ON A CLASS OF HEREDITARY CROSSED-PRODUCT ORDERS

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Abstract. In this brief note, we revisit a class of crossed-product orders over discrete valuation rings introduced by D. E. Haile. We give simple but useful criteria, which involve only the two-cocycle associated with a given crossed-product order, for determining whether such an order is a hereditary order or a maximal order.

Let \( V \) be a discrete valuation ring (DVR), with quotient field \( F \), and let \( K/F \) be a finite Galois extension, with group \( G \), and let \( S \) be the integral closure of \( V \) in \( K \). Let \( f \in \mathbb{Z}^2(G,U(K)) \) be a normalized two-cocycle. If \( f(G \times G) \subseteq S^\# \), then one can construct a “crossed-product” \( V \)-algebra \( A_f = \sum_{\sigma \in G} Sx_{\sigma} \), with the usual rules of multiplication \( (x_\sigma s = \sigma(s)x_\sigma) \) for all \( s \in S, \sigma \in G \) and \( x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma \tau} \). Then \( A_f \) is associative, with identity \( 1 = x_1 \), and center \( V = Vx_1 \).

Further, \( A_f \) is a \( V \)-order in the crossed-product \( F \)-algebra \( \Sigma_f = \sum_{\sigma \in G} Kx_\sigma = (K/F,G,f) \).

Two such cocycles \( f \) and \( g \) are said to be cohomologous over \( S \) (respectively cohomologous over \( K \)), denoted by \( f \sim_S g \) (respectively \( f \sim_K g \)), if there are elements \( \{c_\sigma \mid \sigma \in G\} \subseteq U(S) \) (respectively \( \{c_\sigma \mid \sigma \in G\} \subseteq K^\# \) ) such that \( g(\sigma,\tau) = c_\sigma \sigma(c_\tau)c_{\sigma \tau}^{-1}f(\sigma,\tau) \) for all \( \sigma,\tau \in G \). Following [1], let \( H = \{\sigma \in G \mid f(\sigma,\sigma^{-1}) \in U(S)\} \). Then \( H \) is a subgroup of \( G \). On \( G/H \), the left coset space of \( G \) by \( H \), one can define a partial ordering by the rule \( \sigma H \leq \tau H \) if \( f(\sigma,\sigma^{-1}\tau) \in U(S) \). Then “\( \leq \)” is well-defined and depends only on the cohomology class of \( f \) over \( S \). Further, \( H \) is the unique least element. We call this partial ordering on \( G/H \) the graph of \( f \).

Such a setup was first formulated by Haile in [1], with the assumption that \( S \) is unramified over \( V \), wherein, among other things, conditions equivalent to such orders being maximal orders were considered. This is the class of crossed-product orders we shall study in this paper, always assuming that \( S \) is unramified over \( V \). We emphasize the fact that, since we do not require that \( f(G \times G) \subseteq U(S) \), this
theory constitutes a drastic departure from the classical theory of crossed-product orders over DVRs, such as can be found in [2].

Let us now fix additional notation to be used in the rest of the paper, most of it borrowed from [1] as before. If \( M \) is a maximal ideal of \( S \), let \( D_M \) be the decomposition group of \( M \), let \( K_M \) be the decomposition field, and let \( S_M \) be the localization of \( S \) at \( M \). The two-cocycle \( f : G \times G \rightarrow S^\# \) yields a two-cocycle \( f_{M} : D_M \times D_M \rightarrow S^\#_M \), determined by the restriction of \( f \) to \( D_M \times D_M \) and the inclusion of \( S^\# \) in \( S^\#_M \). Then \( A_{fM} = \sum_{\sigma \in D_M} S_M x_{\sigma} \) is a crossed-product order in \( \Sigma_f = \sum_{\sigma \in D_M} K x_{\sigma} = (K/K_M, D_M, f_M) \). In addition, we can obtain a twist of \( f \), described in [1] pp. 137-138] and denoted by \( \tilde{f} \), which depends on the choice of a maximal ideal \( M \) of \( S \), and the choice of a set of coset representatives of \( D_M \) in \( G \).

