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A DERIVATION OF THE HARDY-RAMANUJAN FORMULA FROM AN ARITHMETIC FORMULA

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Abstract. We re-prove the Hardy-Ramanujan asymptotic formula for the partition function without using the circle method. We derive our result from recent work of Bruinier and Ono on harmonic weak Maass forms.

1. Introduction

The partition function $p(n)$ counts the number of ways to write n as a non-increasing sequence of positive integers. Hardy and Ramanujan [6] created the circle method to prove the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}. \tag{1.1}$$

Rademacher [7], and later Selberg [8] independently, pushed the circle method further to obtain the exact formula

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{m=1}^{\infty} \sqrt{m} A_m(n) \frac{d}{dn} \left[\frac{\sinh\left(\frac{\pi}{m}\sqrt{\frac{2}{3}\left(1-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}} \right],$$

or equivalently,

$$\begin{aligned}(24n-1)p(n) = \sum_{m=1}^{\infty} 2\sqrt{\frac{3}{m}} A_m(n) \left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right) e^{\frac{\pi\sqrt{24n-1}}{6m}} \\ + 2\sqrt{\frac{3}{m}} A_m(n) \left(1 + \frac{6m}{\pi\sqrt{24n-1}}\right) e^{-\frac{\pi\sqrt{24n-1}}{6m}},\end{aligned} \tag{1.2}$$

where $A_m(n)$ is a Kloosterman sum (see [7]).

We use the algebraic formula for $p(n)$ of Bruinier and Ono [1] (Theorem 4 below) to prove an asymptotic formula which captures the main terms of (1.2) without using the circle method. In particular, we prove an asymptotic formula of the form

$$(24n-1)p(n) = \sum_{m=1}^{N} c_m \left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right) e^{\frac{\pi\sqrt{24n-1}}{6m}} + O\left(h(1-24n)e^{\frac{\pi\sqrt{24n-1}}{6(N+1)}}\right),$$

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where the c_m depend on the congruence classes of $n \pmod m$ and $h(1-24n)$ is the usual class number of discriminant $1-24n$. We define a sequence of functions

$$c_m(n) : \mathbb{Z}/m\mathbb{Z} \to \mathbb{R}$$

and implicitly provide a straightforward algorithm to compute $c_m(n)$. In particular, $c_m(n)$ is a finite sum of $12m^{\text{th}}$ roots of unity. For example,

$$c_1(n) = 2\sqrt{3}, \qquad c_2(n) = 2(-1)^n \left(\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12}\right)\right),$$

$$c_3(n) = \begin{cases} 4\cos\left(\frac{\pi}{18}\right) & \text{if } n \equiv 0 \pmod 3 \\ 4\cos\left(\frac{11\pi}{18}\right) & \text{if } n \equiv 1 \pmod 3 \\ 4\cos\left(\frac{23\pi}{18}\right) & \text{if } n \equiv 2 \pmod 3 \end{cases},$$

$$c_4(n) = \begin{cases} 2\cos\left(\frac{\pi}{24}\right) + 2\cos\left(\frac{7\pi}{24}\right) & \text{if } n \equiv 0 \pmod 4 \\ 2\cos\left(\frac{5\pi}{24}\right) + 2\cos\left(\frac{13\pi}{24}\right) & \text{if } n \equiv 1 \pmod 4 \\ -2\cos\left(\frac{\pi}{24}\right) - 2\cos\left(\frac{7\pi}{24}\right) & \text{if } n \equiv 2 \pmod 4 \\ -2\cos\left(\frac{5\pi}{24}\right) - 2\cos\left(\frac{13\pi}{24}\right) & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

Theorem 1. *Let $N \geq 1$ and $1 \leq r \leq \operatorname{lcm}[1, 2, \ldots, N]$ and take $c_m(r)$ as defined in Section 4. If $n \equiv r \pmod{\operatorname{lcm}[1, 2, \ldots, N]}$ and $n \geq 6N^2$, then*

$$(24n-1)p(n) = \sum_{m=1}^{N} c_m(r)\left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right)e^{\frac{\pi\sqrt{24n-1}}{6m}} + O\left(h(1-24n)e^{\frac{\pi\sqrt{24n-1}}{6(N+1)}}\right),$$

where the implied constant is absolute.

