilde{v}, \tilde{p}) \in D^{1,r}(\mathbb{R}^3)^3 \times L^r(\mathbb{R}^3)$, $1 < r < \infty$, is another solution to (1.1)*, then*

$$\tilde{v} = v + \alpha\, \mathrm{e}_3 \tag{1.4}$$

for some $\alpha \in \mathbb{R}$.

Remark 1.2. In [KNP10, Theorem 2.1 and Proposition 3.2] it is stated that a solution $(v,p) \in D^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ to (1.1) with $f \in D_0^{-1,q}(\mathbb{R}^3)^3$ satisfies

$$|\mathcal{R}\partial_3 v|_{-1,q} + |\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)|_{-1,q} \leq C_3 |f|_{-1,q}$$

with C_3 *independent* of $\mathcal{T}$. However, going more carefully through the relevant proofs of [KNP10], in particular those in Appendix 2, one finds that the constant C_3 does, in fact, *depend* on $\mathcal{T}$ exactly in the way shown in (1.3).

Before giving a proof of Theorem 1.1, we first recall some standard notation. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$. For $m \in \mathbb{N}$ and $1 < q < \infty$ we use $D^{m,q}(\mathbb{R}^3)$ to denote the homogeneous Sobolev space with seminorm $|\cdot|_{m,q}$, *i.e.*,

$$|v|_{m,q} := \bigg(\sum_{|\alpha|=m} \int_{\mathbb{R}^3} |\partial^\alpha v(x)|^q \,\mathrm{d}x \bigg)^{\frac{1}{q}}, \; D^{m,q} := \{v \in L^1_{loc}(\mathbb{R}^3) \mid |v|_{m,q} < \infty\}.$$

We put $D_0^{m,q}(\mathbb{R}^3) := \overline{C_0^\infty(\mathbb{R}^3)}^{|\cdot|_{m,q}}$. We introduce homogeneous Sobolev spaces of negative order as the dual spaces $D_0^{-m,q}(\mathbb{R}^3) := \big(D_0^{m,q'}(\mathbb{R}^3)\big)'$ and denote their norms by $|\cdot|_{-m,q}$. Here, and throughout the paper, $q' := q/(q-1)$ denotes the Hölder conjugate of q. For functions $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we let $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$, etc.; that is, unless otherwise indicated, differential operators act in the spatial variable x only. We use $\mathcal{F}f = \widehat{f}$ to denote the Fourier transformation. We put $\mathrm{B}_m := \{x \in \mathbb{R}^3 \mid |x| < m\}$. Finally, note that constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

2. Proof of main theorem

As in [GK11a] we make use of an idea going back to [Gal03] and transform solutions to (1.1) into time-periodic solutions to the classical time-dependent Oseen and Stokes problem. For this purpose, we introduce the rotation matrix corresponding to the angular velocity $\mathcal{T}\,\mathrm{e}_3$:

$$Q(t) := \begin{pmatrix} \cos(\mathcal{T}t) & -\sin(\mathcal{T}t) & 0 \\ \sin(\mathcal{T}t) & \cos(\mathcal{T}t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We split the proof into several lemmas. We begin by recalling the following result; see [Gal02] or [Sil04].

Lemma 2.1. *Let $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$. For any $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$ there is a solution*

$$(v, p) \in D^{1,2}(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \tag{2.1}$$

to

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = \operatorname{div} h & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{2.2}$$

that satisfies

$$\|\nabla v\|_2 + \|p\|_2 \leq C_4 \|h\|_2, \tag{2.3}$$

with C_4 independent of $\mathcal{R}$ and $\mathcal{T}$. Moreover

$$(v, p) \in \bigcap_{m=1}^{\infty} D^{m+1,2}(\mathbb{R}^3)^3 \times D^{m,2}(\mathbb{R}^3). \tag{2.4}$$

In the next lemma we establish suitable L^q-estimates of the solution introduced above.

Lemma 2.2. *Let $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$. Let $1 < q < \infty$ and $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. The solution (v, p) from Lemma 2.1 satisfies*

$$\|\nabla v\|_q + \|p\|_q \leq C_5 \|h\|_q, \tag{2.5}$$

with C_5 independent of $\mathcal{R}$ and $\mathcal{T}$.