We also define \( F : G \times G \rightarrow S^\# \) by \( F(\sigma, \tau) = f(\sigma, \sigma^{-1} \tau) \) for \( \sigma, \tau \in G \). While \( \tilde{f} \) is a two-cocycle, \( F \) is not.

If \( B \) is a \( V \)-order of \( \Sigma_f \) containing \( A_f \), then by [1] Proposition 1.3], \( B = A_g = \sum_{\sigma \in G} S y_{\sigma} \) for some two-cocycle \( g : G \times G \rightarrow S^\# \), with \( g \sim_K f \). Moreover, the proof of [1] Proposition 1.3] shows that \( y_{\sigma} = k_{\sigma} x_{\sigma} \) for some \( k_{\sigma} \in K^\# \), with \( k_1 = 1 \), whence \( g \) is also a normalized two-cocycle.

We begin with a technical result.

**Sublemma.** Let \( \tau \in G \). If \( I_\tau = \prod_{f(\tau, \tau^{-1}) \notin M} M \), where \( M \) denotes a maximal ideal of \( S \), then \( I_\tau^{-1} = I_{\tau^{-1}} \).

**Proof.** We have

\[
I_\tau^{-1} = \prod_{f(\tau, \tau^{-1}) \notin M} M^{-1} = \prod_{f(\tau, \tau^{-1}) \notin M} M^{\tau^{-1}} = \prod_{f(\tau^{-1}, \tau) \notin M^\tau} M^{\tau^{-1}} = I_{\tau^{-1}}.
\]

\( \square \)

**Theorem.** The crossed-product order \( A_f \) is hereditary if and only if \( f(\tau, \tau^{-1}) \notin M^2 \) for all \( \tau \in G \) and every maximal ideal \( M \) of \( S \).

**Proof.** The theorem obviously holds if \( H = G \), in which case \( A_f \) is an Azumaya algebra over \( V \), so let us assume from now on that \( H \neq G \).

Suppose \( A_f \) is hereditary. First, assume \( A_f \) is a maximal order and \( S \) is a DVR. Let \( v \) be the valuation corresponding to \( S \) with value group \( \mathbb{Z} \). Then by [1] Theorem 2.3], \( H \) is a normal subgroup of \( G \) and \( G/H \) is cyclic. Further, there exists \( \sigma \in G \) such that \( v(f(\sigma, \sigma^{-1})) \leq 1 \), \( G/H = \langle \sigma H \rangle \), and the graph of \( f \) is the chain \( H \leq \sigma H \leq \sigma^2 H \leq \cdots \leq \sigma^m H \), where \( m = |G/H| \). Choose \( j \) maximal such that \( 1 \leq j \leq m - 1 \) and \( v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq j \). We always have \( \sigma H \leq \sigma^j H \); but if \( j < m - 1 \), then we also have \( \sigma^j H \leq \sigma^{j+1} H \). Hence if \( j < m - 1 \), then, from the cocycle identity \( f^{\sigma^j}(\sigma, \sigma^{-j}, \sigma^{-1}) f(\sigma^j, \sigma^{-1}) = f(\sigma^j, \sigma) f(\sigma^{j+1}, \sigma^{-j} \sigma^{-1}) \), we conclude that \( v(f(\sigma^j, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq j + 1 \), a contradiction. So we must have \( j = m - 1 \), so that \( v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq m - 1 \). If \( \tau \) is an arbitrary element of \( G \), then \( \tau = \sigma^h \) for some \( h \in H \) and some integer \( i, 0 \leq i \leq m - 1 \). Therefore, by [1] Lemma 3.6], \( v(f(\tau, \tau^{-1})) = v(F(\sigma^i h, 1)) = v(F(\sigma^i, 1)) = v(f(\sigma^i, \sigma^{-i})) \leq 1 \); that is, \( f(\tau, \tau^{-1}) \notin J(S)^2 \).

