MUTATION CLASSES OF SKEW-SYMMETRIZABLE 3 × 3 MATRICES

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Abstract. Mutation of skew-symmetrizable matrices is a fundamental operation that first arose in Fomin-Zelevinsky’s theory of cluster algebras; it also appears naturally in many different areas of mathematics. In this paper, we study mutation classes of skew-symmetrizable 3 × 3 matrices and associated graphs. We determine representatives for these classes using a natural minimality condition, generalizing and strengthening results of Beineke-Brustle-Hille and Felikson-Shapiro-Tumarkin. Furthermore, we obtain a new numerical invariant for the mutation operation on skew-symmetrizable matrices of arbitrary size.

1. Introduction

Mutation of skew-symmetrizable matrices is a fundamental operation that first arose in Fomin-Zelevinsky’s theory of cluster algebras; it also appears naturally in many different areas of mathematics. Mutation can also be naturally viewed as an operation on certain graphs, called diagrams. In this paper, we study mutation classes of skew-symmetrizable 3 × 3 matrices and their diagrams. We determine representatives for these classes using a natural minimality condition, generalizing and strengthening results of [3], [5]. Furthermore, we obtain a new numerical invariant for the mutation operation on skew-symmetrizable matrices of arbitrary size.

To state our results, we need some terminology. Let us recall that an integer matrix \( B \) is skew-symmetrizable if \( DB \) is skew-symmetric for some diagonal matrix \( D \) with positive diagonal entries.

For any matrix index \( k \), the mutation of a skew-symmetrizable matrix \( B \) at \( k \) is another skew-symmetrizable matrix \( \mu_k(B) = B' \):

\[
B' = \begin{cases} 
B'_{i,j} = -B_{i,j} & \text{if } i = k \text{ or } j = k \\
B'_{i,j} = B_{i,j} + \text{sgn}(B_{i,k})[B_{i,k}B_{k,j}]_+ & \text{else}
\end{cases}
\]

(where we use the notation \( [x]_+ = \max\{x, 0\} \) and \( \text{sgn}(x) = x/|x| \) with \( \text{sgn}(0) = 0 \)). Mutation is an involutive operation, so repeated mutations give rise to the mutation-equivalence relation on skew-symmetrizable matrices.

On the other hand, motivated by the Dynkin diagram construction in the theory of Kac-Moody algebras [9], for any skew-symmetrizable \( n \times n \) matrix \( B \), a directed...
graph $\Gamma(B)$, called the diagram of $B$, is associated in \[7\] as follows: the vertices of $\Gamma(B)$ are the indices $1, 2, \ldots, n$ such that there is a directed edge from $i$ to $j$ if and only if $B_{ij} > 0$, and this edge is assigned the weight $|B_{ij}B_{ji}|$. Let us note that if $B$ is not skew-symmetric, then the diagram $\Gamma(B)$ does not determine $B$ as there could be several different skew-symmetrizable matrices whose diagrams are equal; however, if a skew-symmetrizing matrix $D$ is fixed, then $\Gamma(B)$ determines $B$. In any case, we use the general term diagram to mean the diagram of a skew-symmetrizable matrix. Then the mutation $\mu_k$ can naturally be viewed as a transformation on diagrams (see Section \[2\] for a description). In the particular case where the vertex $k$ is a source (resp. sink), i.e. all incident edges are oriented away (resp. towards) $k$, then $\mu_k$ acts by only reversing all edges incident to $k$; in that case we also call $\mu_k$ a reflection (as in classical Bernstein-Gelfand-Ponomarev reflection functors). Note also that if $B$ is skew-symmetric, then the diagram $\Gamma(B)$ may be viewed as a quiver and the corresponding mutation operation is also called quiver mutation. There are several categorical interpretations of the quiver mutation; we refer to \[10\] for a survey.

Given the appearance of the mutation operation in many different areas of mathematics, it is natural to study properties of the mutation classes of skew-symmetrizable matrices and the associated diagrams. Currently, a description of these classes is known for finite and affine types \[2, 12\]. There is also a classification for the so-called finite mutation type diagrams \[6\]. In this paper, we consider the next basic case of size 3 skew-symmetrizable matrices, which is crucial to understand the mutation operation in general size. To be able to state our results, let us recall a little bit more terminology. By a subdiagram of $\Gamma$, we always mean a diagram obtained from $\Gamma$ by taking an induced (full) directed subgraph on a subset of vertices and keeping all its edge weights the same as in $\Gamma$. By a cycle we mean a subdiagram whose vertices can be labeled by elements of $\mathbb{Z}/m\mathbb{Z}$ so that the edges between them are precisely $\{i, i + 1\}$ for $i \in \mathbb{Z}/m\mathbb{Z}$. We call a diagram $\Gamma$ mutation-acyclic if it is mutation-equivalent to an acyclic diagram (i.e. a diagram which has no oriented cycles at all); otherwise we call it mutation-cyclic. Now we can state our first main result:

**Theorem 1.1.** Suppose that $M$ is a mutation class of diagrams with 3 vertices. For any $\Gamma$ in $M$, let $s(\Gamma)$ denote the sum of the square roots of the weights in $\Gamma$. Then there is a diagram $\Gamma_0$ in $M$ such that $s(\Gamma_0)$ is minimal. Furthermore, we have the following:

(i) If $M$ is a mutation class of mutation-cyclic diagrams, then $\Gamma_0$ is unique up to a change of orientation which reverses all edges (and up to an enumeration of vertices).

(ii) If $M$ is the mutation class of an acyclic diagram, then $\Gamma_0$ is acyclic and it is unique up to a reflection at a source or sink (and up to an enumeration of vertices).

