INTRINSIC VOLUMES AND LINEAR CONTRACTIONS

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Abstract. It is shown that intrinsic volumes of a convex body decrease under linear contractions.

Let $C \subset \mathbb{R}^N$ be a convex body and $B^N_2$ the Euclidean ball in $\mathbb{R}^N$. The Steiner formula expresses the volume of the Minkowski sum $C + \varepsilon B^N_2$ in terms of the intrinsic volumes $V_0, V_1, \ldots, V_N$ of $C$:

$$\text{vol}_N(C + \varepsilon B^N_2) = \sum_{n=0}^{N} \omega_n V_{N-n}(C) \varepsilon^n.$$ 

Here $\text{vol}_N(\cdot)$ denotes $N$-dimensional Lebesgue measure and $\omega_n = \text{vol}_n(B^N_2)$. Of particular interest are $V_1$, $V_{N-1}$ and $V_N$, which are multiples of the mean-width, surface area and volume, respectively. We refer the reader to [5] for background on intrinsic volumes. In addition to their role in convex geometry, intrinsic volumes also appear in connection with Gaussian processes; see, e.g., [9], [10] and the references therein.

The purpose of this note is to prove the following.

**Proposition 1.1.** Let $C \subset \mathbb{R}^N$ be a convex body and let $S$ be a linear contraction; i.e., $\|Sx\|_2 \leq \|x\|_2$ for each $x \in \mathbb{R}^N$. Then for $n = 1, \ldots, N$,

$$V_n(SC) \leq V_n(C).$$

The case of $V_1$ and arbitrary contractions (not necessarily linear) is well-studied [6 Theorem 2 in §5], [1 Theorem 1]; see also [2, p. 177]. Of course for $V_N$ one has $V_N(SC) = |\det(S)| \text{vol}_N(C)$. For other intrinsic volumes, we were unable to find Proposition 1.1 in the literature but noticed that it follows from some results in [4] and thought it was worthwhile to show the details.

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Particularly useful for our purpose is the Gaussian representation of intrinsic volumes, as in [10]; see also [8]. If \( \Gamma_{N,n} = [\gamma_{ij}] \) is an \( n \times N \) matrix with independent \( N(0,1) \) Gaussian entries, then the \( n \)-th intrinsic volume of \( C \subset \mathbb{R}^N \) is given by

\[
V_n(C) = \frac{(2\pi)^{n/2}}{\omega_n n!} \mathbb{E} \text{vol}_n(\Gamma_{N,n}C).
\]

As in [4], we say that a function \( F : (\mathbb{R}^n)^N \to \mathbb{R}^+ \) satisfies Groemer’s Convexity Condition, or simply (GCC), if for every \( z \in \mathbb{R}^n \) and for every \( y_1, \ldots, y_N \in z^\perp \) the function \( F_Y : \mathbb{R}^N \to \mathbb{R}^+ \) defined by

\[
F_Y(t) = F(y_1 + t_1 z, \ldots, y_N + t_N z)
\]

is even and convex. The latter definition was motivated by isoperimetric-type problems for random convex sets in [3]. In particular, by adapting [3, Lemma 3], it was shown in [4, Proposition 4.1] that for a convex body \( C \subset \mathbb{R}^N \), the function \( F : (\mathbb{R}^n)^N \to \mathbb{R}^+ \) defined by

\[
F(x_1, \ldots, x_N) = \text{vol}_n([x_1 \ldots x_N]C),
\]

where \([x_1 \ldots x_N]\) denotes the \( n \times N \) matrix with columns \( x_1, \ldots, x_N \), viewed as a linear operator from \( \mathbb{R}^N \) to \( \mathbb{R}^n \), satisfies (GCC). The latter property fits well with symmetrization techniques and can be used in various isoperimetric-type problems for the volume of random (and non-random) sets [4, Theorem 1.1].

For our present purpose, we require less than the (GCC) condition. In fact, we will use only the following consequence.

**Lemma 1.2.** If \( F : (\mathbb{R}^n)^N \to \mathbb{R}^+ \) satisfies (GCC), then for any \( x_1, \ldots, x_N \in \mathbb{R}^n \) and any \( 1 \leq j \leq N \), the function

\[
R \ni s \mapsto F(x_1, \ldots, sx_j, \ldots, x_N)
\]

is convex.

The lemma is immediate since the restriction of a convex function to a line is itself convex.

Additionally, we will make use of the following elementary lemma (the proof is given in [4, Lemma 3.7]).

**Lemma 1.3.** Let \( \rho : \mathbb{R}^n \to \mathbb{R}^+ \) be a function such that

\[
R \ni s \mapsto \rho(sx)
\]

is convex for each \( x \in \mathbb{R}^n \). If \( X \) is a symmetric random vector with values in \( \mathbb{R}^n \), then

\[
R^+ \ni s \mapsto \mathbb{E} \rho(sX)
\]

is an increasing function.

**Proof of Proposition 1.1.** As noted above, the function \( F : (\mathbb{R}^n)^N \to \mathbb{R}^+ \) defined according to [4] satisfies (GCC). Let \( g_1, \ldots, g_N \) denote the columns of the Gaussian random matrix \( \Gamma_{N,n} \). If \( g_1, \ldots, g_N \) are fixed, then

\[
R \ni s \mapsto F(g_1, \ldots, g_{j-1}, sg_j, g_{j+1}, \ldots, g_N)
\]
is convex by Lemma 1.2. Letting $E_j$ denote expectation with respect to $g_j$ and applying Lemma 1.3, we have that

$$\mathbb{R}^+ \ni s \mapsto E_j F(g_1, \ldots, g_{j-1}, sg_j, g_{j+1}, \ldots, g_N)$$

is an increasing function.

Suppose first that $S$ is represented by the $N \times N$ diagonal matrix $S = \text{diag}(1, \ldots, 1, s_j, 1, \ldots, 1)$, where $s_j \in [0, 1]$ is in the $j$th-column. Then

$$E_j F(g_1, \ldots, g_{j-1}, s_j g_j, g_{j+1}, \ldots, g_N) \leq E_j F(g_1, \ldots, g_{j-1}, g_j, g_{j+1}, \ldots, g_N)$$

and hence

$$(2\pi)^{-n/2} \omega_n n! V_n(SC) = E \text{vol}_n (\Gamma_{N,n}SC) = E F(g_1, \ldots, g_{j-1}, s_j g_j, g_{j+1}, \ldots, g_N) \leq E F(g_1, \ldots, g_{j-1}, g_j, g_{j+1}, \ldots, g_N) = (2\pi)^{-n/2} \omega_n n! V_n(C).$$

In the general case, using singular value decomposition, one writes $S = UDV^T$, where $D$ is the diagonal matrix $\text{diag}(s_1, \ldots, s_N)$, and $U$ and $V$ are orthogonal. Since $S$ is a contraction, its singular values satisfy $0 \leq s_i \leq 1$ for $i = 1, \ldots, N$.

To conclude, we use the fact that intrinsic volumes are invariant under orthogonal transformations and apply the latter argument iteratively. 

\[\square\]

Remark 1.4. The latter proof uses ideas from [7, Lemma 2.7].

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