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CONJUGACY IN THOMPSON'S GROUP F

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ABSTRACT. We complete the program begun by Brin and Squier of characterising conjugacy in Thompson's group F using the standard action of F as a group of piecewise linear homeomorphisms of the unit interval.

1. INTRODUCTION

The object of this paper is to extend the methods of Brin and Squier described in [3] to solve the conjugacy problem in Thompson's group F.

Let $f:(a,b)\to(a,b)$ be a piecewise linear order-preserving homeomorphism of the open interval (a,b); the points at which f is not locally affine are called the *nodes* of f. We write $\mathrm{PLF}^+(a,b)$ to denote the group of all piecewise linear order-preserving homeomorphisms of the open interval (a,b) which have finitely many nodes. Thompson's group F is the subgroup of $\mathrm{PLF}^+(0,1)$ defined as follows: an element f of $\mathrm{PLF}^+(0,1)$ lies in F if and only if the nodes of f lie in the ring of dyadic rational numbers, $\mathbb{Z}\left[\frac{1}{2}\right]$, and $f'(x)$ is a power of 2 whenever x is not a node.

In [3] Brin and Squier analysed conjugacy in $\mathrm{PLF}^+(a,b)$ for (a,b) any open interval. For (a,b) equal to $(0,1)$ we can restate their primary result [3, Theorem 5.3] as follows: we have a simple quantity Σ on $\mathrm{PLF}^+(0,1)$ such that two elements f and g of $\mathrm{PLF}^+(0,1)$ are conjugate if and only if $\Sigma_f=\Sigma_g$. If f and g are elements of F, then Σ_f and Σ_g can be computed and compared using a simple algorithm. Brin and Squier comment on their construction of Σ that "Our goal at the time was to analyze the conjugacy problem in Thompson's group F." In this paper we achieve Brin and Squier's goal by defining a quantity Δ on F such that the following theorem holds.

Theorem 1.1. *Let $f,g\in F$. Then f and g are conjugate in F if and only if*

$$(\Sigma_f,\Delta_f)=(\Sigma_g,\Delta_g).$$

This is not the first solution of the conjugacy problem in F. In particular the conjugacy problem in F was first solved by Guba and Sapir in [5] using diagram groups. More recently, Belk and Matucci [1, 2] have another solution using strand diagrams. Kassabov and Matucci [6] also solved the conjugacy problem and the simultaneous conjugacy problem. Our analysis is different from all of these as we build on the geometric invariants introduced by Brin and Squier.

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Note that Theorem 1.1 reduces the study of roots and centralizers in F (first completed in [5]) to a set of easy computations; we leave this to the interested reader.

We will not prove Theorem 1.1 directly. Rather we prove the following proposition, which, given [3, Theorem 5.3], implies Theorem 1.1.

Proposition 1.2. *Two elements f and g of F that are conjugate in* $\mathrm{PLF}^+(0,1)$ *are conjugate in F if and only if $\Delta_f = \Delta_g$.*

Our paper is structured as follows. In §2 we introduce some important background concepts, including the definition of Σ. In §3 we define Δ. In §4 we prove Proposition 1.2. In §5 and §6 we outline formulae which can be used to calculate Δ.

2. The definition of Σ

Let f be a member of Thompson's group F, embedded in $\mathrm{PLF}^+(0,1)$. Following Brin and Squier [3] we define the invariant Σ_f to be a tuple of three quantities, Σ_1, Σ_2 and Σ_3, which depend on f.

The first quantity, Σ_1, is a list of integers relating to values of the *signature of* f, ϵ_f. We define ϵ_f as follows:

$$\epsilon_f : (0,1) \to \{-1,0,1\}, \quad x \mapsto \begin{cases} 1, & f(x) > x; \\ 0, & f(x) = x; \\ -1, & f(x) < x. \end{cases}$$

If f is an element of $\mathrm{PLF}^+(0,1)$, then there is a sequence of open intervals

$$I_1, I_2, \ldots, I_m, \quad I_j = (p_{j-1}, p_j), \quad p_0 = 0, \quad p_m = 1, \tag{2.1}$$

such that ϵ_f is constant on each interval, and the values of ϵ_f on two consecutive intervals differ. We define $\Sigma_1 = (\epsilon_f(x_1), \ldots, \epsilon_f(x_m))$, where $x_i \in I_i$ for $i = 1, \ldots, m$.