Although it is not immediately obvious from the definitions, we must have $c_m(n) = 2\sqrt{\frac{3}{m}}A_m(n)$.

Since $h(1-24n)$ grows subexponentially, (1.1) is a corollary to Theorem 1:

Corollary 2.

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

Proof. Take $N = 1$ in the theorem. Then since $1/\sqrt{24n-1} = o(1)$, we have

$$p(n) = \frac{2\sqrt{3}}{24n-1} \cdot e^{\frac{\pi\sqrt{24n-1}}{6}} + o\left(e^{\frac{\pi\sqrt{24n-1}}{6}}\right).$$

The conclusion follows. □

In Section 2 we establish notation related to binary quadratic forms. These preliminaries are necessary for Section 3 in which we recall Bruinier and Ono's work [1] on harmonic weak Maass forms. Finally, in Section 4 we prove Theorem 1.

2. Binary quadratic forms

We set notation and recall the fundamentals of positive definite quadratic forms. There are many good introductions to this classical material; see [4] for an example.

An integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with *discriminant* $b^2 - 4ac$ is called *primitive* if $\gcd(a, b, c) = 1$. We will often write $Q = [a, b, c]$. Throughout this article, we restrict our attention to positive definite forms with

discriminant $1-24n$ for positive integers n. It is elementary to see that for such forms $0 \equiv ac \not\equiv b \pmod 2$ and

$$\begin{aligned} 3 \mid b &\iff ac \equiv 2 \pmod 6, \\ 3 \nmid b &\iff ac \equiv 0 \pmod 6. \end{aligned}$$

Let $\mathbb{H}$ denote the upper half of the complex plane. The *principal root* of $Q = [a,b,c]$ is the unique point $\alpha = -\frac{b}{2a} + \frac{\sqrt{24n-1}}{2a} i \in \mathbb{H}$ such that $Q(\alpha, 1) = 0$. Matrices $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ act on $z \in \mathbb{H}$ via the left action

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \circ z = \frac{rz+s}{tz+u}$$

and on forms via the right action

$$Q \circ \begin{pmatrix} r & s \\ t & u \end{pmatrix} (x,y) = Q(rx+sy, tx+uy),$$

which preserves the discriminant. If α is the principal root of Q and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma \circ \alpha$ is the principal root of $Q \circ \gamma^{-1}$.

We say that $Q = [a,b,c]$ is $\mathrm{SL}_2(\mathbb{Z})$*-reduced* if $|b| \le a \le c$. Every positive definite form is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a reduced form. With the exception of $[a,b,a] \sim [a,-b,a]$ and $[a,a,c] \sim [a,-a,c]$, no distinct reduced forms are $\mathrm{SL}_2(\mathbb{Z})$-equivalent. The (finite) number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of primitive binary quadratic forms of discriminant $1-24n$ is called the class number $h(1-24n)$. A form is reduced if and only if its principal root lies in the closure of the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$,

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \le \operatorname{Re} z < \frac{1}{2} \text{ and } |z| > 1 \right\} \cup \left\{ z \in \mathbb{H} : |z| = 1 \text{ and } -\frac{1}{2} \le \operatorname{Re} z \le 0 \right\}.$$

Let $\mathcal{Q}_n^1$ denote the set of primitive forms of discriminant $1-24n$ whose principal roots lie in $\mathcal{F}$. Observe that $\mathcal{Q}_n^1$ does not double count any $\mathrm{SL}_2(\mathbb{Z})$-equivalence class.

