MINIMAL $C^1$-DIFFEOMORPHISMS OF THE CIRCLE WHICH ADMIT MEASURABLE FUNDAMENTAL DOMAINS

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Abstract. We construct, for each irrational number $\alpha$, a minimal $C^1$-diffeomorphism of the circle with rotation number $\alpha$ which admits a measurable fundamental domain with respect to the Lebesgue measure.

1. INTRODUCTION

The concept of ergodicity is important not only for measure-preserving dynamical systems but also for systems which admit a natural quasi-invariant measure. Given a probability space $(X, \mu)$ and a transformation $T$ of $X$, $\mu$ is said to be quasi-invariant if the push forward $T_*\mu$ is equivalent to $\mu$. In this case $T$ is called ergodic with respect to $\mu$, if a $T$-invariant Borel subset in $X$ is either null or conull.

A diffeomorphism of a differentiable manifold always leaves the Riemannian volume (also called the Lebesgue measure) quasi-invariant, and one can ask if a given diffeomorphism is ergodic with respect to the Lebesgue measure (below ergodic for short) or not. Answering a question of A. Denjoy [D], A. Katok (see for instance Chap. 12.7, p. 419, [KH]) and independently M. Herman (Chap. VII, p. 86, [H]) showed that a $C^1$-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational. Contrarily Oliveira and da Rocha [OR] constructed a minimal $C^1$-diffeomorphism of the circle which is not ergodic.

At the opposite extreme of the ergodicity lies the concept of measurable fundamental domains. Given a transformation $T$ of a standard probability space $(X, \mu)$ leaving $\mu$ quasi-invariant, a Borel subset $C$ of $X$ is called a measurable fundamental domain if $T^nC$ ($n \in \mathbb{Z}$) is mutually disjoint and the union $\bigcup_{n \in \mathbb{Z}} T^nC$ is conull. In this case any Borel function on $C$ can be extended to a $T$-invariant measurable function on $X$, and an ergodic component of $T$ is just a single orbit. The purpose of this paper is to show the following theorem.
**Theorem 1.1.** For any irrational number $\alpha$, there is a minimal $C^1$-diffeomorphism of the circle with rotation number $\alpha$ which admits a measurable fundamental domain with respect to the Lebesgue measure.

Sections 2, 3 and 4 are devoted to the proof of Theorem 1.1.

Let us mention an important remark and a further question.

**Remark 1.2.** In 2.1 of [DKN2], it is indicated how to construct examples of $C^1$-actions of the $n$-adic Thompson groups ($n \geq 10$) which are minimal but not ergodic. According to the referee, these actions admit measurable fundamental domains.

**Question 1.3.** Does there exist a minimal nonergodic $C^1_\tau$-diffeomorphism ($0 < \tau < 1$)? More generally for any $d \geq 2$ and $\tau > d^{-1}$, any free $\mathbb{Z}^d$-action by $C^1+\tau$-diffeomorphisms on $S^1$ is known to be minimal [DKN1]. Do there exist nonergodic actions? The method of this paper does not seem to be applicable to these problems.

**2. A Measurable Fundamental Domain for a Lipschitz Homeomorphism**

We regard the circle $S^1$ as $\mathbb{R}/\mathbb{Z}$. Suppose $R$ denotes the rotation by $\alpha$.

**Claim 2.1.** For any irrational number $\alpha$, we can construct a Cantor set $C \subset S^1$ so that $R^n C \cap R^m C = \emptyset$ for any integers $n \neq m$.

We will give a proof for the claim in Section 4. Here we show how to construct such a Cantor set for an easy case, namely, $\alpha = (\sqrt{5} - 1)/2$.

Define a Cantor set $C$ in the circle by

$$C = \left\{ \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{3k}} \mid \varepsilon_k = 0 \text{ or } 1 \right\} \mod \mathbb{Z}.$$  

Note that all numbers in $C$ are well approximable by rational numbers.