Let $\mathrm{fix}(f)$ be the set of fixed points of f and observe that the points $p_0, \ldots, p_m$ from (2.1) all lie in $\mathrm{fix}(f)$. We say that the interval I_j is a *bump domain* of f if ϵ_f is nonzero on that interval. Our next two invariants consist of lists with entries for each bump domain of f.

If k is a piecewise linear map from one interval (a, b) to another, then the *initial slope* of k is the derivative of k at any point between a and the first node of k, and the *final slope* of k is the derivative of k at any point between the final node of k and b. The invariant Σ_2 is a list of positive real numbers. The entry for a bump domain $I_j = (p_{j-1}, p_j)$ is the value of the initial slope of f in I_j.

To define Σ_3 we need the notion of a *finite function*; this is a function $[0, 1) \to \mathbb{R}^+$ which takes the value 1 at all but finitely many values. The invariant Σ_3 is a list of equivalence classes of finite functions. We calculate the entry for a bump domain $I_j = (p_{j-1}, p_j)$ as follows. Suppose first of all that $\Sigma_1 = 1$ in I_j. Define, for $x \in I_j$, the *slope ratio* $f^*(x) = \frac{f'_+(x)}{f'_-(x)}$. Thus $f^*(x) = 1$ except when x is a node of f. Now define

$$\phi_{f,j} : I_j \to \mathbb{R}, \quad x \mapsto \prod_{n=-\infty}^{\infty} f^*(f^n(x)).$$

Since f has only finitely many nodes, only finitely many terms of this infinite product are distinct from 1. Let p be the smallest node of f in I_j and let p_* be the smallest node of f in I_j such that $\phi_{f,j}(p_*) \neq 1$ (such a node must exist). Define, for $s \in [0,1)$,

$$\psi_{f,j}(s) = \phi_f(\lambda^s(r - p_{j-1}) + p_{j-1}).$$

Here λ is the entry in Σ_2 corresponding to I_j and r is any point in the interval $(0,p)$ which satisfies the formula $r = f^n(p_*)$ for n some negative integer.

Note that $\psi_{f,j}$ is a finite function; furthermore, in our definition of $\psi_{f,j}$, we have chosen a value for r which guarantees that $\psi_{f,j}(0) \neq 1$; we can do this by virtue of [3, Lemma 4.4].

The entry for Σ_3 corresponding to I_j is the equivalence class $[\psi_{f,j}]$, where two finite functions c_1 and c_2 are considered equivalent if $c_1 = c_2 \circ \rho$, where ρ is a translation of $[0,1)$ modulo 1. If $f(x) < x$ for each $x \in I_j$, then the entry for Σ_3 corresponding to I_j is the equivalence class $[\psi_{f^{-1},j}]$.

3. The definition of Δ

The quantity Δ will also be a list, this time a list of equivalence classes of tuples of real numbers. To begin with we need the concept of a *minimum cornered function.*

3.1. The minimum cornered function.

Take $f \in \mathrm{PLF}^+(0,1)$; in this subsection we focus on the restriction of f to one of its bump domains $D = (a,b)$. We adjust one of the definitions of Brin and Squier [3]: for us, a *cornered function* in $\mathrm{PLF}^+(0,1)$ is an element l which has a single bump domain (a,b) and which satisfies the following property: Σ_1 takes the value 1 (resp. -1) in relation to (a,b) and there exists a point $x \in (a,b)$ such that all nodes of l which lie in (a,b) lie in $(x, l(x))$ (resp. $(l(x), x)$). We will sometimes abuse notation and consider such a cornered function as an element of $\mathrm{PLF}^+(a,b)$. A cornered function is shown in Figure 1.