Proof. Assume first that $q > 2$. Let $T > 0$. For $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ put

$$\begin{aligned} &u(x,t) := Q(t) v\big(Q(t)^\top x - \mathcal{R} t\, \mathrm{e}_3\big), \quad \mathfrak{p}(x,t) := p\big(Q(t)^\top x - \mathcal{R} t\, \mathrm{e}_3\big), \\ &H(x,t) := Q(t) h\big(Q(t)^\top x - \mathcal{R} t\, \mathrm{e}_3\big) Q(t)^\top. \end{aligned}$$

Then

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} = \operatorname{div} H & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = v(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.6}$$

By using classical multiplier theory like, for example, in [Lad69, Chap. 4, Sec. 5, Theorem 6], it is straightforward to show that the Cauchy problem

$$\begin{cases} \partial_t u_1 - \Delta u_1 = \operatorname{div} H - \nabla \mathfrak{p} & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_1 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim\limits_{t\to 0^+} \|u_1(\cdot, t)\|_6 = 0 \end{cases}$$

has a solution with $u_1 \in L^r\big(\mathbb{R}^3 \times (0,T)\big)^3$ for all $1 < r < \infty$, and

$$\|\nabla u_1\|_{L^r(\mathbb{R}^3 \times (0,T))} \leq c_1 \|H\|_{L^r(\mathbb{R}^3 \times (0,T))},$$

with c_1 independent of T. Put

$$u_2(x,t) := (4\pi t)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y|^2/4t}\, v(y)\, \mathrm{d}y. \tag{2.7}$$

An elementary calculation shows that $u_2 \in L^6\big(\mathbb{R}^3 \times (0,T)\big)$, $\partial_t u_2, \nabla u_2, \nabla^2 u_2 \in L^6_{loc}\big(\mathbb{R}^3 \times (0,T)\big)$, and that u_2 solves

$$\begin{cases} \partial_t u_2 - \Delta u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim_{t\to 0^+} \|u_2(\cdot,t) - v(\cdot)\|_6 = 0. \end{cases}$$

Taking derivatives on both sides in (2.7) and applying Young's inequality, we obtain

$$\|\nabla u_2(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq c_2\, t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\, \|\nabla v\|_2,$$

with c_2 independent of T. We claim that $u = u_1 + u_2$ in $\mathbb{R}^3 \times (0,T)$. This follows from the fact that $u_1 + u_2$ satisfies (2.6) combined with a uniqueness argument, for example [GK11b, Lemma 3.6]. Recalling that $q > 2$ by assumption, we can now estimate

$$\begin{aligned} (T-1)\|\nabla v\|_q^q &= \int_1^T \int_{\mathbb{R}^3} |\nabla u(x,t)|^q \,\mathrm{d}x\mathrm{d}t \\ &\leq c_3\bigg(\|\nabla u_1\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_1^T \|\nabla u_2(\cdot,t)\|_q^q \,\mathrm{d}t\bigg) \\ &\leq c_4\bigg(\|H\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_1^T t^{-\frac{3q}{2}(\frac{1}{2}-\frac{1}{q})}\|\nabla v\|_2^q \,\mathrm{d}t\bigg) \\ &\leq c_5\big(T\|h\|_q^q + T^{1-\varepsilon}\|\nabla v\|_2^q\big), \end{aligned}$$

for some $\varepsilon \in (0,1)$ and c_5 independent of T, $\mathcal{R}$, and $\mathcal{T}$. Dividing both sides by T, and subsequently letting $T \to \infty$, we conclude that $\|\nabla v\|_q^q \leq c_5\|h\|_q^q$. Finally, we deduce directly from (2.2), applying div on both sides in $(1.1)_1$, that $\Delta p = \operatorname{div}\operatorname{div} h$, which implies that $\|p\|_q \leq c_6\|h\|_q$, with c_6 independent of $\mathcal{R}$ and $\mathcal{T}$. Hence (2.5) follows in the case $q > 2$.