Note that in this theorem part (ii) generalizes and strengthens \[5, Theorem 9.1\] (which claims uniqueness up to an arbitrary change of orientation for quivers). In fact, part (ii) establishes a special case (rank three) of a standard conjecture of cluster algebra theory \[8, Conjecture 4.14 (4)\], which states that mutation-equivalent acyclic diagrams can be obtained from each other by a sequence of reflections at sources or sinks. For quivers, this conjecture was obtained in \[4, Corollary 4\] using categorical methods. We use more elementary algebraic-combinatorial methods.
Let us also note that a numerical criterion to check whether a given diagram with three vertices is mutation-acyclic has been obtained by the author in [13]. (This criterion is recalled in Theorem 2.6.)

We also characterize Γ₀ using a “local” property, generalizing [3, Lemma 2.1] and [5, Theorem 9.1(3))]:

**Theorem 1.2.** Suppose that M is a mutation class of diagrams with 3 vertices. Let Γ₀ be the diagram in M such that s(Γ₀) is minimal as in Theorem 1.1. Then we have the following:

(i) Γ₀ is the unique, up to the same conditions as in Theorem 1.1, diagram in M such that, for each vertex i, we have s(Γ₀) ≤ s(μᵢ(Γ₀)).

(ii) For each Γ in M, there is a (possibly empty) sequence {μᵢ} of mutations with Γ₀ = μ₁...μₙ(Γ) such that for Γᵢ₋₁ = μ₁...μₙ(Γ) we have s(Γᵢ₋₁) < s(Γᵢ); here i = 1, ..., n with Γₙ = Γ. Furthermore, if Γ is mutation-cyclic, then the sequence {μᵢ} is uniquely determined: specifically, the vertex i is the vertex which is not incident to the edge with maximal¹ weight in Γᵢ₊₁, i = 0, 1, ..., n − 1.

Conversely, for any maximal sequence {μᵢ : i = 1, ..., n} such that s(Γᵢ₋₁) < s(Γᵢ) with Γᵢ₋₁ = μ₁...μₙ(Γ) and Γₙ = Γ, we have Γ₀ = μ₁...μₙ(Γ).

We also obtain the following result, which gives a new numerical invariant for the mutation of diagrams with any number of vertices.

**Theorem 1.3.** Suppose that Γ is a diagram with n vertices. For any vertex i in Γ, let δᵢ = δᵢ(Γ) be the greatest common divisor of the weights of the edges which are incident to i. Let δ(Γ) = (δ₁, δ₂, ..., δₙ) be the ordered sequence of these greatest common divisors such that δ₁ ≥ δ₂ ≥ ... ≥ δₙ. Then for any Γ' which is mutation-equivalent to Γ, we have δ(Γ) = δ(Γ').

Note that if Γ is the diagram of a skew-symmetric matrix, then the same conclusion holds if δᵢ is defined as the greatest common divisor of the radicals of the weights of the edges which are incident to the vertex i. (Equivalently, in the quiver notation that represents skew-symmetric matrices, the conclusion of the theorem holds if δᵢ is defined as the greatest common divisor of the number of arrows in the edges which are incident to the vertex i.)

We prove our results in Section 3 after some preparation in Section 2.

2. Preliminaries

In this section, we will recall some more terminology and prove some statements that we will use to prove our results. First, let us recall that the diagram of a skew-symmetrizable (integer) matrix has the following property:

(2.1) the product of weights along any cycle is a perfect square, i.e. the square of an integer.

Thus we can use the term diagram to mean a directed graph, with no loops or two-cycles, such that the edges are weighted with positive integers satisfying (2.1). Let us note that if an edge in a diagram has weight equal to one, then we do not specify its weight in the picture.

¹We will show that Γᵢ₊₁ has a unique edge with maximal weight (Lemma 3.3).
For any vertex $k$ in a diagram $\Gamma$, the associated mutation $\mu_k$ changes $\Gamma$ as follows [7]:

- The orientations of all edges incident to $k$ are reversed, their weights intact.
- For any vertices $i$ and $j$ which are connected in $\Gamma$ via a two-edge oriented path going through $k$ (see Figure 1), the direction of the edge $\{i, j\}$ in $\mu_k(\Gamma)$ and its weight $\gamma'$ are uniquely determined by the rule

$$\pm \sqrt{\gamma} \pm \sqrt{\gamma'} = \sqrt{\alpha \beta},$$

where the sign before $\sqrt{\gamma}$ (resp., before $\sqrt{\gamma'}$) is “+” if $i, j, k$ form an oriented cycle in $\Gamma$ (resp., in $\mu_k(\Gamma)$), and is “−” otherwise. Here either $\gamma$ or $\gamma'$ can be equal to 0, which means that the corresponding edge is absent.

- The rest of the edges and their weights in $\Gamma$ remain unchanged.

This operation is involutive, i.e. $\mu_k(\mu_k(\Gamma)) = \Gamma$, so it defines an equivalence relation on the set of all diagrams. More precisely, two diagrams are called mutation-equivalent if they can be obtained from each other by applying a sequence of mutations. The mutation class of a diagram $\Gamma$ is the set of all diagrams which are mutation-equivalent to $\Gamma$. If $B$ is a skew-symmetrizable matrix, then $\Gamma(\mu_k(B)) = \mu_k(\Gamma(B))$ (see Section 1 for the definition of $\mu_k(B)$). Let us note that if $B$ is not skew-symmetric, then the diagram $\Gamma(B)$ does not determine $B$ as there could be several different skew-symmetrizable matrices whose diagrams are equal; however, if a skew-symmetrizing matrix $D$ is fixed, then $\Gamma(B)$ determines $B$, so the mutation class of $\Gamma(B)$ determines that of $B$ (the matrix $\mu_k(B)$ shares the same skew-symmetrizing matrix $D$ with $B$ [7]).