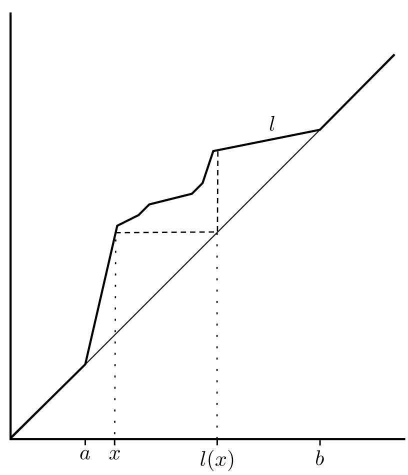

Figure 1. A cornered function.

We say that a cornered function l *corresponds to* a finite function c if $\psi_l = c$ (we drop the subscript i here, since there is only one bump domain). Roughly speaking, this means that the first node of l corresponds to $c(0)$. For a given initial slope λ

there is a unique cornered function in $\mathrm{PLF}^+(a,b)$ such that $\psi_l = c$ (this follows from [3, Prop. 4.9]; it can also be deduced from the proof of Lemma 5.2).

Now let $c : [0,1) \to \mathbb{R}$ be a finite function such that $[c]$ is the entry in Σ_3 associated with D. Within this equivalence class $[c]$ we can define a *minimum finite function* c_m as follows. First define $C = \{c_1 \in [c] \mid c_1(0) \neq 1\}$ and define an ordering on C as follows. Let $c_1, c_2 \in C$ and let x be the smallest value such that $c_1(x) \neq c_2(x)$. Write $c_1 < c_2$ provided $c_1(x) < c_2(x)$. We define c_m to be the minimum function in C under this ordering.

The graphs of two finite functions c_L and c_R are shown in Figure 2; c_L on the left and c_R on the right. These two functions are equivalent, because the graph of c_L can be obtained by translating the graph of c_R horizontally modulo 1. Neither c_L nor c_R is a minimum finite function in their equivalence class, because $c_L(0) = c_R(0) > 1$ and there are other equivalent finite functions that take values less than 1 at 0. Since $c_R(x) < c_L(x)$ for the smallest number x at which c_L and c_R differ, we see that $c_R < c_L$.

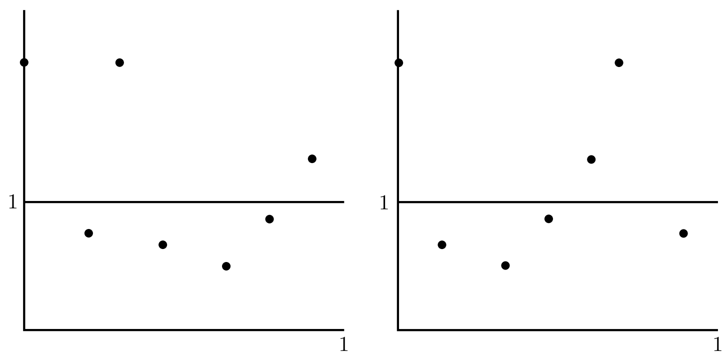

FIGURE 2. Two equivalent finite functions.

Suppose that λ is the entry in Σ_2 associated with D. Suppose that l is the cornered function in $\mathrm{PLF}^+(a,b)$, with initial slope λ, which corresponds to c_m. We say that l is the *minimum cornered function* associated with f over D.

3.2. **The quantity Δ.** Now let us fix f to be in F and define Δ accordingly. A *bump chain* is a subsequence $I_t, I_{t+1}, \ldots, I_u$ of (2.1) such that each interval is a bump domain, and of the points $p_{t-1}, p_t, \ldots, p_u$ only p_{t-1} and p_u are dyadic. Thus $I_1, I_2, \ldots, I_m$ can be partitioned into bump chains and open intervals of fixed points of f (which have dyadic numbers as end-points).