The case $q = 2$ is included in Lemma 2.1. Now consider $1 < q < 2$. In this case we will establish (2.5) by a duality argument. Consider for this purpose $\varphi \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. For notational purposes, we put

$$Lv := -\Delta v - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big), \tag{2.8}$$

$$L^* v := -\Delta v + \mathcal{R}\partial_3 v + \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big). \tag{2.9}$$

As in Lemma 2.1, one can show the existence of a solution (ψ, η), in the class (2.1) and (2.4), to the adjoint problem

$$\begin{cases} L^*\psi + \nabla\eta = \operatorname{div}\varphi & \text{in } \mathbb{R}^3, \\ \operatorname{div}\psi = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.10}$$

By arguments as above, one can also show that

$$\forall r \in (2,\infty): \ \|\nabla\psi\|_r + \|\eta\|_r \leq c_7\|\varphi\|_r, \tag{2.11}$$

with c_7 independent of $\mathcal{R}$ and $\mathcal{T}$. Using the same approximation technique as in [GK11a, Proof of Lemma 2.3], we compute

$$\begin{aligned}
|\int_{\mathbb{R}^3} \nabla v : \varphi \,\mathrm{d}x| &= |\int_{\mathbb{R}^3} v \cdot \operatorname{div}\varphi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} v \cdot L^*\psi \,\mathrm{d}x| \\
&= |\int_{\mathbb{R}^3} Lv \cdot \psi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} \operatorname{div} h \cdot \psi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} h : \nabla\psi \,\mathrm{d}x| \\
&\le \|h\|_q \|\nabla\psi\|_{q'} \le c_7 \|h\|_q \|\varphi\|_{q'},
\end{aligned} \tag{2.12}$$

where the last estimate follows from (2.11) since $2 < q' < \infty$. Having established (2.12) for arbitrary φ, we conclude that $\|\nabla v\|_q \le c_7 \|h\|_q$. Finally, the estimate $\|p\|_q \le c_8 \|h\|_q$ follows simply from the fact that $\Delta p = \operatorname{div}\operatorname{div} h$. We have thus established (2.5) also in the case $1 < q \le 2$. This concludes the lemma. □

In the next lemma we establish estimates of the lower-order terms on the left-hand side of (1.1).

Lemma 2.3. *Let $\mathcal{R} > 0$ and $\mathcal{T} > 0$. Let $1 < q < \infty$ and $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. The solution (v, p) from Lemma 2.1 satisfies*

$$|\mathcal{R}\partial_3 v|_{-1,q} + |\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)|_{-1,q} \le C_6 \left(1 + \frac{1}{\mathcal{T}^2}\right) \|h\|_q, \tag{2.13}$$

with $C_6 = C_6(\mathcal{R}_0)$.

Proof. Consider first $1 < q \le 2$. For $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ put

$$\begin{aligned}
&u(x,t) := Q(t) v\big(Q(t)^\top x\big), \quad \mathfrak{p}(x,t) := p\big(Q(t)^\top x\big), \\
&H(x,t) := Q(t) h\big(Q(t)^\top x\big) Q(t)^\top.
\end{aligned}$$

Note that u, $\mathfrak{p}$, and H are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in the t variable. We can therefore expand these fields in their Fourier series. More precisely, we have

$$\begin{aligned}
&u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \quad \mathfrak{p}(x,t) = \sum_{k\in\mathbb{Z}} \mathfrak{p}_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \\
&H(x,t) = \sum_{k\in\mathbb{Z}} H_k(x)\,\mathrm{e}^{i\mathcal{T}kt},
\end{aligned}$$

with

$$\begin{aligned}
&u_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} u(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t, \quad \mathfrak{p}_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \mathfrak{p}(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t, \\
&H_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} H(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t.
\end{aligned}$$

As one may easily verify,

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \mathcal{R}\partial_3 u = \operatorname{div} H & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}. \end{cases} \tag{2.14}$$

Replacing in (2.14) u, $\mathfrak{p}$, and H with their respective Fourier series, we find that each Fourier coefficient satisfies

$$\begin{cases} i\mathcal{T}k u_k - \Delta u_k + \nabla \mathfrak{p}_k - \mathcal{R}\partial_3 u_k = \operatorname{div} H_k & \text{in } \mathbb{R}^3, \\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.15}$$