The congruence group $\Gamma_0(6) = \{\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : 6|t\}$ acts on the set of primitive forms $[a,b,c]$ of discriminant $1-24n$ such that $6 \mid a$ and $b \equiv 1 \pmod{12}$. Let $\mathcal{Q}_n^6$ denote the set of $\Gamma_0(6)$-equivalence classes of such forms. The group $\Gamma_0(6)$ has index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(6)] = 12$, and we choose right coset representatives:

$$\begin{aligned} \gamma_\infty &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \gamma_{\frac{1}{3},r} &= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \text{ for } r = 0,1, \\ \gamma_{\frac{1}{2},s} &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \text{ for } s = 0,1,2, \\ \gamma_{0,t} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ for } t = 0,1,2,3,4,5. \end{aligned} \tag{2.1}$$

The proposition on page 505 of [5] says that there is a bijection between $\mathcal{Q}_n^1$ and $\mathcal{Q}_n^6$. In particular, for any $Q \in \mathcal{Q}_n^1$, there exists a unique choice from (2.1) of right coset leader γ_Q such that $[Q \circ \gamma_Q^{-1}] \in \mathcal{Q}_n^6$. It will be expedient to know γ_Q explicitly. The proof of the following lemma obviates the appeal to [5].

TABLE 1. The matrix γ_Q for $Q = [a, b, c] \in \mathcal{Q}_n^1$. Triples $(\overline{a}, \overline{b}, \overline{c})$ which can never occur are omitted or left blank.

		$b \pmod{12}$					
		1	3	5	7	9	11
$a \equiv 0 \pmod 6$	$c \equiv 0 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},0}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,0}$
	$c \equiv 1 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},2}$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,5}$
	$c \equiv 2 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},1}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,4}$
	$c \equiv 3 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},0}$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,3}$
	$c \equiv 4 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},2}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,2}$
	$c \equiv 5 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},1}$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,1}$
$a \equiv 1 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{0,1}$		$\gamma_{0,3}$	$\gamma_{0,4}$		$\gamma_{0,0}$
	$c \equiv 2 \pmod 6$		$\gamma_{0,2}$			$\gamma_{0,5}$	
$a \equiv 2 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{\frac{1}{2},2}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,2}$		$\gamma_{0,0}$
	$c \equiv 1 \pmod 6$		$\gamma_{0,1}$			$\gamma_{\frac{1}{2},1}$	
	$c \equiv 3 \pmod 6$	$\gamma_{\frac{1}{2},2}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,5}$		$\gamma_{0,3}$
	$c \equiv 4 \pmod 6$		$\gamma_{0,4}$			$\gamma_{\frac{1}{2},1}$	
$a \equiv 3 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,3}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,0}$
	$c \equiv 2 \pmod 6$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,1}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,4}$
	$c \equiv 4 \pmod 6$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,5}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,2}$
$a \equiv 4 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{\frac{1}{2},1}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,4}$		$\gamma_{0,0}$
	$c \equiv 2 \pmod 6$		$\gamma_{0,2}$			$\gamma_{\frac{1}{2},2}$	
	$c \equiv 3 \pmod 6$	$\gamma_{\frac{1}{2},1}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,1}$		$\gamma_{0,3}$
	$c \equiv 5 \pmod 6$		$\gamma_{0,5}$			$\gamma_{\frac{1}{2},2}$	
$a \equiv 5 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{0,5}$		$\gamma_{0,3}$	$\gamma_{0,2}$		$\gamma_{0,0}$
	$c \equiv 4 \pmod 6$		$\gamma_{0,4}$			$\gamma_{0,1}$	

Lemma 3. *If $Q = [a, b, c] \in \mathcal{Q}_n^1$, then the matrix γ_Q from* (2.1) *such that $[Q \circ \gamma_Q^{-1}] \in \mathcal{Q}_n^6$ is as given in Table* 1.