Suppose $x \in R^n C \cap R^m C$; then $x - n\alpha, x - m\alpha \in C$. Therefore

$$(-n + m)\alpha \in C + (-C) := \left\{ \sum_{k \geq 3} \frac{\varepsilon_k'}{2^{3k}} \mid \varepsilon_k' = 0 \text{ or } \pm 1 \right\} \mod \mathbb{Z}.$$  

$(-n + m)\alpha$ is badly approximable, while $C + (-C)$ consists of well approximable numbers, which is a contradiction. Therefore, this Cantor set $C$ satisfies the condition for Claim 2.1.

Fix a probability measure $\mu_0$ on $C$ without atoms such that $\text{supp}(\mu_0) = C$. We also choose a sequence $(a_i)_{i \in \mathbb{Z}}$ of positive numbers satisfying $\sum_{i \in \mathbb{Z}} a_i = 1$. Now we can define a probability measure $\mu$ on $S^1$ by

$$\mu := \sum_{i \in \mathbb{Z}} a_i R^i \mu_0.$$  

The Radon-Nikodým derivative $\frac{dR^{-1}_x \mu}{d\mu}$ is equal to $\frac{a_{i+1}}{a_i}$ on the set $R^i C$. Now we assume that $\frac{a_{i+1}}{a_i} \in \left[ \frac{1}{D}, D \right]$ for some $D > 1$. Then it follows that $\frac{dR^{-1}_x \mu}{d\mu} \in L^\infty(S^1, \mu)$. 

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What shall we do?

3.1. What shall we do? To prove the theorem, we are trying to make the Radon-Nikodým derivative \( g = \frac{dR^{-1}}{d\mu} \) continuous on \( S^1 \).

Fix an arbitrary point \( x_0 \in C \). For a positive integer \( i \),

\[
a_i = (a_i/a_{i-1}) \cdots (a_2/a_1)(a_1/a_0)a_0
\]

so that \( \supp J \) for a single point. A simple calculation shows that \( \Phi(x,y) = (R^x, y + \phi(x)) \). Therefore \( \phi = \exp(\phi(x_0))a_0 \). To satisfy \( \sum_{i\in\mathbb{Z}} a_i = 1 \), it suffices to find \( \phi \) so that \( \sum_{i\in\mathbb{Z}} \exp(\phi(x_0)) < \infty \).

3.2. Construction step I. Now we forget about the \( a_i \)'s and the Cantor set \( C \). As a first step, we construct a function \( \phi \in C(S^1) \) satisfying \( \sum_{i\in\mathbb{Z}} \exp(\phi(x_0)) < \infty \) for a single point \( x_0 \), where \( \phi(i) \) are defined by (3.1).

We will define continuous functions \( \phi_n \in C(S^1) \) \( (n \in \mathbb{N}) \) in such a way that \( \sum_{i=1}^{\infty} \|\phi_n\| < \infty \). Then \( \phi = \sum_{i=1}^{\infty} \phi_n \) converges uniformly; thus \( \phi \) is also continuous.

Fix an integer \( n \in \mathbb{N} \). Choose a sufficiently small neighbourhood \( J \) of \( x_0 \) so that \( R^{-2^n}J, \ldots, R^{-1}J, R^J, \ldots, R^{2^n-1}J \) are disjoint. Consider a bump function \( f \) on \( J \) so that \( \supp f \subset J \), \( f(x_0) = (3/4)^n \) and \( 0 \leq f(x) < (3/4)^n \) on \( J \setminus \{x_0\} \). Define \( \phi_n : S^1 \to \mathbb{R} \) by

\[
\phi_n(x) = \begin{cases} 
-f(R^{-i}x) & x \in R^i J, i = 0, 1, \ldots, 2^n - 1, \\
f(R^{-i}x) & x \in R^i J, i = -2^n, -2^n + 1, \ldots, -1, \\
0 & \text{otherwise}
\end{cases}
\]