In the case $k = 0$, (2.15) reduces to the classical Oseen system. By well-known theories (see for example [Gal94, Theorem VII.4.2]),

$$\|\nabla u_0\|_q + \mathcal{R}|\partial_3 u_0|_{-1,q} \le c_1 \|H_0\|_q \le c_2 \|h\|_q, \tag{2.16}$$

with c_2 independent of $\mathcal{R}$ and $\mathcal{T}$. Now consider $k \neq 0$. By Minkowski's integral inequality and Lemma 2.2, we find that

$$\|\nabla u_k\|_q \le \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \left(\int_{\mathbb{R}^3} |\nabla u(x,t)|^q \, \mathrm{d}x \right)^{1/q} \mathrm{d}t = \|\nabla v\|_q \le C_5 \|h\|_q,$$

and similarly $\|\mathfrak{p}_k\|_q \le C_5 \|h\|_q$. We can thus conclude from (2.15) that

$$|\mathcal{T}k| |u_k|_{-1,q} \le \|\nabla u_k\|_q + \|\mathfrak{p}_k\|_q + \mathcal{R}|\partial_3 u_k|_{-1,q} \le c_3 \|h\|_q + \mathcal{R}|\partial_3 u_k|_{-1,q}, \tag{2.17}$$

with c_3 independent of $\mathcal{R}$ and $\mathcal{T}$.[1] A simple interpolation argument[2] yields

$$|\partial_3 u_k|_{-1,q} \le c_4 (\varepsilon |u_k|_{-1,q} + \varepsilon^{-1} \|\nabla u_k\|_q) \tag{2.18}$$

for all $\varepsilon > 0$. We now choose $\varepsilon = |\mathcal{T}k|/(2\mathcal{R}c_4)$ in (2.18) and apply the resulting estimate in (2.17). It follows that

$$|u_k|_{-1,q} \le c_5 \frac{1}{|\mathcal{T}k|} \left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}k|} \right) \|h\|_q \quad (k \neq 0), \tag{2.19}$$

with c_5 independent of $\mathcal{R}$ and $\mathcal{T}$. We observe at this point that $v(x) = u(x,0) = \sum_{k\in\mathbb{Z}} u_k(x)$ and put

$$v_1 := v - u_0. \tag{2.20}$$

We then define

$$U(x,t) := Q(t) v_1 \big(Q(t)^\top x \big) = u(x,t) - u_0 = \sum_{k \neq 0} u_k(x) \, \mathrm{e}^{i\mathcal{T}kt}.$$

The first equality above follows from the fact that $Q(t) u_0 (Q(t)^\top x) = u_0(x)$ for all $t \in \mathbb{R}$, which one easily verifies directly from the definition of u_0. Now let $\varphi \in C_0^\infty(\mathbb{R}^3)^3$ and put $\varphi(x,t) := Q(t)\varphi(Q(t)^\top x)$. Since φ is smooth and $2\pi/\mathcal{T}$-periodic in t, we can write φ in terms of its Fourier series:

$$\Phi(x,t) = \sum_{k\in\mathbb{Z}} \Phi_k(x) \, \mathrm{e}^{i\mathcal{T}kt}, \quad \Phi_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \Phi(x,t) \, \mathrm{e}^{-i\mathcal{T}kt} \, \mathrm{d}t.$$

---

[1]Since u_k solves the resolvent-like system (2.15), known theory implies that $|\partial_3 u_k|_{-1,q}$ is finite. One can also show this directly by applying ∂_3 to both sides of (2.15), which shows that $\partial_3 u_k$ satisfies the same system. Repeating the preceding arguments of the proof with $(\partial_3 u_k, \partial_3 \mathfrak{p}_k)$ in the role of $(u_k, \mathfrak{p}_k)$, and likewise substituting $(\partial_3 u, \partial_3 \mathfrak{p})$ for $(u, \mathfrak{p})$ and $(\partial_3 v, \partial_3 p)$ for (v, p), it follows that $\nabla^2 \partial_3 u_k, \nabla \partial_3 \mathfrak{p}_k \in D_0^{-1,q}(\mathbb{R}^3)$. Returning to (2.15), one then finds $\partial_3 u_k \in D_0^{-1,q}(\mathbb{R}^3)$.