In this paper, we will mainly consider diagrams with exactly three vertices. Therefore it will be convenient for us to use special notation for these diagrams, generalizing that used in [3] and [5]:

**Definition 2.1.** Suppose that $\Gamma$ is a three-vertex diagram with weights $\alpha, \beta$ and $\gamma$. Let $a = \sqrt{\alpha}, b = \sqrt{\beta}$ and $c = \sqrt{\gamma}$. We call $a, b, c$ the radical weights of $\Gamma$ and use the following notation: if $\Gamma$ is acyclic we write $\Gamma = (a, b, c)^{-}$; if $\Gamma$ is cyclic we write $\Gamma = (a, b, c)$, without considering any particular ordering. We denote by $s(\Gamma)$ the sum of the radical weights of $\Gamma$. By the definition of a diagram, the product of the radical weights is an integer by (2.1).

Note that if $B$ is a skew-symmetric matrix, then the radical weights of $\Gamma(B)$ are equal to the positive entries of $B$. Also note that this notation does not uniquely determine $\Gamma$. Nevertheless it is convenient for us because it behaves well under the mutation operation:

**Proposition 2.2.** Suppose that $\Gamma$ is a diagram with three vertices. Then we have the following:

(i) If $\Gamma = (a, b, c)^{-}$ and $k$ is a vertex which is a source or sink in $\Gamma$, then $\mu_k(\Gamma) = (a, b, c)^{-}$.
(ii) Suppose that $\Gamma = (a, b, c)^-$ and $k$ is a vertex which is neither a source nor sink in $\Gamma$. Also assume that $k$ is not incident to the edge with radical weight $c$. Then $\mu_k(\Gamma) = (a, b, c + ab)$

(iii) Suppose that $\Gamma = (a, b, c)$ and $k$ is the vertex which is not incident to the edge with radical weight $c$. Then $\mu_k(\Gamma) = (a, b, ab - c)$ (resp. $\mu_k(\Gamma) = (a, b, c - ab)^-$) provided $c < ab$ (resp. $c \geq ab$).

The proposition follows from the definition of the mutation operation and the following technical statement, providing skew-symmetrization by conjugation:

**Lemma 2.3** ([7 Proposition 8.1]). Let $B$ be a skew-symmetrizable (integer) matrix. Then there exists a diagonal matrix $H$ with positive diagonal entries such that $HBH^{-1}$ is skew-symmetric. Furthermore, the matrix $S(B) = (S_{ij}) = HBH^{-1}$ is uniquely determined by $B$. Specifically, the matrix entries of $S(B)$ are given by

\[
S_{ij} = \text{sgn}(B_{ij})\sqrt{|B_{ij}B_{ji}|}.
\]

Furthermore, for any matrix index $k$, we have $S(\mu_k(B)) = \mu_k(S(B))$.

The matrix $H$ can be taken as $D^{1/2}$, where $D$ is a skew-symmetrizing matrix for $B$.

Let us also record the following statements for convenience; they can be checked easily using the definition of the mutation operation:

**Proposition 2.4.** Suppose that $\Gamma$ is a diagram with three vertices. Then we have the following:

(i) $s(\Gamma) = s(\mu_k(\Gamma))$ if and only if $\Gamma$ and $\mu_k(\Gamma)$ have the same weights; furthermore:

(a) the diagrams $\Gamma$ and $\mu_k(\Gamma)$ are both cyclic or both acyclic;
(b) if $\Gamma$ is acyclic, then the vertex $k$ is a source or sink in $\Gamma$.

(ii) $s(\Gamma) > s(\mu_k(\Gamma))$ if and only if the edge which is not incident to the vertex $k$ has smaller weight in $\mu_k(\Gamma)$ than in $\Gamma$ (the weights of the remaining edges are equal).

**Proposition 2.5.** Suppose that $\Gamma$ is a three-vertex diagram which has an edge whose weight is less than 4. Then $\Gamma$ is mutation-acyclic.

**Proof.** Suppose that $\Gamma = (a, b, c)$ is cyclic such that $c \leq a, b$ (so $c < 2$; thus $c = 1, \sqrt{2}$ or $\sqrt{3}$). Let us also assume, without loss of generality, that $a \leq b$. Let $i$ (resp. $j$) be the vertex which is not incident to the edge with radical weight $b$ (resp. $a$). If $c = 1$, then $\mu_i(\Gamma) = (a, b - a, c)^-$ is acyclic. Let us assume now that $c = \sqrt{2}$. If $ac \leq b$, then $\mu_i(\Gamma) = (a, b - ac, c)^-$ is acyclic; otherwise $\mu_j\mu_i(\Gamma) = (bc - a, ac - b, c)^-$ is acyclic (because $bc - a > 0$ for our assumption $a \leq b$ and $c = \sqrt{2} > 1$). For $c = \sqrt{3}$ we use a similar argument: if $ac \leq b$, then $\mu_i(\Gamma) = (a, ac - b, c)^-$ is acyclic; otherwise either $\mu_j\mu_i(\Gamma) = (bc - 2a, ac - b, c)^-$ is acyclic or (i.e. if $bc - 2a < 0$) $\mu_i\mu_j\mu_i(\Gamma) = (2a - bc, 2b - ac, c)^-$ is acyclic (note $2b - ac > 0$ because $b \geq a$ and $2 > c$). This completes the proof. \(\square\)