**Lemma 3.1.** \( \phi_n(i)(x_0) \leq 0 \) for any \( i \in \mathbb{Z} \).
Proof. The equality for the first case is trivial. Define an increasing sequence
\((m_k)_{k \in \mathbb{Z}}\) by \(m_0 = 0\) and \(\{m_k | k \in \mathbb{Z}\} = \{m \in \mathbb{Z} | R^m x_0 \in J\} \). Since \(R^{2^n} J, \ldots, R^{2^n-1} J\) are disjoint, \(m_{k+1} - m_k \geq 2^{n+1}\) for any \(k \in \mathbb{Z}\). Using this sequence, \(R^m x_0 \in R^i J\) if and only if \(m = m_k + i\) for some \(k\). Therefore,

\[
\phi^{(i+1)}(x_0) = \begin{cases} 
\phi^{(i)}(x_0) & m_{k-1} + 2^n \leq i < m_k - 2^n, \\
\phi^{(i)}(x_0) + f(R^{m_k} x_0) & m_k - 2^n \leq i < m_k, \\
\phi^{(i)}(x_0) - f(R^{m_k} x_0) & m_k \leq i < m_k + 2^n
\end{cases}
\]

for some \(k\); see also Figure 1. Induction for \(|k|\) shows that

\[
\phi^{(i)}(x_0) = \begin{cases} 
-2^n \cdot (3/4)^n & m_{k-1} + 2^n \leq i \leq m_k - 2^n, \\
-2^n \cdot (3/4)^n + (i - (m_k - 2^n)) f(R^{m_k} x_0) & m_k - 2^n \leq i \leq m_k, \\
-2^n \cdot (3/4)^n + ((m_k + 2^n) - i) f(R^{m_k} x_0) & m_k \leq i \leq m_k + 2^n.
\end{cases}
\]
Since \( f(R^{m_1}x_0) \leq (3/4)^n \), the inequality \( \phi^{(i)}(x_0) \leq 0 \) also holds.
\[ \square \]

Therefore, if \( 2^n \leq |i| < 2^{n+1} \), \( \phi^{(i)}(x_0) \leq \phi^{(i)}_{n+1}(x_0) = -|i|(3/4)^{n+1} \leq -2^n \cdot (3/4)^{n+1} = -3/4 \cdot (3/2)^n \). Finally, \( \sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x_0)) \leq 1 + \sum_{n=0}^{\infty} 2^{n+1} \exp(-3/4 \cdot (3/2)^n) = M < \infty \).

3.3. Construction step II. We will execute the same argument for the Cantor set \( C \) instead of the single point \( x_0 \). Since \( R^{-2^n}C, \ldots, C, \ldots, R^{2^n-1}C \) are disjoint compact sets, there exists an \( \varepsilon \)-neighbourhood \( N \) of \( C \) such that \( R^{-2^n}N, \ldots, N, \ldots, R^{2^n-1}N \) are disjoint. Define a bump function \( f \) so that \( \text{supp} f \subset N \), \( f(x) = (3/4)^n \) on \( C \) and \( 0 \leq f(x) < (3/4)^n \) on \( N \setminus C \). Now we apply the same argument as in the previous subsection to obtain the function \( \phi \in C(S^1) \) such that \( \sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x)) < M < \infty \) for any \( x \in C \).

We define a finite measure \( \tilde{\mu} \) on \( S^1 \) by
\[
\tilde{\mu} := \sum_{i \in \mathbb{Z}} (\exp \circ \phi^{(i)} \circ R^{-1}) R_i \mu_0.
\]

Normalize \( \tilde{\mu} \) to obtain a probability measure \( \mu \), namely \( \mu := \frac{\tilde{\mu}}{\int_{S^1} d\mu} \).

Define \( h \) and \( F \) as in section 2. Then
\[
dF^{-1}_* \text{Leb} \quad \frac{d}{d \text{Leb}} = \frac{dR^{-1}_* \mu}{d\mu} \circ h = g \circ h
\]
is a continuous function because \( g(x) = \exp(\phi(x)) \). We have proved Theorem 1.1 up to Claim 2.1.