In [3], conjugating functions in $\mathrm{PLF}^+(0,1)$ are constructed by dealing with one bump domain at a time. We will construct conjugating functions in F by dealing with one bump *chain* at a time. Consider a particular bump chain $D_1, \ldots, D_s$ and let f_j be the restriction of f to $D_j = (a_j, b_j)$.

According to [3, Theorem 4.18], the centralizer of f_j within $\mathrm{PLF}^+(a_j, b_j)$ is an infinite cyclic group generated by a root $\widehat{f_j}$ of f_j. We define λ_j to be the initial slope of $\widehat{f_j}$ and μ_j to be the final slope of $\widehat{f_j}$. (Let m_j be the integer such that $\widehat{f_j}^{\,m_j} = f_j$; then λ_j and μ_j are the positive m_jth roots of the initial and final slopes of f_j.)

Next, let k_j be a member of $\mathrm{PLF}^+(a_j, b_j)$ that conjugates f_j to the associated minimum cornered function, l_j, in $\mathrm{PLF}^+(a_j, b_j)$. Thus k_j is some function satisfying the equality $k_j f_j k_j^{-1} = l_j$. Let α_j be the initial slope of k_j and let β_j be the final slope.

Consider the equivalence relation on $\mathbb{R}^s$ where $(x_1, \ldots, x_s)$ is equivalent to $(y_1, \ldots, y_s)$ if and only if there are integers $m, n_1, \ldots, n_s$ such that

$$\begin{aligned} 2^m x_1 &= \lambda_1^{n_1} y_1, \\ \mu_1^{n_1} x_2 &= \lambda_2^{n_2} y_2, \\ \mu_2^{n_2} x_3 &= \lambda_3^{n_3} y_3, \\ &\vdots \\ \mu_{s-1}^{n_{s-1}} x_s &= \lambda_s^{n_s} y_s. \end{aligned}$$

It is possible to check whether two s-tuples of real numbers are equivalent according to the above relation in a finite amount of time because the quantities λ_i and μ_j are rational powers of 2. We assign to the chain $D_1, \ldots, D_s$ the equivalence class of the s-tuple

$$\left(\frac{\alpha_1}{w_1}, \frac{\alpha_2}{w_2} \frac{w_1}{\beta_1}, \ldots, \frac{\alpha_s}{w_s} \frac{w_{s-1}}{\beta_{s-1}} \right),$$

where $w_j = b_j - a_j$. We define Δ_f to consist of an ordered list of such equivalence classes, one per bump chain.

4. Proof of Proposition 1.2

We prove Proposition 1.2 after the following elementary lemma.

Lemma 4.1. *Let f and g be maps in F, and let h be an element of $\mathrm{PLF}^+(0,1)$ such that $hfh^{-1} = g$. Let $D = (a, b)$ be a bump domain of f and suppose that the initial slope of h in D is an integer power of 2. Then all slopes of h in D are integer powers of 2 and all nodes of h in D occur in $\mathbb{Z}[\frac{1}{2}]$.*

Proof. Let $(a, a+\delta)$ be a small interval over which h has constant slope; suppose that this slope is greater than 1. We may assume that f has initial slope greater than 1; otherwise replace f with f^{-1} and g with g^{-1}. Now observe that $hf^n h^{-1} = g^n$ for all integers n and so $h = g^n h f^{-n}$.

Now, for any $x \in (a, b)$ there is an interval $(x, x+\epsilon)$ and an integer n so that $f^{-n}(x, x+\epsilon) \subset (a, a+\delta)$. Then the equation $h = g^n h f^{-n}$ implies that, where defined, the derivative of h over $(x, x+\epsilon)$ is an integral power of 2. Furthermore any node of h occurring in $(x, x+\epsilon)$ must lie in $\mathbb{Z}[\frac{1}{2}]$, as required.