Proof. Consider $\gamma_{\frac{1}{2},0} = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right)$. We compute that $[a, b, c] \circ \gamma_{\frac{1}{2},0}^{-1} = [A, B, C]$, where

$$A = 9a - 6b + 4c,$$
$$B = -6a + 5b - 4c.$$

Clearly $2 \mid a$, $3 \mid c$ and $b \equiv 5 \pmod{12}$ if and only if $A \equiv 0 \pmod 6$ and $B \equiv 1 \pmod{12}$. The latter conditions are precisely the requirements for $[A, B, C] = [a, b, c] \circ \gamma_{\frac{1}{2},0}^{-1} \in \mathcal{Q}_n^6$. This gives six entries in Table 1. The other 11 matrices from (2.1) are analogous. □

3. HARMONIC WEAK MAASS FORMS

We recall notation from [1]. We adopt the convention that $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. The Maass raising operator (see Section 2 of [2]) R_k acts on functions $f : \mathbb{H} \to \mathbb{C}$ and is defined by

$$R_k = 2i\frac{\partial}{\partial z} + \frac{k}{y} = i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{k}{y}.$$

Given $\gamma = \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define the Petersson slash operator which also acts on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f \mid_k \gamma = (tz+u)^{-k} f\left(\frac{rz+s}{tz+u}\right).$$

The slash operator intertwines with the raising operator

$$R_k\left(f \mid_k \gamma\right) = \left(R_k f\right) \mid_{k+2} \gamma. \tag{3.1}$$

Recall Dedekind's eta function $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, where $q = e^{2\pi i z}$ and the Eisenstein series $E_2(z) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$, where $\sigma(n) = \sum_{d|n} d$. Define the $\Gamma_0(6)$ weight -2 weakly holomorphic modular form

$$F(z) := \frac{1}{2}\cdot\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z)} = q^{-1} - 10 - 29q - \cdots.$$

Set $\zeta_6 := e^{2\pi i/6}$. Fourier expansions of F at the cusps $\frac{1}{3}$, $\frac{1}{2}$ and 0 are given by

$$\begin{aligned} F \mid_{-2} \gamma_{\frac{1}{3},r} &= 2(-1)^r q^{-1/2} + 20 - (-1)^r 34 q^{1/2} + \cdots, \\ F \mid_{-2} \gamma_{\frac{1}{2},s} &= 3\zeta_6^{3-2s} q^{-1/3} + 30 - 87\zeta_6^{3+2s} q^{1/3} + \cdots, \\ F \mid_{-2} \gamma_{0,t} &= 6\zeta_6^{-t} q^{-1/6} - 60 - 174\zeta_6^{t} q^{1/6} - \cdots. \end{aligned} \tag{3.2}$$

For any $Q \in \mathcal{Q}_n^1$, define $h_Q \in \{1,2,3,6\}$ to be the width of the cusp $\gamma_Q \circ \infty$ on the modular curve $X_0(6)$ and define ζ_Q to be the sixth root of unity such that

$$F \mid_{-2} \gamma_Q = h_Q \zeta_Q q^{-1/h_Q} + O(1).$$

Bruinier and Ono define the $\Gamma_0(6)$ weight zero weak Maass form

$$P(z) := \frac{1}{4\pi} R_{-2} F(z) = \left(1 - \frac{1}{2\pi y}\right) q^{-1} + \frac{5}{\pi y} + \left(29 + \frac{29}{2\pi y}\right) q + \cdots.$$

Part of Theorem 1.1 of [1] says:

Theorem 4 (Bruinier-Ono)**.** *For all positive integers n,*

$$(24n-1)p(n) = \sum_{[Q]\in\mathcal{Q}_n^6} P(\alpha_Q),$$

where α_Q denotes the principal root of Q.