Determining whether a given diagram is mutation-acyclic or not is a natural problem in the theory of cluster algebras and related topics. For diagrams with three vertices, a numerical criterion for being mutation-acyclic has been obtained by the author in [13], using the notion of a quasi-Cartan companion. For the convenience of the reader, we will recall this criterion. First let us recall that
a quasi-Cartan companion of a skew-symmetrizable matrix $B$ is a symmetrizable matrix $A$ whose diagonal entries are equal to 2 and whose off-diagonal entries differ from the corresponding entries of $B$ only by signs. A quasi-Cartan companion $A$ of a skew-symmetrizable matrix $B$ is called admissible if it satisfies the following sign condition: for any cycle $Z$ in $\Gamma(B)$, the product $\prod_{(i,j) \in Z} (-A_{i,j})$ over all edges of $Z$ is negative if $Z$ is oriented and positive if $Z$ is nonoriented. The main examples of admissible companions are the generalized Cartan matrices: if $\Gamma(B)$ is acyclic, i.e. has no oriented cycles at all, then the quasi-Cartan companion $A$ with $A_{i,j} = -|B_{i,j}|$, for all $i \neq j$, is admissible. However, for an arbitrary skew-symmetrizable matrix $B$, an admissible quasi-Cartan companion may not exist; if it exists it is unique up to simultaneous sign changes in rows and columns. For any skew-symmetrizable matrix $B$ of size 3, an admissible quasi-Cartan companion exists and it determines whether its diagram $\Gamma(B)$ is mutation-acyclic:

**Theorem 2.6** ([13] Theorem 2.6). Suppose that $B$ is a skew-symmetrizable matrix of size 3 and let $A$ be an admissible quasi-Cartan companion of $B$. Then $\Gamma(B)$ is mutation-acyclic if and only if one of the following holds:

1. $\det(A) > 0$ and $A$ is positive\(^2\)
2. $\det(A) = 0$ and $A$ is semipositive of corank 1,
3. $\det(A) < 0$.

Let us note that parts (i) and (ii) occur if and only if $\Gamma(B)$ is mutation-equivalent to a Dynkin and an extended Dynkin diagram respectively ([2] [12]) (here a Dynkin diagram is an orientation of a Dynkin graph). Let us also mention that the main ingredient in proving the theorem is an extension of the mutation operation to quasi-Cartan companions; we refer to [12] Section 2 for details.

The previous Theorem 2.6 was obtained in [13] as a nontrivial generalization of a characterization in [3] for skew-symmetric matrices of size 3, using a polynomial called the Markov constant. More explicitly, for a skew-symmetric $3 \times 3$ matrix $B$ with $\Gamma(B) = (x,y,z)$, the associated Markov constant is defined as $C(B) = C(x,y,z) = x^2 + y^2 + z^2 - xyz$. Then skew-symmetric matrices with mutation-acyclic diagrams can be characterized as follows:

**Theorem 2.7** ([3] Theorem 1.1). Suppose that $B$ is a skew-symmetric (integer) matrix such that $\Gamma(B) = (x,y,z)$ is cyclic. Then the following are equivalent:

1. $\Gamma(B)$ is mutation-acyclic.
2. The Markov constant satisfies $C(x,y,z) > 4$ or $\min\{x,y,z\} < 2$.
3. The Markov constant satisfies $C(x,y,z) > 4$ or the triple $(x,y,z)$ is in the following list (where we assume $x \geq y \geq z$):
   a) $C(x,y,z) = 0$: $(x,y,z) = (0,0,0)$,
   b) $C(x,y,z) = 1$: $(x,y,z) = (1,0,0)$,
   c) $C(x,y,z) = 2$: $(x,y,z) = (1,1,0)$ or $(1,1,1)$,
   d) $C(x,y,z) = 4$: $(x,y,z) = (2,0,0)$ or $(2,1,1)$.

Note that a generalization of this theorem to skew-symmetrizable matrices is not immediate because the Markov constant is not defined for non-skew-symmetric matrices; it is also not defined for skew-symmetric matrices whose diagrams are acyclic. It was observed in [13] that for a skew-symmetric matrix $B$ of size 3 and

\(^2\)A is called (semi)positive if $DA$ is positive (semi)definite, where $D$ is a symmetrizing matrix of $A$. 

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an admissible quasi-Cartan companion $A$ of $B$, we have $\det A = 2(4-C(B))$, leading to Theorem 2.6. For skew-symmetrizable matrices of arbitrary size, it seems that one needs to consider the admissible quasi-Cartan companion itself rather than just its determinant; see [2] for a conjecture.

3. Proofs of main results

First we will prove some lemmas that we use to prove our theorems. In the proofs we assume, without loss of generality, that all diagrams are connected. The following statement generalizes [3, Lemma 2.1c].

Lemma 3.1. Suppose that $\Gamma$ is a three-vertex diagram. If there are vertices $i \neq j$ such that $s(\mu_i(\Gamma)) < s(\Gamma)$ and $s(\mu_j(\Gamma)) \leq s(\Gamma)$, then $\Gamma$ has an edge whose weight is less than 4 and it is mutation-acyclic.

Proof. Let us first note that $\Gamma$ is not acyclic (otherwise for any vertex $k$, we have $s(\mu_k(\Gamma)) \geq s(\Gamma)$), so we suppose that $\Gamma = (a, b, c)$ is cyclic. We assume that $i$ is not incident to the edge with radical weight $a$ and $j$ is not incident to the edge with radical weight $b$. Let us first assume that $\mu_i(\Gamma)$ and $\mu_j(\Gamma)$ are both cyclic.