If h does not have slope greater than 1, then apply the same argument to h^{-1} using the equation $h^{-1}gh = f$. $\square$

We have two elements f and g of F and a third element h of $\mathrm{PLF}^+(0,1)$ such that $hfh^{-1} = g$. We use the notation for f described in the previous section, such as the quantities I_j, p_j, f_j, $\widehat{f}_j$, k_j, l_j, α_j, β_j, w_j, λ_j, and μ_j. We need exactly the same quantities for g, and we distinguish the quantities for g from those for f by adding a $'$ after each one. In particular, we choose a bump chain $D_1, \ldots, D_s$ of f and define $D_i' = h(D_i)$ for $i = 1, \ldots, s$. Note that $D_1', \ldots, D_s'$ are bump domains but need not form a bump chain for g according to our assumptions, because h is not necessarily a member of F.

Let the function $h_i = h|_{D_i}$ have initial slope γ_i and final slope δ_i. Let u be the member of $\mathrm{PLF}^+(0,1)$ which, for $i = 1,\dots,m$, is affine when restricted to I_i and maps this interval onto I_i'. Notice that, restricted to D_i', ul_iu^{-1} is a cornered function which is conjugate to l_i' (by the map $k_i'h_ik_i^{-1}u^{-1}$) and which satisfies $\psi_{l_i'} = \psi_{ul_iu^{-1}}$. Therefore $ul_iu^{-1} = l_i'$. Combine this equation with the equations $k_if_ik_i^{-1} = l_i$, $k_i'g_ik_i'^{-1} = l_i'$, and $h_if_ih_i^{-1} = g_i$ to yield

$$(k_i^{-1}u^{-1}k_i'h_i)f_i(k_i^{-1}u^{-1}k_i'h_i)^{-1} = f_i.$$

Therefore $k_i^{-1}u^{-1}k_i'h_i$ is in the centralizer of f, so there is an integer N_i such that

$$h_i = (k_i')^{-1}uk_i\widehat{f_i}^{N_i}$$

for each $i = 1,\dots,s$. Then by comparing initial and final slopes in this equation we see that

$$\gamma_i = \lambda_i^{N_i}\frac{\alpha_i}{w_i}\frac{w_i'}{\alpha_i'}, \quad \delta_i = \mu_i^{N_i}\frac{\beta_i}{w_i}\frac{w_i'}{\beta_i'} \tag{4.1}$$

for $i = 2,\dots,s$. We are now in a position to prove Proposition 1.2.

Proof of Proposition 1.2. Suppose that $h \in F$. Then there are integers $M_1,\dots,M_s$ such that $\gamma_1 = 2^{M_1}$ and $\gamma_i = \delta_{i-1} = 2^{M_i}$ for $i = 2,\dots,s$. Substituting these values into (4.1) we see that

$$2^{M_1}\frac{\alpha_1'}{w_1'} = \lambda_1^{N_1}\frac{\alpha_1}{w_1}, \quad \mu_{i-1}^{N_{i-1}}\frac{\alpha_i'}{w_i'}\frac{w_{i-1}'}{\beta_{i-1}'} = \lambda_i^{N_i}\frac{\alpha_i}{w_i}\frac{w_{i-1}}{\beta_{i-1}},$$

for $i = 2,\dots,s$, as required.

Conversely, suppose that $\Delta_f = \Delta_g$. We modify h so that it is a member of F. If I_j is an interval of fixed points of f, then modify h_j so that it is any piecewise linear map from I_j to I_j' whose slopes are integer powers of 2 and whose nodes occur in $\mathbb{Z}[\frac{1}{2}]$. (It is straightforward to construct such maps; see [4, Lemma 4.2].)

Now we modify h on a bump chain $D_1,\dots,D_s$. Since $\Delta_f = \Delta_g$ we know that there are integers m and $n_1,\dots,n_s$ such that, for $i = 2,\dots,s$,

$$2^m\frac{\alpha_1}{w_1} = \lambda_1^{n_1}\frac{\alpha_1'}{w_1'}, \quad \mu_{i-1}^{n_{i-1}}\frac{\alpha_i}{w_i}\frac{w_{i-1}}{\beta_{i-1}} = \lambda_i^{n_i}\frac{\alpha_i'}{w_i'}\frac{w_{i-1}'}{\beta_{i-1}'}. \tag{4.2}$$