Ono has informed us that an upcoming article [3] will use the theory of elliptic curves with complex multiplication to compute the polynomials whose roots are the singular moduli for $P(z)$ modulo primes p. By the Chinese Remainder Theorem, they then obtain a fast algorithm for deriving these polynomials exactly.

4. Proof of Theorem 1

The key to the proof of Theorem 1 is that if $P(\alpha_Q)$ is evaluated using a Fourier expansion centered at the cusp closest – on $X_0(6)$ – to α_Q, then the first term of the Fourier expansion dominates the tail.

Lemma 5. *Let γ be one of the 12 matrices from (2.1) and suppose that*

$$(F \mid_{-2} \gamma)(z) = h\zeta q^{-1/h} + a(0) + a(1)q^{1/h} + \cdots,$$

where h is the width of the cusp $\gamma \circ \infty$ on the modular curve $X_0(6)$ and $\zeta \in \mathbb{C}$. Then for $z \in \mathcal{F}$,

$$(P \mid_0 \gamma)(z) = \zeta \left(1 - \frac{h}{2\pi y}\right) e^{-2\pi i z/h} + O(1),$$

where the implied constant is absolute.

Proof. The intertwining property (3.1) of R_{-2} implies that

$$\begin{aligned} P \mid_0 \gamma &= \left(\frac{1}{4\pi} R_{-2} F\right) \mid_0 \gamma \\ &= \frac{1}{4\pi} R_{-2} \left(F \mid_{-2} \gamma\right) \\ &= \frac{1}{4\pi} R_{-2} \left(h\zeta q^{-1/h} + \left(F \mid_{-2} \gamma - h\zeta q^{-1/h}\right)\right) \\ &= \zeta \left(1 - \frac{h}{2\pi y}\right) e^{-2\pi i z/h} + \frac{1}{4\pi} R_{-2} \left(F \mid_{-2} \gamma - h\zeta q^{-1/h}\right). \end{aligned}$$

Thus, it suffices to show that the second term is bounded by a constant depending only on γ, since there are only 12 possible choices for γ.

Suppose $z \in \mathcal{F}$. Then clearly

$$z \in \mathcal{U} := \left\{ z \in \mathbb{H} : -\frac{h}{2} \leq \operatorname{Re} z < \frac{h}{2}, \operatorname{Im} z \geq \frac{\sqrt{3}}{2} \right\}.$$

Recall that the Fourier expansion of the meromorphic modular form $F(z)$ at a cusp of width h is in fact a Laurent expansion of a meromorphic function on a punctured disc in the local variable $q_h := q^{1/h} = e^{2\pi i z/h}$. Hence, $f(q_h) := F \mid_{-2} \gamma - h\zeta q_h^{-1}$ is bounded on the closure of the punctured disc. Since $F(z)$ is holomorphic for all $z \in \mathbb{H}$, the q_h-series expansion of f converges for all $|q_h| < 1$. In particular, $f(q_h)$ is bounded on $\left\{ q_h : |q_h| \leq e^{-\pi\sqrt{3}/h} \right\}$.

Now $-\frac{1}{2\pi i}\frac{d}{dz} = \frac{1}{h} q_h \frac{d}{dq_h}$, and hence

$$-\frac{1}{2\pi i}\frac{d}{dz}\left(F \mid_{-2} \gamma - h\zeta q^{-1/h}\right) = \frac{1}{h} q_h \frac{d}{dq_h} f(q_h)$$

is holomorphic, and hence bounded, for $|q_h| \leq e^{-\pi\sqrt{3}/h}$, and hence for $z \in \mathcal{F} \subset \mathcal{U}$. Additionally, $1/(2\pi y)$ is bounded on $\mathcal{U}$. Hence, for $z \in \mathcal{U}$, we have

$$\frac{1}{4\pi} R_{-2} \left(F \mid_{-2} \gamma - h\zeta q^{-1/h}\right) = O(1).$$

