BIALYNICKI-BIRULA DECOMPOSITION OF
DELIGNE-MUMFORD STACKS

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Abstract. This short note considers the Bialynicki-Birula decomposition of Deligne-Mumford stacks under one-dimensional torus actions and extends a result of Oprea.

1. Introduction

In this short note, we consider actions of one-dimensional tori on tame Deligne-Mumford stacks which are smooth and proper over an algebraically closed field. We extend a result of Oprea [Opr06] to show that in the aforementioned case, if the stack has a scheme for a coarse moduli space, or if it is toric, then it admits a Bialynicki-Birula decomposition and often a corresponding decomposition of cohomology. A history of the result can be found in [Bro05, Theorem 3.2].

2. Notation and terminology

Let $k$ be an algebraically closed field of arbitrary characteristic, fixed in what follows.

In this note, an algebraic stack will be a stack $X$ fibered over $(\text{Sch}/k)$ in the étale topology, such that the diagonal mapping $\Delta : X \to X \times X$ is representable, separated and quasi-compact, and such that there exists a smooth, surjective $k$-morphism $U \to X$ from a $k$-scheme $U$, which will be called an atlas. Deligne-Mumford stacks are those admitting étale atlases. Proper Deligne-Mumford stacks are those admitting a finite, surjective morphism from a proper $k$-scheme. Tame Deligne-Mumford stacks are those with linearly reductive geometric stabilizer groups.

An affine fibration is a flat morphism $p : E \to X$ which is étale locally a trivial bundle of affine spaces. This definition weakens the definition of a vector bundle by relaxing the requirement that the transition functions be linear.

Let $T$ be a one-dimensional torus over $k$ with an isomorphism to $\mathbb{G}_m$. Let the fixed points of an action of $T$ on a stack $X$ be denoted by $X_T$ [Rom05].

If an algebraic stack has the form of a quotient $[X/G]$ of a normal toric variety $X$ over an algebraically closed field of characteristic zero by a subgroup $G$ of the torus of $X$, then it is called a toric stack [GS11a].

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1933
3. Bialynicki-Birula decomposition of Deligne-Mumford stacks

Oprea proved a form of the Bialynicki-Birula decomposition [BB73] for Deligne-Mumford stacks assuming there exists a $T$-equivariant, affine, étale atlas [Opr06, Proposition 5].

Proposition 3.1 (Oprea). Let $X$ be a smooth, proper Deligne-Mumford stack over an algebraically closed field $k$ with a $T$-action that admits a $T$-equivariant, affine, étale atlas $U \to X$. Let $F = \coprod F_i$ be the decomposition of the fixed substack into connected components. Then $X$ decomposes into disjoint, locally closed, $T$-equivariant substacks $X_i$ which are $T$-equivariant affine fibrations over $F_i$.

Oprea expected the existence of an atlas to be a general fact [Opr06, Section 2]. Here we show that the desired atlas exists under somewhat general conditions.

Proposition 3.2. Let $X$ be a tame, irreducible Deligne-Mumford stack, smooth and proper over $k$, whose generic stabilizer is trivial and whose coarse moduli space is a scheme. Furthermore, let an action of $T$ on $X$ be given such that $T$ acts trivially on its fixed locus. Then there exists a $T$-equivariant, affine, étale atlas $U \to X$.

Remark 3.3. If an algebraic group $G$ acts algebraically on a Deligne-Mumford stack $X$, then the action descends to the coarse moduli space of $X$ by the universal property of the coarse moduli space of $G \times_k X$.

Remark 3.4. If $T$ does not act trivially on its fixed locus, it can be made to do so by a reparametrization of the action. Since $T$-invariance is not affected by this change, the decomposition for the new action will be a decomposition for the original action.

Proof. Since $X$ is reduced and irreducible with a scheme for a coarse moduli space, the Keel-Mori theorem [KM97, Proposition 4.2] shows that the coarse moduli space is in fact a normal, proper variety over $k$. It inherits a $T$-action by Remark 3.3.

Since it is proper, any collection of open, $T$-invariant neighborhoods containing the fixed point locus covers it and contains a finite subcover. They can be chosen to be affine by Sumihiro’s theorem [Sum74, Corollary 2]. So it suffices to find the desired atlas for an arbitrary fixed $k$-point $x \in X$. To this end, let $Y$ be the pullback of $X$ to an open, affine, $T$-invariant neighborhood of the image of $x$ in the coarse moduli space.