Consider the piecewise linear map $h_i' : D_i \to h_i(D_i)$ given by $h_i' = h_i\widehat{f_i}^{-n_i-,N_i}$. The initial slope γ_i' of h_i' is $\gamma_i\lambda_i^{-n_i-N_i}$ and the final slope $\delta_i' = \delta_i\mu^{-n_i-N_i}$. From (4.1) and (4.2) we see that

$$\gamma_1' = 2^{-m}, \quad \gamma_i' = \delta_{i-1}'$$

for $i = 2,\dots,s$. We modify h by replacing h_i with h_i' on D_i. Then h does not have a node at any of the end-points of $D_1,\dots,D_s$ other than the first and last end-point. By Lemma 4.1, the nodes of h_1 occur in $\mathbb{Z}[\frac{1}{2}]$ and the slopes of h_1 are all powers of 2. Since the initial slope of h_1 coincides with the final slope of h_1, the same can be said of h_2. Similarly, for $i = 2,\dots,s$, the initial slope of h_i coincides with the final slope of h_{i-1}. We repeat these modifications for each bump chain of f; the resulting conjugating map is a member of F. $\square$

5. Calculating α_i and β_i

It may appear that, in order to calculate Δ, it is necessary to construct various conjugating functions. In particular, to calculate α_i one might have to construct the function in $\mathrm{PLF}^+(a_i, b_i)$ which conjugates f_i to the conjugate minimum cornered function in $\mathrm{PLF}^+(a_i, b_i)$.

It turns out that this is not the case. The values for α_i and β_i can be calculated simply by looking at the entries in Σ_1, Σ_2 and Σ_3 which correspond to D_i. In this section we give a formula for α_i; we then observe how to use the formula for α_i to calculate β_i.

In what follows we take f to be a function in $\mathrm{PLF}^+(a, b)$ such that $f(x) \neq x$ for $x \in (a, b)$. Let l be the minimum cornered function which is conjugate to f in $\mathrm{PLF}^+(a, b)$.

5.1. **Calculating** α_i**.** Suppose first that $f(x) > x$ for $x \in (a, b)$. Let y_j, for $j = 0, \ldots, t$, be the points at which the finite function ψ_f does not take the value 1; let ψ_f take the positive value z_j at the point y_j and assume that $0 = y_0 < y_1 < \cdots < y_t < 1$. We will denote ψ_f by c_t and define $c_j = c_t(x + y_{j+1})$. Then c_j is a translation of c_t under which y_j is mapped to the last point of c_j which does not take the value 1.

Let u_j be the cornered function corresponding to c_j and let x_j be the final node of u_j. Note that u_j is conjugate to f and, for j equal to some integer n, u_j equals l, the minimum cornered function. Define the *elementary function* $h_{x,r}$ to be the function which is affine on $(0, x)$ and $(x, 1)$ and which has slope ratio r at x. We define ζ_j to be the initial slope of the elementary function h_{x_j, z_j}.

Let p be the first node of f and let q be the first node of u_t.

Lemma 5.1. *There exists k in $\mathrm{PLF}^+(a, b)$ such that $kfk^{-1} = l$ and the initial slope of k is*

$$(\zeta_t \zeta_{t-1} \ldots \zeta_{n+1}) \left(\frac{q - a}{p - a} \right).$$

Note that, in the formula just given, p and q stand for the x-coordinates of the corresponding nodes. Before we prove Lemma 5.1 we observe that we can calculate values for the ζ_j and q simply by looking at Σ_2 and Σ_3 and using the following lemma:

Lemma 5.2. *Let l be a cornered function in $\mathrm{PLF}^+(a, b)$ with initial slope $\lambda > 1$, and suppose that the corresponding finite function c takes the value 1 at all points in $[0, 1)$ except $0 = s_0 < s_1 < \cdots < s_k < 1$, at which $c(s_i) = z_i$. Then the first node q_0 of l is given by the formula*