We maintain the assumption that \( A_f \) is a maximal order, but we now drop the condition that \( S \) is a DVR. By [1] Theorem 3.16], there exists a twist of \( f \), say \( \tilde{f} \), such that \( f \sim_S \tilde{f} \). By [1] Corollary 3.11], for any maximal ideal \( M \) of \( S \), \( A_{fM} \) is a maximal order in \( \Sigma_{fM} \), hence \( f_M(\tau, \tau^{-1}) \notin M^2 \forall \tau \in D_M \) by the preceding
crossed-product order for all $\tau$. Thus, $A_f$ is not a maximal order, then it is the intersection of finitely many maximal orders, say $A_{f_1}, A_{f_2}, \ldots, A_{f_l}$. Note that

$$A_{f_i} = \sum_{\tau \in G} S y_{\tau}^{(i)} = \sum_{\tau \in G} S k_{\tau}^{(i)} x_{\tau},$$

for some $k_{\tau}^{(i)} \in K$. Fix a $\sigma \in G$, and a maximal ideal $N$ of $S$. Let $v_N$ be the valuation corresponding to $N$, with value group $\mathbb{Z}$. Since

$$S = \bigcap_{i=1}^l S k_{\sigma}^{(i)},$$

there exists $i_0$ such that $v_N(k_{\sigma}^{(i_0)}) = 0$. Let $y = f_{i_0}$ and, for $\tau \in G$, let $k_\tau = k_{\sigma}^{(i_0)}$ and $y_\tau = y_{(i_0)}^{(i_0)}$, so that $A_g = \sum_{\tau \in G} S k_\tau x_\tau = \sum_{\tau \in G} S y_{\tau}$. By \cite{1} Proposition 3.1, $J(A_g) = \sum_{\tau \in G} I_{\tau} x_\tau$ and $J(A_g) = \sum_{\tau \in G} J_{\tau} y_{\tau}$, where

$$I_{\tau} = \prod_{\sigma \in G} M$$ and $$J_{\tau} = \prod_{\sigma \in G} M,$$

and $M$ denotes a maximal ideal of $S$. Since $A_f$ is a hereditary $V$-order in $\Sigma_f$ and $A_f \subseteq A_g \subseteq \Sigma_f$, we have $J(A_g) \subseteq J(A_f)$, from which we conclude that $J_{\sigma^{-1}} y_{\sigma^{-1}} \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}}$ and so $J_{\sigma^{-1}} k_{\sigma^{-1}} \subseteq I_{\sigma^{-1}}$. We have $y_{\sigma^{-1}} x_{\sigma^{-1}} = k_{\sigma^{-1}} x_{\sigma^{-1}} y_{\sigma^{-1}} = J_{\sigma^{-1}} k_{\sigma^{-1}} x_{\sigma^{-1}} k_{\sigma^{-1}} x_{\sigma^{-1}} = J_{\sigma^{-1}} f(\sigma^{-1}, \sigma) = (k_{\sigma^{-1}} f(\sigma^{-1}, \sigma)).$

On the other hand, $y_{\sigma^{-1}} x_{\sigma^{-1}} = J_{\sigma^{-1}} g(\sigma^{-1}, \sigma) = J_{\sigma^{-1}} g(\sigma^{-1}, \sigma)$. Since $A_f$ is a maximal order and therefore $g(\sigma^{-1}, \sigma) \not\in M^2$ for every maximal ideal $M$ of $S$, we see that $J_{\sigma^{-1}} g(\sigma^{-1}, \sigma) = J(V) S$ and so $y_{\sigma^{-1}} x_{\sigma^{-1}} = J(V) S$. Therefore $J(V) S \subseteq k_{\sigma} I_{\sigma} f(\sigma, \sigma^{-1})$. Since $v_N(k_{\sigma}) = 0$, we conclude that $f(\sigma, \sigma^{-1}) \not\in N^2$, and so $f(\tau, \tau^{-1}) \not\in M^2 \forall \tau \in G$ and any maximal ideal $M$ of $S$.