Then $\mu_i(\Gamma) = (-a + bc, b, c)$ and $\mu_j(\Gamma) = (a, -b + ac, c)$. The conditions of the lemma imply that $bc < 2a$ and $ac \leq 2b$ (Proposition 2.4(ii)). Multiplying the first inequality by $c$, we have $bc^2 < 2ac$; on the other hand, by the second inequality, we have $2ac \leq 4b$, implying that $c^2 < 4$. Then by Proposition 2.5 the diagram $\Gamma$ is mutation-acyclic.

Let us now assume that one of $\mu_i(\Gamma)$ or $\mu_j(\Gamma)$ is cyclic; without loss of generality, suppose that $\mu_i(\Gamma)$ is cyclic. Then $a \geq bc$. If $\mu_j(\Gamma)$ is not acyclic, then by the condition of the lemma we have $ac \leq 2b$. Then $ac^2 \leq 2bc \leq 2a$, implying that $c^2 \leq 2$. Similarly if $\mu_i(\Gamma)$ is acyclic, then $ac \leq b$, implying $a \leq ac^2 \leq bc \leq a$, so $c = 1$ (and $a = b$). In any case, by Proposition 2.5 the diagram $\Gamma$ is mutation-acyclic. This completes the proof. \qed

In view of the previous lemma, the following is a special case of Theorem 1.1(ii):

Lemma 3.2. Suppose $\Gamma$ is a three-vertex diagram which has an edge whose weight is less than 4. If $\Gamma$ is cyclic, then it has a vertex $k$ such that $s(\Gamma) > s(\mu_k(\Gamma))$; in particular, $s(\Gamma)$ is not minimal.

Proof. Suppose that $\Gamma = (a, b, c)$ with $a^2 < 4$ (so $a = 1, \sqrt{2}, \sqrt{3}$). If the conclusion of the lemma is not satisfied, then we have $bc \geq 2a$, $ac \geq 2b$ and $ab \geq 2c$ (here $\mu_i(\Gamma)$ is cyclic for any vertex $i$ in $\Gamma$). Note that from the last inequality we have $b \geq 2c/a$. On the other hand, from the second inequality, we have $c \geq 2b/a$, implying $b \geq 4b/a^2$; however, this is not possible because $a^2 < 4$. This completes the proof. \qed

Let us now show the following special case of Theorem 1.2(ii):

Lemma 3.3. Suppose that $\Gamma$ is mutation-cyclic and has a vertex $i$ such that $s(\mu_i(\Gamma)) < s(\Gamma)$ (in particular, $s(\Gamma)$ is not minimal). Then $i$ is the unique vertex with this property. Furthermore $\Gamma$ has a unique edge $e$ with maximal weight: the vertex $i$ is the vertex which is not incident to $e$.

Note that the statement may not be true if $\Gamma$ is mutation-acyclic (e.g., for $\Gamma = (1, 3, 5)$).
Proof. The first part of the statement follows from Lemma 3.1. For the second part, suppose $\Gamma = (a, b, c)$ and assume that $i$ is the vertex which is not incident to the edge with radical weight $c$. Then, by the first part, $ba < 2c$ but $bc \geq 2a$ and $ac \geq 2b$. Multiplying the second inequality by $a$ we have $abc \geq 2a^2$. Since $2c^2 > abc$, we have $2c^2 > 2a^2$; thus $c > a$. Similarly $c > b$. This completes the proof. □

3.1. Proof of Theorems 1.1 and 1.2. We will first prove Theorem 1.1 and Theorem 1.2(i) both at the same time. For this, let us first note that the mutation class of $\Gamma$ obviously contains a diagram $\Gamma_0$ such that $s(\Gamma_0)$ is minimal. To show its uniqueness, note that if $s(\Gamma)$ is minimal, then it satisfies:

(*) $s(\Gamma) \leq s(\mu_i(\Gamma))$ for any vertex $i$.

We will show that, in the mutation class of $\Gamma$, there is a unique, up to a change of orientation as described in the statement of Theorem 1.1, diagram $\Gamma_0$ such that $s(\Gamma_0)$ is minimal. To show its uniqueness, note that if $s(\Gamma)$ is minimal, then it satisfies:

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(*) $s(\Gamma) \leq s(\mu_i(\Gamma))$ for any vertex $i$.
so the conclusion of the theorem holds, or $\Gamma_2 = \mu_k(\Gamma_1) = (a, a, a^2 - 1)$. Let us note that, in the case where $\Gamma_2 = \mu_k(\Gamma_1) = (a, a, a^2 - 1)$, we have the following: if $a > 1$, then $\Gamma_2 = \mu_k(\Gamma_1)$ does not have any edge whose weight is less than 4, contradicting (3.2) (here $a \geq 2$ because $a$ is an integer in this case); if $a = 1$, then $\Gamma_2 = \mu_k(\Gamma_1)$ is equal to $\Gamma_0$ up to an enumeration of vertices as claimed.

Suppose now that $\Gamma_0$ is not a tree, say $\Gamma_0 = (1, a, b)^-$. Note that if the vertex 1 is not incident to any edge whose weight is 1, then $\Gamma_1 = \mu_1(\Gamma_0) = (a, b, 1 + ab)$ with $a, b \geq 2$, contradicting our assumption that $\Gamma_1$ has an edge whose weight is less than 4 (3.2). Thus, in $\Gamma_0$, we can assume that the vertex 1 is incident to an edge of weight 1. The remaining radical weights are $a, b$ (so $\Gamma_0$ has a source or sink whose incident edges have radical weights $a, b$).

To proceed, let us first consider the subcase where $a$ or $b$ is equal to one; without loss of generality, say $a = 1$. Then it can be checked easily that, for $i = 1, \ldots, n - 1$, we have $\Gamma_i = (1, 1, a + 1)$ or $\Gamma_i = (1, a, a + 1)$ (so $\Gamma_n = \Gamma_0 = (1, 1, a)^-$) and $\Gamma_0$ is equal to $\Gamma_0$ up to an enumeration of vertices possibly after a reflection at a source or sink.