4. Construction of Cantor set for general \( \alpha \)

We are going to prove Claim 2.1 for a general irrational number \( \alpha \). For a real number \( x \) and a function \( p : \mathbb{N} \to \mathbb{N} \), define the approximation constant \( c_p(x) \) by
\[
c_p(x) := \liminf_{q \to \infty} \left( p(q) \cdot \text{dist}(x, \frac{1}{q} \mathbb{Z}) \right).
\]
A real number \( x \) is said to be \( p \)-approximable if \( c_p(x) = 0 \). Note that \( x \) is well approximable if \( x \) is \( p \)-approximable for \( p(q) = q^2 \), so this is a generalization of well-approximability.

It is clear that \( c_p(x) = 0 \) if \( x \) is a rational number. On the other hand, for any irrational number \( x \) we can find a function \( p \) satisfying \( c_p(x) > 0 \). Moreover, we will show the following lemma.

**Lemma 4.1.** For a given irrational number \( \alpha \), we can find a function \( p \) such that \( c_p(m\alpha) \geq 1 \) for any nonzero integer \( m \).

**Proof.** Since \( c_p(-mx) = c_p(mx) \), it is enough to show the lemma for the case \( m \in \mathbb{N} \). Let us start for any natural numbers \( n \) and \( q \) by taking a natural number \( p_n(q) \) so that \( p_n(q) \cdot \text{dist}(n\alpha, \frac{1}{q} \mathbb{Z}) \geq 1 \). Then define a function \( p \) by
\[
p(q) = \max_{1 \leq n \leq q} p_n(q).
\]
By this construction $p(q) \geq p_m(q)$ for any $q \geq m$. Therefore
\[
c_p(m\alpha) = \liminf_{q \to \infty} \left( p(q) \cdot \text{dist}(m\alpha, \frac{1}{q}Z) \right)
\geq \liminf_{q \to \infty} \left( p_m(q) \cdot \text{dist}(m\alpha, \frac{1}{q}Z) \right)
\geq 1.
\]
\[\square\]

For this function $p$, we inductively take an increasing sequence $q_0, q_1, \ldots$ of natural numbers satisfying the following conditions:

\[q_0 = 1, q_n|q_{n+1}, q_n/q_{n+1} \leq 1/3\] and $p(q_n)/q_{n+1} \leq 2^{-n}$. Define a Cantor set $C$ by
\[
C := \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_n} \mid \varepsilon_n = 0 \text{ or } 1 \right\}.
\]

This Cantor set $C$ consists of $p$-approximable numbers. We can also show the following lemma.

**Lemma 4.2.** For any $\beta \in C - C$, the approximation constant $c_p(\beta)$ is equal to 0, where
\[
C - C = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon'_n}{q_n} \mid \varepsilon'_n = 0 \text{ or } \pm 1 \right\}.
\]

**Proof.**
\[
p(q_i) \cdot \text{dist}(\beta, \frac{1}{q_i}Z) \leq p(q_i) \left| \sum_{n=i+1}^{\infty} \frac{\varepsilon'_n}{q_n} \right|
\leq p(q_i) \sum_{n=i+1}^{\infty} \frac{1}{q_n} = p(q_i) \sum_{n=i+1}^{\infty} \frac{q_i+1}{q_n} \leq \frac{1}{2i} \sum_{k=0}^{\infty} \left( \frac{1}{3} \right)^k = \frac{3}{2^{i+1}}.
\]

Thus
\[
c_p(\beta) = \liminf_{q \to \infty} \left( p(q) \cdot \text{dist}(\beta, \frac{1}{q}Z) \right)
\leq \liminf_{i \to \infty} \left( p(q_i) \cdot \text{dist}(\beta, \frac{1}{q_i}Z) \right)
\leq \liminf_{i \to \infty} \frac{3}{2^{i+1}}
= 0.
\]

Therefore $c_p(\beta) = 0$. \[\square\]

Claim 2.1 follows from Lemma 4.1 and Lemma 4.2, so we have proved Theorem 1.1 for the general case.

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