Conversely, suppose that $f(\tau, \tau^{-1}) \not\in M^2$ for every maximal ideal $M$ of $S$ and every $\tau \in G$. Let $B = O_l(J(A_f))$, the left order of $J(A_f)$; that is, $B = \{x \in \Sigma_f \mid x J(A_f) \subseteq J(A_f)\}$. Since $\Sigma_f \supseteq B \supseteq A_f$, $B = \sum_{\tau \in G} S k_\tau x_\tau$, for some $k_\tau \in K^\#$. For each $\tau \in G$, we have $S \subseteq S k_\tau$, and we will now show that $S = S k_\tau$. As above, write $J(A_f) = \sum_{\tau \in G} I_{\tau} x_\tau$, with $I_{\tau} = \prod_{\sigma \in G} M$, where the product is taken over all maximal ideals $M$ of $S$ for which $f(\tau, \tau^{-1}) \not\in M$. Observe that $J(V) S = I_{\tau} f(\tau, \tau^{-1}) = k_{\tau} I_{\tau} f(\tau, \tau^{-1})$. Since $f(\tau, \tau^{-1}) \not\in M^2$ for every maximal ideal $M$ of $S$, we must have $I_{\tau} f(\tau, \tau^{-1}) = J(V) S$, and so $J(V) S \subseteq k_{\tau} J(V) S \supseteq J(V) S$ and thus $S = S k_{\tau}$, as desired. This shows that $O_l(J(A_f)) = A_f$ and $A_f$ is hereditary.

Not only can this criterion enable one to rapidly determine whether or not the crossed-product order $A_f$ is hereditary, the utility of the theorem above is now demonstrated by the ease with which the following corollaries of it are obtained.

**Corollary 1.** The crossed-product order $A_f$ is hereditary if and only if $f(\tau, \gamma) \not\in M^2$ for all $\tau, \gamma \in G$ and every maximal ideal $M$ of $S$.

**Proof.** This follows from the cocycle identity $f^\tau(\tau^{-1}, \gamma) f(\tau, \gamma) = f(\tau, \tau^{-1})$.
In other words, the order $A_f$ is hereditary if and only if the values of the two-cocycle $f$ are all square-free.

Since $A_f$ is a maximal order if and only if it is hereditary and primary, by combining our result and results in \cite{1}, we immediately have the following.

**Corollary 2.** Given a crossed-product order $A_f$,

1. it is a maximal order if and only if for every maximal ideal $M$ of $S$, $f(\tau, \tau^{-1}) \not\in M^2$ for all $\tau \in G$, and there exists a set of right coset representatives $g_1, g_2, \ldots, g_r$ of $D_M$ in $G$ (i.e., $G$ is the disjoint union $\bigcup_i D_M g_i$) such that for all $i$, $f(g_i, g_i^{-1}) \not\in M$.
2. if $S$ is a DVR, then it is a maximal order if and only if $f(\tau, \tau^{-1}) \not\in J(S)^2$ for all $\tau \in G$.

**Proof.** In either case, the primarity of $A_f$ is guaranteed by \cite{1} Theorem 3.2 (see also \cite{1} Proposition 2.1(b)] when $S$ is a DVR). □

The Theorem above can readily be put to effective use with the crossed-product orders in \cite{1} §4, for example. In that section, all the crossed-product orders involved are primary orders, and the two-cocycles are given in tabular form, with the values factorized into primes of $S$. Using our criterion, it now becomes a straightforward process to determine which of those orders are maximal orders and which are not, by simply consulting, in each case, the given table of values for the two-cocycle; the table whose entries are all square-free represents a maximal order. This determination can be made with little effort! In fact, if one knows that the crossed-product order $A_f$ is a primary order, then determining whether or not it is a maximal order could even be easier, as the following result shows.