□

We let α_Q denote the principal root of Q. Now by the explicit bijection $\mathcal{Q}_n^1 \leftrightsquigarrow \mathcal{Q}_n^6$ and Theorem 4, we have

$$(24n-1)p(n) = \sum_{[Q] \in \mathcal{Q}_n^6} P(\alpha_Q) = \sum_{Q \in \mathcal{Q}_n^1} P(\gamma_Q \circ \alpha_Q) = \sum_{Q \in \mathcal{Q}_n^1} (P \mid_0 \gamma_Q)(\alpha_Q). \tag{4.1}$$

The widths of the cusps $\gamma_\infty \circ \infty$, $\gamma_{\frac{1}{3},r} \circ \infty$, $\gamma_{\frac{1}{2},s} \circ \infty$, and $\gamma_{0,t} \circ \infty$, are, respectively, 1, 2, 3, and 6. Observe from Table 1 that for any $Q = [a,b,c] \in \mathcal{Q}_n^1$, the product

$$a \cdot h_Q \equiv 0 \pmod 6.$$

We group the summands in (4.1) accordingly and apply Lemma 5 using $\alpha_Q = -\frac{b}{2a} + \frac{\sqrt{24n-1}}{2a} i$:

$$\begin{aligned}
(24n-1)p(n) &= \sum_{m=1}^{\infty} \sum_{\substack{[a,b,c]\in\mathcal{Q}_n^1 \\ a\cdot h_{[a,b,c]}=6m}} (P \mid_0 \gamma_Q)(\alpha_Q) \\
&= \sum_{m=1}^{\infty} \left(\sum_{\substack{[a,b,c]\in\mathcal{Q}_n^1 \\ a\cdot h_{[a,b,c]}=6m}} \zeta_{[a,b,c]} e^{\pi i b/6m} \right) \left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right) e^{\frac{\pi\sqrt{24n-1}}{6m}} \\
&\quad + O(h(1-24n)) \\
&= \sum_{m=1}^{N} \left(\sum_{\substack{[a,b,c]\in\mathcal{Q}_n^1 \\ a\cdot h_{[a,b,c]}=6m}} \zeta_{[a,b,c]} e^{\pi i b/6m} \right) \left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right) e^{\frac{\pi\sqrt{24n-1}}{6m}} \\
&\quad + O\left(h(1-24n) e^{\frac{\pi\sqrt{24n-1}}{6(N+1)}}\right).
\end{aligned}$$

The truncation in the last step omits at most $h(1-24n)$ summands, each of which is clearly bounded by $e^{\frac{\pi\sqrt{24n-1}}{6(N+1)}} \geq 1$. Hence the implied constants are still absolute. It remains to consider the finite sums

$$\widehat{c_m}(n) := \sum_{\substack{[a,b,c]\in\mathcal{Q}_n^1 \\ a\cdot h_{[a,b,c]}=6m}} \zeta_{[a,b,c]} e^{\pi i b/6m}.$$

As an example, we first consider $\widehat{c_1}(n)$. It is easy to determine that the forms $[a,b,c] \in \mathcal{Q}_n^1$ with $a \leq 6$ are exactly:

$$\begin{array}{rc}
 & [1,1,6n] \\
 & [2,1,3n] \\
 & [2,-1,3n] \\
 & [3,1,2n] \\
 & [3,-1,2n] \\
\text{if } n \text{ even} & \left[4,1,\frac{3n}{2}\right] \\
 & \left[4,-1,\frac{3n}{2}\right] \\
\text{if } n \text{ odd} & \left[4,3,\frac{3n+1}{2}\right] \\
 & \left[4,-3,\frac{3n+1}{2}\right]
\end{array}
\qquad
\begin{array}{rc}
\text{if } n \equiv 0 \pmod 5 & \left[5,1,\frac{6n}{5}\right] \\
 & \left[5,-1,\frac{6n}{5}\right] \\
\text{if } n \equiv 3 \pmod 5 & \left[5,3,\frac{6n+2}{5}\right] \\
 & \left[5,-3,\frac{6n+2}{5}\right] \\
\text{if } n \equiv 4 \pmod 5 & \left[5,5,\frac{6n+6}{5}\right] \\
 & [6,1,n] \\
 & [6,-1,n] \\
 & [6,5,n+1] \\
 & [6,-5,n+1].
\end{array}$$