Consider the frame bundle of $Y$ with total space $FY$. Each $k$-point $y \in Y$ lies in the image of an étale, representable morphism from a quotient stack of the form $[U/G]$ for an affine, irreducible scheme $U$ with an action of the stabilizer group $G$ of $y$ and containing a point $u$ fixed by the $G$-action and mapping to $y$ [KM97]. By the tameness hypothesis, $G$ is linearly reductive. Since $Y$ has trivial generic stabilizer, $G$ acts faithfully on $U$. Applying [BB73, Theorem 2.4] to $U$ and $T_uU$ shows that $G$ acts faithfully on $T_yY$. So the total space $FY$ is an algebraic space, and $Y = [FY/GL_n]$, where $n$ is the dimension of $X$. But $Y$ has an affine coarse moduli space, so $FY$ is, in fact, an affine scheme (cf. [EHKV01, Remark 4.3]).

The action of $T$ on $Y$ induces an action on $TY$ and hence an action on $FY$, which is an open, $T$-invariant substack of $TY^{\oplus n}$. Let $p : FY \to Y$ be the projection. An atlas will be defined by finding a $T$-equivariant étale slice $U \to FY$ over the fixed point $y$. 

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One may assume that a fixed point \( f \in FY \) lies over \( y \) by modifying the action as follows. Choose a basis of the tangent space \( T_yY \) which diagonalizes the \( T \)-action, i.e., so that \( t : (v_1, \ldots, v_n) \mapsto (t^{a_1}v_1, \ldots, t^{a_n}v_n) \). Then the induced \( T^n \)-action on \( FY \) can be used to define a twisted \( T \)-action on \( FY \) by

\[
T \to T^n, \\
t \mapsto (t^{-a_1}, \ldots, t^{-a_n}).
\]

The projection from the frame bundle remains \( T \)-equivariant after twisting the action, but now the action fixes the frame \( f \) formed by the basis vectors.

The proof finishes by arguing as in [Ols06] Lemma 3. The torus \( T \) acts on the tangent space \( T_fFY \) at \( f \), with \( T_fFY \to T_yY \) surjective and \( T \)-equivariant. By the linear reductivity of \( T \), \( T_yY \) may be identified with some \( T \)-invariant subspace \( N \subset T_fFY \), compatibly with the \( T \)-actions. By a theorem of Białynicki-Birula [BB73, Theorem 2.1], there exists a reduced and irreducible, closed, \( T \)-invariant subscheme \( Z \) of \( FY \) containing \( f \) as a nonsingular point such that \( T_fZ = N \).

Taking \( U \) to be the largest open subscheme of \( Z \) on which the restriction of \( p \) is étale ensures that \( U \) is \( T \)-invariant. By applying Sumihiro’s theorem again, one may shrink \( U \) to an affine, \( T \)-invariant neighborhood of \( f \) whose image in \( Y \) contains \( y \).

\[\tag{3.1}\]

**Theorem 3.5.** Let \( X \) be a tame Deligne-Mumford stack, smooth and proper over \( k \), whose coarse moduli space is a scheme, and let an action of \( T \) on \( X \) be given. Then \( X \) admits a Białynicki-Birula decomposition.

**Proof.** One first reduces to the case that \( X \) is irreducible by decomposing each irreducible component separately and combining to give a decomposition of all of \( X \). Applying Remark 3.1, one may suppose \( T \) acts trivially on its fixed locus. By [Ols07] Proposition 2.1, there exists a rigidification \( \overline{X} \) of \( X \) which has trivial generic stabilizer. In particular, there is an étale, proper morphism \( X \to \overline{X} \) which forms a \( G \)-gerbe for a finite group \( G \). The \( T \)-action descends to \( \overline{X} \) using the universal property of rigidification, so Proposition 3.2 guarantees the existence of a \( T \)-equivariant, affine, étale atlas of \( \overline{X} \). Let the substacks \( F_i \) and \( X_i \), together with \( T \)-equivariant affine fibrations \( \overline{X}_i \to F_i \), be defined according to Proposition 3.1.

Let the decomposition of \( X \) be defined by pulling back along \( X \to \overline{X} \). Pulling back again by the affine fibration \( \overline{X}_i \to F_i \) forms the diagram:

\[
\begin{align*}
\overline{X}_i \times_{\overline{F}_i} F_i & \longrightarrow F_i \longrightarrow X_i \longrightarrow X \\
\overline{X}_i & \longrightarrow \overline{F}_i \longrightarrow \overline{X}_i \longrightarrow \overline{X}.
\end{align*}
\]

So \( F = \bigsqcup_i F_i \) is a decomposition of the fixed substack into connected components [Rom05]. Furthermore, all morphisms are \( T \)-equivariant.