Let us now assume that $a$ and $b$ are greater than one. We may assume without loss of generality that (in $\Gamma_0$) the vertex 1 is not incident to the edge with radical weight $a$ (otherwise it is incident to the edge with radical weight $b$; then we exchange the letters $a$ and $b$). Let us also assume that $\Gamma_0 = (1, a, b)^-$ such that the edge 2 $\rightarrow$ 1 has radical weight 1, the edge 1 $\rightarrow$ 3 has radical weight $b$ and the edge 2 $\rightarrow$ 3 has radical weight $a$. Then $\Gamma_1 = \mu_1(\Gamma_0) = (1, a + b, b)$. Now, the diagram $\Gamma_2$ is obtained by mutating $\Gamma_1$ at a vertex $v \neq 1$ such that $v$ is incident to an edge with weight one (otherwise $\mu_v(\Gamma_1)$ does not have any edge whose weight is less than 4), so $v$ is the edge 1 $\rightarrow$ 2 in $\Gamma_1$; thus $v = 2$. Then $\Gamma_2 = \mu_2(\Gamma_1) = (1, a + b, a)$. Similarly, the diagram $\Gamma_3$ is obtained by mutating $\Gamma_2$ at a vertex $w \neq 2$ such that $w$ is incident to an edge with weight one. Then $w = 1$ (because the vertex 3 is incident to the edges with radical weights $a, b, a$, which are greater than or equal to 2). Then $\Gamma_3 = \mu_1(\Gamma_1) = (1, b, a)^-$, where the edge 1 $\rightarrow$ 2 has radical weight 1, the edge 3 $\rightarrow$ 1 has radical weight $a$ and the edge 3 $\rightarrow$ 2 has radical weight $b$. Thus $\Gamma_0 = \Gamma_3 = (1, b, a)^-$. Note then that $\Gamma_0'$ can be obtained from $\Gamma_0$ by first applying the reflection at the vertex 3, then exchanging (renumbering) the vertices 1 and 2. This completes the case.

Case 2 ($\Gamma_0$ is not skew-symmetric and has an edge of weight one in $\Gamma_0$). Let us denote this edge by $e$. As in the previous case, let us first suppose that $\Gamma_0 = (a, 1, 0)^-$ is a tree. Then $\Gamma_1 = \mu_1(\Gamma_0) = (a, a, 1)$ (recall that 1 is the vertex which is neither a source nor a sink in $\Gamma_0$). For convenience, let us first assume that $a \geq 2$. Then, for any vertex $k \neq 1$, either $\Gamma_2 = \mu_k(\Gamma_1)$ is equal to $\Gamma_0$ up to an enumeration of vertices, so the conclusion of the theorem holds, or $\Gamma_2 = \mu_k(\Gamma_1) = (a, a, a^2 - 1)$, which does not have any edge whose weight is less than 4, contradicting (3.2). Let us now assume that $a < 2$, so $a = \sqrt{2}$ or $a = \sqrt{3}$. If $a = \sqrt{2}$, then for any vertex $k \neq 1$ either $\Gamma_2 = \mu_k(\Gamma_1)$ is equal to $\Gamma_0$ up to an enumeration of vertices (so the conclusion of the theorem holds) or $\Gamma_2 = \mu_k(\Gamma_1) = (\sqrt{2}, \sqrt{2}, 1)$. Similarly, in the case where $\Gamma_2 = \mu_k(\Gamma_1) = (\sqrt{2}, \sqrt{2}, 1)$, we have $\Gamma_3 = \mu_j(\Gamma_2)$ for some $j \neq k$; furthermore $\Gamma_3$ is equal to $\Gamma_0$ or it is equal to $\Gamma_2$ up to an enumeration of vertices. Continuing by induction, we have the following conclusion (when $a = \sqrt{2}$): for all $i = 1, \ldots, n - 1$, $\Gamma_i = (\sqrt{2}, \sqrt{2}, 1)$ and $\Gamma_0'$ is equal to $\Gamma_0$ up to an enumeration of vertices. If $a = \sqrt{3}$, ...
by a similar argument, we have the following: for all \( i = 1, \ldots, n-1 \), \( \Gamma_i = (\sqrt{3}, \sqrt{3}, 1) \) or \( \Gamma_i = (\sqrt{3}, \sqrt{3}, 2) \), and \( \Gamma'_0 \) is equal to \( \Gamma_0 \) up to an enumeration of vertices.

Let us now suppose that \( \Gamma_0 = (a, b, \frac{a+b}{2})^{-} \) is not a tree. Note that since \( \Gamma_0 \) is not skew-symmetric, the numbers \( a, b \) are not integers (but square-roots of integers). For convenience, we consider in subcases:

**Subcase 2.1.** \( a \) or \( b \) is equal to \( \sqrt{2} \).

Let us assume, without loss of generality, that \( a = \sqrt{2} \). (Note then that \( b = m\sqrt{2} \), where \( m \) is an integer.) Then, by similar arguments as in Case 1 above, it follows that \( \Gamma_i, 1 \leq i \leq n-1 \) (in fact \( n \leq 5 \)), belongs to one of the following types (in the notation of Definition \( \ref{def:types} \)):

- \( (\sqrt{2}, b, \sqrt{2}b + 1) \);
- \( (\sqrt{2}, b + \sqrt{2}, \sqrt{2}b + 1) \);
- \( (\sqrt{2} + b, \sqrt{2}, 1) \);
- \( (\sqrt{2} + b, 1, b) \) such that \( \Gamma_0' \) can be obtained from \( \Gamma_0 \) possibly after enumerating the vertices and reflecting at a source or sink.