For each of these, Table 1 indicates the γ_Q for which $Q \circ \gamma_Q^{-1} = [A, B, C] \in \mathcal{Q}_n^6$. For example,

$$\left.\begin{aligned} [1,1,6n] \circ \gamma_{0,1}^{-1} &= [6n, 1, 1] \\ [2,1,3n] \circ \gamma_{\frac{1}{2},-1}^{-1} &= [84+12n, -71-12n, 15+3n] \\ [3,1,2n] \circ \gamma_{\frac{1}{3},0}^{-1} &= [126+18n, -71-12n, 10+2n] \\ [6,1,n] \circ \gamma_{\infty} &= [6, 1, n] \end{aligned}\right\} \in \mathcal{Q}_n^6. \tag{4.2}$$

For each of these four forms, we see that $ah = 6$. Moreover, it is easy to check that for all other forms, we have $ah \geq 12$. That is, for all $Q \in \mathcal{Q}_n^6$ other than the four listed in (4.2), we have

$$P(\alpha_Q) = O\left(e^{\pi \frac{\sqrt{24n-1}}{12}}\right).$$

Finally, we compute that if $n \geq 2$, we have

$$\widehat{c_1}(n) = \zeta_6^{-1} \cdot e^{\pi i/6} + \zeta_6^{3-2(-1)} \cdot e^{\pi i/6} + (-1)^0 \cdot e^{\pi i/6} + 1 \cdot e^{\pi i/6} = 4\cos\left(\frac{\pi}{6}\right) = 2\sqrt{3}.$$

More generally, we have the following lemma.

Lemma 6. *If $n \geq 6m^2$, then $\widehat{c_m}(n)$ only depends on $n \pmod m$.*

Proof. For a given m, we consider forms with $a = m$, $2m$, $3m$, and $6m$. From Table 1, we see that $h_{[a,b,c]}$ depends only on $a \pmod 6$ and $b \pmod{12}$. Thus, for each of the four choices of a, we consider all b with $|b| \leq a$ and such that $b \pmod{12}$ would yield the proper width $h_{[a,b,*]} = 6m/a$. We stress that the set of pairs of a and b which we consider depends only on m.

For each pair a and b, the only possible c for which $b^2 - 4ac = 1 - 24n$ is obviously

$$c = \frac{6n + \left(\frac{b^2-1}{4}\right)}{a}.$$

Thus, $[a, b, c] \in \mathcal{Q}_n^1$ if and only if $6n + \left(\frac{b^2-1}{4}\right) \equiv 0 \pmod a$ and $6n + \left(\frac{b^2-1}{4}\right) \geq a^2$. The latter condition is satisfied whenever $n \geq 6m^2$. By (3.2) and Lemma 3, if $[a, b, c] \in \mathcal{Q}_n^1$, then $\zeta_{[a,b,c]}$ depends on $c \pmod h$. Thus, the contribution of the pair a and b, if any, to $\widehat{c_m}(n)$ depends on $6n$ modulo $ah = 6m$. That is, $\widehat{c_m}(n)$ depends only on $n \pmod m$. □

In light of the previous lemma, $\widehat{c_m}(n)$ induces (in the obvious way) the function

$$c_m(n) : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$$

referred to in Section 1. To see that $c_m(n)$ in fact maps to $\mathbb{R}$, we take real parts of the asymptotic expansion for $\mathrm{Re}\left((24n-1)p(n)\right) = (24n-1)p(n)$. This concludes the proof of Theorem 1.

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