[2]In fact, the inequality is an obvious consequence of the following one:

$$\|u\|_{q,\mathbb{R}^3}^2 \le c \, |u|_{-1,q,\mathbb{R}^3} \, |u|_{1,q,\mathbb{R}^3}, \tag{*}$$

which, by the argument of [Gal94, Lemma VII.4.3], is enough to prove for $u \in C_0^\infty(\mathbb{R}^3)$. By the Calderón-Zygmund theorem, it is easy to see that the function $\psi = \nabla(\mathcal{E} * u)$, with $\mathcal{E}$ a fundamental solution to Laplace's equation, satisfies $\operatorname{div} \psi = u$, $\|\nabla^2 \psi\|_q \le c|u|_{1,q}$, $\|\psi\|_q \le c|u|_{-1,q}$, so that (*) follows from the classical Nirenberg's inequality $\|\operatorname{div} \psi\|_q^2 \le c\|\psi\|_q \|\nabla^2 \psi\|_q$.

We now compute, using Parseval's identity and (2.19),

$$\begin{aligned}
|\int_{\mathbb{R}^3} v_1(x)\cdot\varphi(x)\,\mathrm{d}x| &= |\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\int_{\mathbb{R}^3} U(x,t)\cdot\Phi(x,t)\,\mathrm{d}x\mathrm{d}t| \\
&= |\int_{\mathbb{R}^3}\sum_{k\neq 0} u_k(x)\cdot\Phi_k(x)\,\mathrm{d}x| \\
&\leq \sum_{k\neq 0} |u_k|_{-1,q}\|\nabla\Phi_k\|_{q'} \\
&\leq c_5\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\|h\|_q\sum_{k\neq 0}\frac{1}{|\mathcal{T}k|}\|\nabla\Phi_k\|_{q'} \\
&\leq c_5\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\frac{1}{\mathcal{T}}\|h\|_q\Big(\sum_{k\neq 0}\frac{1}{|k|^q}\Big)^{\frac{1}{q}}\Big(\sum_{k\neq 0}\|\nabla\Phi_k\|_{q'}^{q'}\Big)^{\frac{1}{q'}}.
\end{aligned}$$

Recalling that $1<q\leq 2$, we employ the Hausdorff-Young inequality to estimate

$$\Big(\sum_{k\neq 0}\|\nabla\Phi_k\|_{q'}^{q'}\Big)^{\frac{1}{q'}} \leq \Big(\int_{\mathbb{R}^3}\Big[\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}|\nabla\Phi(x,t)|^q\,\mathrm{d}t\Big]^{\frac{q'}{q}}\mathrm{d}x\Big)^{\frac{1}{q'}}.$$

Applying Minkowski's integral inequality to the right-hand side above, we obtain

$$\Big(\sum_{k\neq 0}\|\nabla\Phi_k\|_{q'}^{q'}\Big)^{\frac{1}{q'}} \leq \Big(\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\Big[\int_{\mathbb{R}^3}|\nabla\Phi(x,t)|^{q'}\,\mathrm{d}x\Big]^{\frac{q}{q'}}\mathrm{d}t\Big)^{\frac{1}{q}} = \|\nabla\varphi\|_{q'}.$$

We thus conclude that

$$|\int_{\mathbb{R}^3} v_1(x)\cdot\varphi(x)\,\mathrm{d}x| \leq c_6\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\frac{1}{\mathcal{T}}\|h\|_q\|\nabla\varphi\|_{q'},$$

and consequently, since φ is arbitrary,

$$|v_1|_{-1,q} \leq c_7\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\frac{1}{\mathcal{T}}\|h\|_q, \tag{2.21}$$

with c_7 independent of $\mathcal{R}$ and $\mathcal{T}$. By the same interpolation argument as in (2.18), we estimate

$$|\partial_3 v_1|_{-1,q} \leq c_8(|v_1|_{-1,q}+\|\nabla v_1\|_q). \tag{2.22}$$

Now combining (2.5), (2.16), (2.20), (2.21), and (2.22), we obtain

$$\forall q\in(1,2]:\ |\mathcal{R}\partial_3 v|_{-1,q} \leq c_9\Big(1+\frac{1}{\mathcal{T}^2}\Big)\|h\|_q, \tag{2.23}$$

with $c_9 = c_9(\mathcal{R}_0)$.