**Subcase 2.2.** \( a \) or \( b \) is equal to \( \sqrt{3} \).

Let us assume, without loss of generality, that \( a = \sqrt{3} \). Then, by similar arguments as in Case 1 above, it follows that \( \Gamma_i, 1 \leq i \leq n-1 \) (in fact \( n \leq 6 \)), is of one of the following types:

- \( (\sqrt{3}, b, 1 + \sqrt{3}b) \);
- \( (\sqrt{3}, \sqrt{3}b + 2b + \sqrt{3}b, 1 + \sqrt{3}b) \);
- \( (\sqrt{3}, \sqrt{3}b + 2 + \sqrt{3}b, 2 + \sqrt{3}b) \);
- \( (\sqrt{3}, \sqrt{3}b, b + 1) \);
- \( (b, \sqrt{3} + b, 1) \) such that \( \Gamma_0' \) can be obtained from \( \Gamma_0 \) possibly after enumerating the vertices and reflecting at a source or sink.

**Subcase 2.3.** \( a, b \geq 2 \).

Note that if the vertex 1 is not incident to \( e \), then \( \Gamma_1 = \mu_1(\Gamma_0) = (a, b, ab + 1) \), so \( \Gamma_1 \) does not have any edge whose weight is less than four, contradicting our assumption. Thus in this case the vertex 1 is incident to \( e \). Then \( \Gamma_i, 1 \leq i \leq n-1 \) (in fact \( n = 3 \)), is of type \( (1, a + b, b) \) or \( (1, a + b, a) \) such that \( \Gamma_0' \) can be obtained from \( \Gamma_0 \) by a reflection at a source or sink and renumbering the vertices if necessary.

**Case 3 (\( \Gamma_0 \) is not skew-symmetric and the minimal edge weight is equal to 2).** Let us write \( \Gamma_0 = (\sqrt{2}, a, b)^- \), where \( a, b \geq \sqrt{2} \); if \( \Gamma_0 \) is a tree, then we take \( b = 0 \).

**Subcase 3.1.** \( a \) or \( b \) is equal to \( \sqrt{2} \).

Let us assume, without loss of generality, that \( a = \sqrt{2} \). If \( \Gamma_0 = (\sqrt{2}, \sqrt{2}, 0)^- \) is a tree, then it is easily checked (under the assumption \( \ref{assumption:tree} \)) that \( \Gamma_i = (\sqrt{2}, \sqrt{2}, 2), \quad i = 1, \ldots, n-1 \), and \( \Gamma'_0 \) is as required in the conclusion of the uniqueness claims in the theorems. Let us now assume that \( \Gamma_0 \) is not a tree. (Note then that \( b \) is an integer.) Then \( \Gamma_i, 1 \leq i \leq n-1 \) (in fact \( n = 4 \)), belongs to one of the following types:

- \( (\sqrt{2}, \sqrt{2} + \sqrt{2}b, b) \);
- \( (\sqrt{2}, \sqrt{2} + \sqrt{2}b, b + 2) \);
- \( (\sqrt{2}, \sqrt{2}, b + 2) \) such that \( \Gamma_0' \) can be obtained from \( \Gamma_0 \) by a reflection at a source or sink and renumbering the vertices if necessary.

**Subcase 3.2.** \( a \) or \( b \) is equal to \( \sqrt{3} \).

Let us assume, without loss of generality, that \( a = \sqrt{3} \). If \( \Gamma_0 \) is a tree, then it is easily checked that, for \( 1 \leq i \leq n-1 \), \( \Gamma_i = (\sqrt{3}, \sqrt{6}, \sqrt{3}) \); \( 2\sqrt{2}, \sqrt{6}, \sqrt{3} \) and the uniqueness conclusion of the theorems is satisfied. Let us now assume that \( \Gamma_0 \) is not a tree. (Note then that \( b = \sqrt{2}\sqrt{3}m \), where \( m \) is an integer.) Then \( \Gamma_i, 1 \leq i \leq n-1 \) (in fact \( n \leq 6 \)), is of one of the following types:

- \( (\sqrt{3}, \sqrt{2} + \sqrt{3}b, b) \);
- \( (\sqrt{3}, \sqrt{2} + \sqrt{3}b, \sqrt{6} + 2b) \);
- \( (\sqrt{3}, 2\sqrt{2} + b\sqrt{3}, \sqrt{6} + 2b) \);
- \( (\sqrt{3}, 2\sqrt{2} + b\sqrt{3}, \sqrt{6} + b) \);
(\sqrt{3}, \sqrt{2}, \sqrt{6} + b); (\sqrt{2}, \sqrt{3} + \sqrt{2}b, b); (\sqrt{2}, \sqrt{3} + \sqrt{2}b, \sqrt{6} + b) such that \( \Gamma' \) can be obtained from \( \Gamma_0 \) by a reflection at a source or sink and renumbering the vertices if necessary.

**Subcase 3.3.** \( a, b \geq 2 \).

Note that the vertex 1 is not incident to the edge with radical weight \( \sqrt{2} \), because otherwise \( \Gamma_1 = \mu_1(\Gamma_0) = (\sqrt{2} + ab, a, b) \) does not have any edge whose weight is less than four, contradicting our assumption. Then, by similar arguments as in Case 1, it follows that \( \Gamma_1, 1 \leq i \leq n - 1 \) (in fact \( n = 4 \)), belongs to one of the following types: \( (\sqrt{2}, a, b + \sqrt{2}a); (\sqrt{2}, a + \sqrt{2}b, b + \sqrt{2}a); (\sqrt{2}, a + \sqrt{2}b, b) \) such that \( \Gamma'_0 \) can be obtained from \( \Gamma_0 \) by a reflection at a source or sink and renumbering the vertices if necessary.