**Corollary 3.** Suppose the crossed-product order $A_f$ is primary. Then it is a maximal order if and only if there exists a maximal ideal $M$ of $S$ such that $f(\tau, \tau^{-1}) \not\in M^2$ for all $\tau \in D_M$.

**Proof.** This follows from \cite{1} Corollary 3.11 and Proposition 2.1(b)]. □

Let $L$ be an intermediate field of $F$ and $K$, let $G_L$ be the Galois group of $K$ over $L$, let $U$ be a valuation ring of $L$ lying over $V$, and let $T$ be the integral closure of $U$ in $K$. Then one can obtain a two-cocycle $f_{L,U}$ : $G_L \times G_L \to T^\#$ from $f$ by restricting $f$ to $G_L \times G_L$ and embedding $S^\#$ in $T^\#$. As before, $A_{f_{L,U}} = \sum_{\tau \in G_L} T x_\tau$ is a $U$-order in $G_{f_{L,U}} = \sum_{\tau \in G_L} K x_\tau = (K/L, G_L, f_{L,U})$.

**Corollary 4.** Suppose the crossed-product order $A_f$ is hereditary. Then $A_{f_{L,U}}$ is a hereditary order in $G_{f_{L,U}}$ for each intermediate field $L$ of $F$ and $K$ and for every valuation ring $U$ of $L$ lying over $V$.

This leads to the following.

**Corollary 5.** Suppose the crossed-product order $A_f$ is hereditary. Then $A_{f_m}$ is a maximal order in $G_{f_m}$ for each maximal ideal $M$ of $S$.

**Proof.** The order $A_{f_m}$ is always primary, by \cite{1} Proposition 2.1(b)]. □

The following example illustrates two limitations of our theory, however.

**Example.** We give two crossed-product orders $A_{f_1}$ and $A_{f_2}$ with $f_1 \sim_K f_2$ and the graphs of $f_1$ and $f_2$ identical, but $A_{f_1}$ is hereditary while $A_{f_2}$ is not. Also, we give an example to demonstrate that the converse of Corollary 5 does not always hold.
Let $F = \mathbb{Q}(x)$, and let $K = \mathbb{Q}(i)(x)$. Then the Galois group $G = \langle \sigma \rangle$ is a group of order two, where $\sigma$ is induced by the complex conjugation on $\mathbb{Q}(i)$. If $V = \mathbb{Q}[x]/(x^2 + 1)$, then $S$ has two maximal ideals, namely $M_1 = (x + i)S$ and $M_2 = (x - i)S$, and $D_{M_1} = D_{M_2} = \{1\}$. Let $f_1, f_2 : G \times G \to S^\#$ be two-cocycles defined by $f_j(1, 1) = f_j(1, \sigma) = f_j(\sigma, 1) = 1$ and $f_1(\sigma, \sigma) = (x^2 + 1)x$, $f_2(\sigma, \sigma) = (x^2 + 1)^2x$.

Then $f_1 \sim_K f_2$, and the subgroup of $G$ associated with either cocycle is $H = \{1\}$, so that the graphs of $f_1$ and $f_2$ are identical. Clearly, $A_{f_1}$ is hereditary but $A_{f_2}$ is not. We conclude that the property that a crossed-product order $A_f$ is hereditary is not an intrinsic property of the graph of $f$.

Also, if we set $f = f_2$, we see that $A_{f_M} = S_M$ for each maximal ideal $M$ of $S$, and therefore $A_{f_M}$ is a maximal order in $\Sigma_{f_M} = K$ for each maximal ideal $M$ of $S$, and yet $A_f$ is not even hereditary (cf. [1] Corollary 3.11 and [2] Theorem 1]). This is the case because $A_f$ is not primary, and also because $f(G \times G) \not\subseteq U(S)$. □

References


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