Now consider $2<q<\infty$. Let $\varphi\in C_0^\infty(\mathbb{R}^3)^3$. Recall (2.8) and (2.9). By [GK11a, Lemma 2.1] there is a solution $(\psi,\eta)\in D^{1,2}(\mathbb{R}^3)^3\cap L^6(\mathbb{R}^3)^3\times L^6(\mathbb{R}^3)$ to

$$\begin{cases} L^*\psi+\nabla\eta=\varphi & \text{in } \mathbb{R}^3, \\ \operatorname{div}\psi = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{2.24}$$

satisfying (2.4). Moreover, since Δ commutes with L^*, $(\Delta\psi,\Delta\eta)$ satisfies

$$\begin{cases} L^*\Delta\psi+\nabla\Delta\eta=\operatorname{div}\nabla\varphi & \text{in } \mathbb{R}^3, \\ \operatorname{div}\Delta\psi = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.25}$$

Repeating the argument from above leading to (2.23), we also obtain

$$\forall r \in (1,2]: \ |\mathcal{R}\partial_3\Delta\psi|_{-1,r} \le c_{10}\Big(1+\frac{1}{\mathcal{T}^2}\Big)\|\nabla\varphi\|_r, \tag{2.26}$$

with $c_{10}=c_{10}(\mathcal{R}_0)$. As in (2.12), we compute

$$\int_{\mathbb{R}^3}\partial_3 v\cdot\varphi\,\mathrm{d}x=\int_{\mathbb{R}^3}\partial_3 v\cdot L^*\psi\,\mathrm{d}x=-\int_{\mathbb{R}^3}Lv\cdot\partial_3\psi\,\mathrm{d}x=-\int_{\mathbb{R}^3}\operatorname{div}h\cdot\partial_3\psi\,\mathrm{d}x.$$

Put[3] $\Theta_i := \mathcal{F}^{-1}\big[\frac{\xi_j}{|\xi|^2}\widehat{h_{ij}}(\xi)\big]$, $i=1,2,3$. Then $\Theta\in L^r(\mathbb{R}^3)^3$ for all $r\in(3/2,\infty)$, $\|\nabla\Theta\|_q\le c_{11}\|h\|_q$, and $\Delta\Theta=\operatorname{div}h$. It follows that

$$|\int_{\mathbb{R}^3}\partial_3 v\cdot\varphi\,\mathrm{d}x|=|\int_{\mathbb{R}^3}\Theta\cdot\partial_3\Delta\psi\,\mathrm{d}x|\le\|\nabla\Theta\|_q|\partial_3\Delta\psi|_{-1,q'}\le c_{12}\|h\|_q|\partial_3\Delta\psi|_{-1,q'}.$$

Since $q'\in(1,2)$, we deduce by (2.26) that

$$|\int_{\mathbb{R}^3}\partial_3 v\cdot\varphi\,\mathrm{d}x|\le c_{13}\Big(1+\frac{1}{\mathcal{T}^2}\Big)\|h\|_q\|\nabla\varphi\|_{q'}.$$

We conclude $|\mathcal{R}\partial_3 v|_{-1,q}\le c_{14}(1+\mathcal{T}^{-2})\|h\|_q$, with $c_{14}=c_{14}(\mathcal{R}_0)$.

Since $\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v-\mathrm{e}_3\wedge v\big)=\Delta v-\nabla p+\mathcal{R}\partial_3 v+\operatorname{div}h$, the estimates already obtained in (2.5) together with the estimate for $\mathcal{R}\partial_3 v$ above imply that

$$|\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v-\mathrm{e}_3\wedge v\big)|_{-1,q}\le c_{15}\Big(1+\frac{1}{\mathcal{T}^2}\Big)\|h\|_q,$$

with $c_{15}=c_{15}(\mathcal{R}_0)$. We have thus established (2.13) completely. □

We can now finalize the proof of the main theorem.