In what follows, let \( i \) be fixed. It remains to prove the existence of an affine fibration, and diagram (3.1) shows it will suffice to supply a \( T \)-equivariant isomorphism \( \overline{X}_i \times_{\overline{F}_i} F_i \to X_i \) over \( \overline{X}_i \). This can be done by specifying such an isomorphism, unique up to a canonical 2-isomorphism, over an étale, \( T \)-equivariant atlas of \( \overline{X}_i \), and then applying the descent property of the stack \( \text{Hom}_{\overline{X}_i}(\overline{X}_i \times_{\overline{F}_i} F_i, X_i) \) [Ols06].
First, étale, $T$-equivariant atlases forming the following pullback will be defined:

$$
\begin{array}{ccc}
W & \rightarrow & P \\
\downarrow & & \downarrow \\
\bar{X}_i & \rightarrow & \bar{F}_i.
\end{array}
$$

There is an $\text{Out}(G)$-torsor on $\bar{F}_i$ associated to the rigidification gerbe $F_i \rightarrow \bar{F}_i$ whose class $\alpha$ lies in $H^1(\bar{F}_i, \text{Out}(G))$. Base change to the total space of the torsor trivializes $\alpha$ and gives an element of the étale cohomology $\beta \in H^2(\bar{F}_i, Z)$ which classifies the gerbe, where $Z$ is the center of $G$ (cf. [EHKV01, Section 3.1]). Let $P \rightarrow \bar{F}_i$ be an affine, étale atlas that trivializes $\alpha$, $\beta$, and the affine fibration $\bar{X}_i \rightarrow \bar{F}_i$. Also, let $W := P \times_{\bar{F}_i} \bar{X}_i$, as in diagram (3.2).

By the Künneth formula for the algebraic fundamental group [Ray71, Proposition 4.6], the atlas $W$ trivializes the $\text{Out}(G)$-torsor associated to the gerbe $X_i \rightarrow \bar{X}_i$. Then [Art73, Corollary 2.2] shows that the morphism $P \times A^n \cong W \rightarrow P$ induces an isomorphism $H^2(P, Z) \cong H^2(W, Z)$, implying that the classifying element of $X_i \times \bar{X}_i \rightarrow W \rightarrow W$ vanishes. The isomorphism $X_i \times \bar{X}_i \rightarrow F_i \times \bar{F}_i$ where $W$ of trivial gerbes can be chosen, uniquely up to a canonical 2-isomorphism, to be the isomorphism over $W$ which extends the identity morphism of $F_i \times \bar{F}_i$ over $P$. This follows from the triviality of the affine fibration $W \rightarrow P$ and the Künneth formula. The isomorphism will also be $T$-equivariant by similar reasoning, since the identity is $T$-equivariant.

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\text{□}
\end{flushright}

Remark 3.6. If $X$ is a tame Deligne-Mumford stack, smooth and proper over $k$, with a projective coarse moduli space and a $T$-action, then the induced decomposition forms a filtration, and a lemma of Oprea [Opr06, Lemma 6] implies that the Betti numbers of the stack are calculated by the Betti numbers of the fixed points.

In what follows, $T$ may be a torus of arbitrary dimension.

Proposition 3.7. Let $\text{char} \ k = 0$, and let $X$ be a normal algebraic space, separated and of finite type over $k$, with an action of $T$ which gives a dense, open embedding of $T$ in $X$. Then $X$ is a scheme and hence a toric variety.

Proof. First, let $k = C$. The scheme locus of the normalized blowup at the closure of any nondense $T$-orbit forms a toric variety whose image includes the orbit. The associated fans give, in each $T$-orbit, a limit point of a $G_m$-orbit of $1 \in T \hookrightarrow X$ for a subtorus $G_m$ of $T$. Then $X$ is a finite union of $T$-orbits of such points and hence a scheme [Hau00, Theorem 1].

For general $k$, one may immediately reduce to the case that $k$ is a subfield of $C$. The pullback of $X$ to $C$ is a toric variety [Hau00], so a theorem [GS11b, Theorem 6.1] implies there exists an étale, representable, surjective morphism $p : [U/GL_n] \rightarrow [X_C/T]$ where $U$ is a quasi-affine scheme over $C$. Let $L \subset C$ be a subfield of definition of $p$ which is finitely generated over $k$, giving a morphism $p_L : [U_L/GL_n] \rightarrow [X_L/T]$ where $U$ is obtained by pulling back $U_L$ to $C$. Then $d = \text{tr} \deg L/K < \infty$, and $p_L$ remains étale, representable and surjective. Writing $L = k(V)$ for a $d$-dimensional affine variety $V$, one may suppose that $U_L$ with its $GL_n$-action is defined over $V$, realizing $p_L$ as the pullback of a dominant, étale morphism $p_V : [U_V/GL_n] \rightarrow [X/T] \times V$ to the generic point of $V$. After excluding points of $U$ lying in the image of the pullback of the relative inertia, $p_V$ becomes
The disjoint union of fibers of $p_V$ over finitely many closed points of $V$ forms an étale, representable, surjective morphism to $[X/T]$. Applying in the reverse direction, one deduces that $X$ is a toric algebraic space and hence a scheme.

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\begin{thebibliography}{99}


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