(5.1)
$$q_0 = a + \frac{(b-a)(1-[\lambda z_0 \cdots z_k])}{[\lambda(1-z_0)]+[\lambda^{s_1+1} z_0(1-z_1)]+\cdots+[\lambda^{s_{k-1}+1} z_0 \cdots z_{k-2}(1-z_{k-1})]+[\lambda^{s_k+1} z_0 \cdots z_{k-1}(1-z_k)]},$$

and the initial slope ζ of the elementary function h_{q_k, z_k}, where q_k is the final node of l, is given by

$$\zeta = \frac{b - a}{\lambda^{s_k}(q_0 - a)(1 - z_k) + (b - a) z_k}. \tag{5.2}$$

Proof. If $q_0, \dots, q_k$ are the nodes of l we have the equations

$$\lambda^{s_i}(q_0 - a) + a = q_i, \quad i = 0, 1, 2, \dots, k. \tag{5.3}$$

Define $q_{k+1} = b$ and let λ_i be the slope of l between the nodes q_{i-1} and q_i for $i = 1, \dots, k+1$. Then $z_i = \lambda_i / \lambda_{i-1}$ for $i > 1$, and we obtain

$$\lambda_i = \lambda z_0 \dots z_{i-1}, \quad i = 1, \dots, k+1. \tag{5.4}$$

If we substitute (5.3) and (5.4) into the equation

$$b - a = \lambda(q_0 - a) + \lambda_1(q_1 - q_0) + \lambda_2(q_2 - q_1) + \cdots + \lambda_{k+1}(b - q_k),$$

then we obtain (5.1). To obtain (5.2), notice that $z_k\zeta$ is the final slope of h_{q_k, z_k}; therefore $b - a = \zeta(q_k - a) + z_k\zeta(b - q_k)$. Substitute the value of q_k from (5.3) into this equation to obtain (5.2). □

Before we prove Lemma 5.1, we make the following observation. Let g be a function such that $g(x) > x$ for all $x \in (a, b)$ and suppose that g has nodes $p_1 < \cdots < p_s$. Now let $h = h_{p_s, g^*(p_s)}$. Then hgh^{-1} has nodes $h(p_1), \dots, h(p_{s-1}), hg^{-1}(p_s)$ with $(hgh^{-1})^*$ taking on values $g^*(p_1), \dots, g^*(p_t)$ at the respective nodes. If $hg^{-1}(p_s) = h(p_i)$ for some i, then $(hgh^{-1})^*$ has value $g^*(p_i)g^*(p_s)$.

Proof of Lemma 5.1. The formula given in Lemma 5.1 arises as follows. We start by finding the conjugator from f to the cornered function u_t; then we cycle through the cornered functions u_j until we get to $u_n = l$. Thus the $\frac{q-a}{p-a}$ part of the formula arises from the initial conjugation to a cornered function, and the ζ_j's arise from the cycling.

Consider this cycling part first and use our observation above on the cornered functions, u_j: we have $h_{x_j, z_j} u_j (h_{x_j, z_j})^{-1} = u_{j-1}$. Thus in order to move from u_t to u_n we repeatedly conjugate by elementary functions with initial gradient $\zeta_t, \dots, \zeta_{n+1}$.

We must now explain why we can use $\frac{q-a}{p-a}$ for the first conjugation which moves from f to u_t. It is sufficient to find a function which conjugates f to u_t and which is linear on $[a, p]$.

Consider the effect of applying an elementary conjugation to a function f that is not a cornered function. Suppose that f has nodes $p_1 < \cdots < p_s$. So $p = p_1$. We consider the effect of conjugation by an elementary function $h = h_{p_s, f^*(p_s)}$ as above. To reiterate, we obtain a function with nodes

$$h(p_1), \dots., h(p_{s-1}), hf^{-1}(p_s).$$

Now observe that, since $f^*(p_s) < 1$, $h(x) > x$ for all x and h is linear on $[a, p_s]$. So clearly h is linear on the required interval. There are three possibilities:

- If $hf^{-1}(p_s) < h(p)$, then f was already a cornered function; in fact $f = u_t$. We are done.
- If $hf^{-1}(p_s) > h(p)$, then we simply iterate. We replace f with hfh^{-1}, p with $h(p)$, etc. We conjugate by another elementary function exactly as before. It is clear that the next elementary conjugation will be linear on $[a, h(p)]$, which is sufficient to ensure that the composition is linear on $[a, p]$.