Proof of Theorem 1.1. Except for the uniqueness statement, Lemmas 2.1–2.3 establish the theorem in the case $f=\operatorname{div}h$ for some $h\in C_0^\infty(\mathbb{R}^3)^{3\times3}$. It remains to extend to the general case $f\in D_0^{-1,q}(\mathbb{R}^3)^3$. Consider therefore $f\in D_0^{-1,q}(\mathbb{R}^3)^3$. Choose a sequence $\{h_n\}_{n=1}^\infty\subset C_0^\infty(\mathbb{R}^3)^{3\times3}$ with $\lim_{n\to\infty}\operatorname{div}h_n=f$ in $D_0^{-1,q}(\mathbb{R}^3)^3$. Let (v_n,p_n) be the solution from Lemma 2.1 corresponding to the right-hand side $\operatorname{div}h_n$. Then choose $\kappa_n\in\mathbb{R}^3$ such that $0=\int_{\mathrm{B}_1}v_n-\kappa_n\,\mathrm{d}x$. From Lemma 2.2 and Poincaré's inequality, it follows that $\{(v_n-\kappa_n,p_n)\}_{n=1}^\infty$ is a Cauchy sequence in the Banach space

$$X_m:=\{(v,p)\in L^1_{loc}(\mathbb{R}^3)^3\times L^1_{loc}(\mathbb{R}^3)\mid\|(v,p)\|_{X_m}<\infty\},$$
$$\|(v,p)\|_{X_m}:=\|\nabla v\|_q+\|p\|_q+\|v\|_{L^q(\mathrm{B}_m)}$$

for all $m\in\mathbb{N}$. Consequently, there is an element $(v,p)\in\bigcap_{m\in\mathbb{N}}X_m$ with the property that $\lim_{n\to\infty}(v_n-\kappa_n,p_n)=(v,p)$ in X_m for all $m\in\mathbb{N}$. Recall (2.8). It follows that $\lim_{n\to\infty}[L(v_n-\kappa_n)+\nabla p_n]=Lv+\nabla p$ in $\mathcal{D}'(\mathbb{R}^3)^3$. By construction, $\lim_{n\to\infty}[Lv_n+\nabla p_n]=f$ in $D_0^{-1,q}(\mathbb{R}^3)^3$. We thus deduce that $\lim_{n\to\infty}L\kappa_n=f-[Lv+\nabla p]$. Consequently, $f-[Lv+\nabla p]=L\kappa$ for some $\kappa\in\mathbb{R}^3$. It follows

[3]Following the summation convention, we implicitly sum over repeated indices.

that $(v+\kappa,p)\in D^{1,q}(\mathbb{R}^3)^3\times L^q(\mathbb{R}^3)$ solves (1.1). Moreover, since (v_n,p_n) satisfies (1.2) and (1.3) for all $n\in\mathbb{N}$, so does $(v+\kappa,p)$. This concludes the first part of the theorem.

To prove the statement of uniqueness, assume that $(\tilde{v},\tilde{p})\in D^{1,r}(\mathbb{R}^3)^3\times L^r(\mathbb{R}^3)$ is another solution to (1.1). Put $w:=v-\tilde{v}$ and $\mathfrak{q}:=p-\tilde{p}$. It immediately follows that $\Delta\mathfrak{q}=0$, which, since $\mathfrak{q}\in L^q(\mathbb{R}^3)+L^r(\mathbb{R}^3)$, implies that $\mathfrak{q}=0$. Now put $U(x,t):=Q(t)w(Q(t)^\top x)$ for $(x,t)\in\mathbb{R}^3\times\mathbb{R}$. Since U is smooth and $2\pi/\mathcal{T}$-periodic in t, we can write U in terms of its Fourier series

$$U(x,t)=\sum_{k\in\mathbb{Z}}U_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\ U_k(x):=\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}U(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t.$$