**Case 4.** \( \Gamma_0 \) is not skew-symmetric and the minimal edge weight is equal to 3). Let us write \( \Gamma_0 = (\sqrt{3}, a, b)^- \), where \( a, b \geq \sqrt{3} \); if \( \Gamma_0 \) is a tree, then we take \( b = 0 \).

If \( \Gamma_0 \) is a tree, then it is easily checked (under the assumption (3.2)) that, for \( 1 \leq i \leq n - 1 \) (in fact, \( n = 5 \)), \( \Gamma_i = (\sqrt{3}, a, \sqrt{3}a) \) or \( \Gamma_i = (\sqrt{3}, 2a, \sqrt{3}a) \) and the uniqueness conclusion of the theorems is satisfied. We now assume that \( \Gamma_0 \) is a cycle (triangle). Suppose first that one of the radical weights \( a, b \) is less than two; without loss of generality, say \( a = \sqrt{3} \). Then, by similar arguments as in Case 1, it follows that, for \( 1 \leq i \leq n - 1 \) (in fact, \( n = 6 \)), the diagram \( \Gamma_i \) belongs to one of the following types: \( (\sqrt{3}, \sqrt{3} + \sqrt{3}b, b); (\sqrt{3}, \sqrt{3} + \sqrt{3}b, 3 + 2b); (\sqrt{3}, 2\sqrt{3} + \sqrt{3}b, 3 + 2b); (\sqrt{3}, \sqrt{3} + \sqrt{3}b, 3 + b); (\sqrt{3}, \sqrt{3} + \sqrt{3}b, 3 + b) \) such that \( \Gamma'_0 \) can be obtained from \( \Gamma_0 \) by a reflection at a source or sink and renumbering the vertices if necessary.

Suppose now that \( a, b \geq 2 \). Note that the vertex 1 is not incident to the edge with radical weight \( \sqrt{3} \), because otherwise \( \Gamma_1 = \mu_1(\Gamma_0) = (\sqrt{3} + ab, a, b) \) does not have any edge whose weight is less than four, contradicting our assumption. Then, by similar arguments as in Case 1, it follows that \( \Gamma_1, 1 \leq i \leq n - 1 \) (in fact, \( n = 6 \)), belongs to one of the following types: \( (\sqrt{3}, a, b + \sqrt{3}a); (\sqrt{3}, 2a + \sqrt{3}b, b + \sqrt{3}a); (\sqrt{3}, a + \sqrt{3}b, 2b + \sqrt{3}a); (\sqrt{3}, a + \sqrt{3}b, b + \sqrt{3}a) \) such that \( \Gamma'_0 \) can be obtained from \( \Gamma_0 \) by a reflection at a source or sink and renumbering the vertices if necessary. This completes the case.

We have completed the proof of Theorem 1.1 and Theorem 1.2(i). Then Theorem 1.2(ii) follows from these statements and Lemma 3.3. This completes the proofs of the theorems.

### 3.2. Proof of Theorem 1.3

It is enough to show the theorem for \( \Gamma' = \mu_k(\Gamma) \), where \( k \) is a vertex in \( \Gamma \). In the proof we will use the following notation: for any two vertices \( i \) and \( j \), we denote by \( \omega_{i,j} \) (resp. \( \omega'_{i,j} \)) the corresponding weight in \( \Gamma \) (resp. \( \Gamma' \)) (note that \( \omega_{i,j} = \omega_{j,i} \), also if the vertices \( i, j \) are not connected in \( \Gamma \), then \( \omega_{i,j} = 0 \), similarly in \( \Gamma' \)). We will show that, for any vertex \( i \), we have \( \delta_i(\Gamma') = \delta_i(\Gamma) \).

For this purpose, let us first note that \( \delta_k(\Gamma') = \delta_k(\Gamma) \) by the definition of the mutation (because the weights of the edges which are incident to \( k \) are not affected). Similarly for any vertex \( i \) which is not adjacent to \( k \), we have \( \delta_i(\Gamma') = \delta_i(\Gamma) \).

To complete the proof, let us now assume that \( i \) is a vertex which is adjacent to \( k \). Then, by the description of the mutation of diagrams in Section 2 for any vertex \( j \), the weight \( \omega'_{i,j} \) is equal to \( \omega_{i,j} \) or \( (\sqrt{\omega_{i,j} + \sqrt{\omega_{j,k} \omega_{k,i}}}^2)^2 = \omega_{i,j} \pm 2\sqrt{\omega_{i,j} \omega_{j,k} \omega_{k,i}} \).

Here note that \( \sqrt{\omega_{i,j} \omega_{j,k} \omega_{k,i}} \) is an integer by the definition of a diagram; furthermore it is divisible by \( \delta_i \) (because \( \delta_i \) divides both \( \omega_{i,j} \) and \( \omega_{k,i} \)). Thus \( \delta_i \) divides \( \omega'_{i,j} \) for
any \( j \), so it divides \( \delta'_i = \delta_i(\Gamma') \). Since \( \mu_k \) is involutive, \( \delta'_i \) divides \( \omega_{i,j} \) for any \( j \), so \( \delta'_i \) divides \( \delta_i \) as well, consequently \( \delta_i = \delta'_i \). (For diagrams of skew-symmetric matrices, the same arguments work if \( \delta_i \) is defined as the greatest common divisor of the radicals of the weights of the edges which are incident to the vertex \( i \)). This completes the proof.

**References**


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