- If $hf^{-1}(p_s) = h(p)$, then we need to check if hfh^{-1} is a cornered function. If so, then $hfh^{-1} = u_t$, the corner function we require. If hfh^{-1} is not a cornered function, then we iterate as above, replacing f with hfh^{-1}. It is possible that $h(p)$ will no longer be the first node of hfh^{-1}, but in this case we replace p by $h(p_2)$. Since $[a, h(p_2] \supset [a, h(p)]$ this is sufficient to ensure that the composition is linear on $[a, p]$.

We can proceed like this until the process terminates at a cornered function. Since conjugating a noncornered function by h preserves ψ we can be sure that we will terminate at u_t as required. What is more the composition of these elementary functions is linear on $[a, p]$. $\square$

Suppose next that $f(x) < x$ for all $x \in (a, b)$ and $kfk^{-1} = l$, a minimum cornered function. Observe that $kf^{-1}k^{-1} = l^{-1}$ and $f^{-1}(x) > x$ for all $x \in (a, b)$. We can now apply the formula in Lemma 5.1, replacing f with f^{-1} and l with l^{-1}, to get a value for the initial slope of k.

5.2. **Calculating β_i.** The method we have used to calculate α_i can also be used to calculate β_i. Define

$$\tau : [a, b] \to [a, b], x \mapsto b + a - x.$$

Now τ is an automorphism of $\mathrm{PLF}^+(a, b)$; the graph of a function, when conjugated by τ, is rotated 180° about the point $(\frac{b+a}{2}, \frac{b+a}{2})$. Consider the function $\tau f \tau$ and let k be the conjugating function from earlier, so that $kfk^{-1} = l$. Then

$$(\tau k \tau)(\tau f \tau)(\tau k \tau)^{-1} = (\tau l \tau).$$

The initial slope of $\tau k \tau$ equals the final slope of k. Thus we can use the method outlined above (replacing f with $\tau f \tau$ and l with $\tau l \tau$) to calculate the initial slope of $\tau k \tau$. Note that, for this to yield β_i, we must make an adjustment to the integer n in the formula in Lemma 5.1: the function $\tau l \tau$ is not necessarily the *minimum* cornered function which is conjugate to $\tau f \tau$. Thus we choose n to ensure that l is minimum rather than $\tau l \tau$.

6. Calculating λ_i and μ_i

Let f be a fixed-point-free element of $\mathrm{PLF}^+(a, b)$. Let $\widehat{f}$ be a generator of the centralizer of f within $\mathrm{PLF}^+(a, b)$. The formula for Δ requires that we calculate the initial slope and the final slope of $\widehat{f}$. It turns out that this is easy, thanks to the work of Brin and Squier [3].

Let $c, c' : [0, 1) \to \mathbb{R}$ be finite functions. We say that c' is the p-th root of c provided that, for all $x \in [0, 1)$, we have $c(x) = c'(px)$. The property of having a p-th root is preserved by the equivalence used to define Σ_3. Thus we may talk about the equivalence class $[c]$ having a p-th root, provided any representative of $[c]$ has a p-th root.

Now [3, Theorem 4.15] asserts that f has a p-th root in $\mathrm{PLF}^+(a, b)$, for p a positive integer, if and only if the single equivalence class in Σ_3 is a p-th power (following Brin and Squier, we say that this class has *p-fold symmetry*). What is more, [3, Theorem 4.18] asserts that $\widehat{f}$ must be a root of f.

Thus if p is the largest integer for which the single class in Σ_3 has p-fold symmetry, then $\widehat{f}$ is the p-th root of f. The initial slope of $\widehat{f}$ is the positive p-th root of the initial slope of f, and the final slope of $\widehat{f}$ is the positive p-th root of the final slope of f.

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