As one may easily verify, U_k satisfies $i\mathcal{T}kU_k-\Delta U_k-\mathcal{R}\partial_3U_k=0$ in $\mathscr{S}'(\mathbb{R}^3)^3$. Thus, a Fourier transformation yields $\big(i(\mathcal{T}k-\mathcal{R}\xi_3)+|\xi|^2\big)\widehat{U_k}=0$. It follows that $U_k=0$ for all $k\neq0$. Moreover, since $\big(-i\mathcal{R}\xi_3+|\xi|^2\big)\widehat{U_0}=0$, it follows that $\mathrm{supp}(\widehat{U_0})\subset\{0\}$. Consequently, since $U_0\in D^{1,q}(\mathbb{R}^3)^3+D^{1,r}(\mathbb{R}^3)^3$, $U_0=b$ for some $b\in\mathbb{R}^3$. It follows that $U(x,t)=b=Q(t)w(Q(t)^\top x)$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^3$. Thus, $Q(t)^\top b$ is t-independent, and so $b=\alpha\,\mathrm{e}_3$ for some $\alpha\in\mathbb{R}$. We conclude that $w(x)=U_0(x)=\alpha\,\mathrm{e}_3$. □

References

[Far06] Reinhard Farwig, *An L^q-analysis of viscous fluid flow past a rotating obstacle*, Tohoku Math. J. (2) **58** (2006), no. 1, 129–147. MR2221796 (2007f:35226)

[FHM04] Reinhard Farwig, Toshiaki Hishida, and Detlef Müller, *L^q-theory of a singular winding integral operator arising from fluid dynamics*, Pac. J. Math. **215** (2004), no. 2, 297–312. MR2068783 (2005f:35078)

[Gal94] Giovanni P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I: Linearized steady problems*, Springer Tracts in Natural Philosophy. 38. New York: Springer-Verlag, 1994. MR1284205 (95i:35216a)

[Gal02] ———, *On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications*, Friedlander, S. et al. (eds.), Handbook of mathematical fluid dynamics. Vol. 1. Amsterdam: Elsevier. 653-791, 2002. MR1942470 (2003j:76024)

[Gal03] ———, *Steady flow of a Navier-Stokes fluid around a rotating obstacle*, Journal of Elasticity **71** (2003), 1–31. MR2042672 (2005c:76030)

[GK11a] Giovanni P. Galdi and Mads Kyed, *A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part I: Strong solutions*, 2011. To appear in Proc. Amer. Math. Soc.

[GK11b] ———, *Steady-state Navier-Stokes flows past a rotating body: Leray solutions are physically reasonable*, Arch. Ration. Mech. Anal. **200** (2011), no. 1, 21–58. MR2781585 (2012c:35324)

[His06] Toshiaki Hishida, *L^q estimates of weak solutions to the stationary Stokes equations around a rotating body*, J. Math. Soc. Japan **58** (2006), no. 3, 743–767. MR2254409 (2007e:35226)

[KNP08] Stanislav Kračmar, Šárka Nečasová, and Patrick Penel, *L^q-approach to weak solutions of the Oseen flow around a rotating body*, Rencławowicz, Joanna et al. (eds.), Parabolic and Navier-Stokes equations. Part 1. Proceedings of the conference, Będlewo, Poland, September 10–17, 2006. Warsaw: Polish Academy of Sciences, Institute of Mathematics. Banach Center Publications 81, Pt. 1, 259-276, 2008. MR2547463 (2010d:35002)

[KNP10] S. Kračmar, S. Nečasová, and P. Penel, *L^q-approach of weak solutions to stationary rotating Oseen equations in exterior domains*, Q. Appl. Math. **68** (2010), no. 3, 421–437. MR2676969 (2012b:76032)

[Lad69] Olga A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Second English edition, Gordon and Breach Science Publishers, New York, 1969. MR0254401 (40:7610)

[Sil04] Ana Leonor Silvestre, *On the existence of steady flows of a Navier-Stokes liquid around a moving rigid body*, Math. Methods Appl. Sci. **27** (2004), no. 12, 1399–1409. MR2069156 (2005f:35251)

Department of Mechanical Engineering and Materials Science, University of Pittsburgh, Pittsburgh, Pennsylvania 15261

E-mail address: `galdi@pitt.edu`

Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D-64289 Darmstadt, Germany

E-mail address: `kyed@mathematik.tu-darmstadt.de`