BINOMIAL ARITHMETICAL RANK OF EDGE IDEALS OF FORESTS

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Abstract. We prove that the binomial arithmetical rank of the edge ideal of a forest coincides with its big height.

1. Introduction

Let $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field $K$ and $I$ a square-free monomial ideal of $S$. The arithmetical rank of $I$ is defined by the minimum number $r$ of elements $a_1, a_2, \ldots, a_r \in S$ which generate $I$ up to radical, that is, $\sqrt{(a_1, a_2, \ldots, a_r)} = I$ holds. We denote it by $ara_I$. The binomial arithmetical rank of $I$ is defined by the minimum number $r$ of binomials or monomials $a_1, a_2, \ldots, a_r \in S$ which generate $I$ up to radical. We denote it by $baira_I$. By Lyubeznik [12] we know that

$$pd S/I \leq ara I \leq baira I,$$

where $pd S/I$ denotes the projective dimension of $S/I$. We are interested in the problem when the equality $ara I = pd S/I$ holds. It is proved for some classes of squarefree monomial ideals; see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16].

In this paper, a graph is assumed to be finite and simple. Denote by $G = (V(G), E(G))$ the graph with the vertex set $V(G)$ and the edge set $E(G)$. We consider the polynomial ring $K[V(G)]$ whose variables are $x_v, v \in V(G)$. The ideal of $K[V(G)]$ generated by the quadratic squarefree monomials $x_u x_v, \{u, v\} \in E(G)$ is called the edge ideal of $G$ and denoted by $I(G)$. We work on the above problem for the edge ideals of forests, graphs with no cycle. This was first discussed by Barile [1] and settled for special forests $G$.

In this paper, we compare the binomial arithmetical rank with the big height instead of the projective dimension. Here the big height of a squarefree monomial ideal $I$, denoted by $bight I$ is the maximum height of the minimal prime ideals of $I$. Note that the inequality $bight I \leq pd S/I$ always holds. Precisely, we prove the following theorem:

**Theorem 1.1.** Let $G$ be a forest. Then

$$bight I(G) = pd K[V(G)]/I(G) = ara I(G) = baira I(G).$$
To obtain Theorem 1.1 through combinatorial interpretations of the big height and binomial arithmetical rank of the edge ideal of a tree, we prove a purely graph-theoretic theorem, which shows an existence of a certain decomposition of a tree. See Theorem 3.2.

2. Tree-like systems and primitive trees

In this section we define some notions on a graph which have connections with the binomial arithmetical rank of an edge ideal.

First we introduce a notion of a tree-like system.

Definition 2.1. Let $G$ be a graph. We call the sequence $\Sigma: T_1, T_2, \ldots, T_r$ of non-empty subsets of $E(G)$ with $\#T_1 = 1$, $\#T_i \leq 2$, $i = 2, 3, \ldots, r$ a tree-like system of $G$ with length $r$ if the following two conditions are satisfied:

(1) The edge set $E(G)$ of $G$ is the disjoint union of $T_1, T_2, \ldots, T_r$.

(2) For each $i$ with $\#T_i = 2$, there exist an integer $j < i$ and two vertices $u, v$ each of which is contained in an edge in $T_i$ such that $\{u, v\} \in T_j$.

Remark 2.2. Barile [1, Definition 1, p. 4681] first introduced a tree-like system, which has little difference from ours.

A fundamental technique in the study of the arithmetical rank is the following lemma due to Schmitt and Vogel [15].

Lemma 2.3 (Schmitt and Vogel [15 Lemma, p. 249]). Let $S$ be a ring and $P_1, P_2, \ldots, P_r$ finite subsets of $S$. Assume that $P_1, P_2, \ldots, P_r$ satisfy the following two conditions:

(1) $\#P_1 = 1$.

(2) For all $i > 1$ and $a, a'' \in P_i$, $a \neq a''$, there exists $j < i$ and $a' \in P_j$ such that $aa'' \in (a')$.

Let $I$ be an ideal generated by $P_1, P_2, \ldots, P_r$. We set

$$q_i = \sum_{a \in P_i} a, \quad i = 1, 2, \ldots, r.$$ 

Then $q_1, q_2, \ldots, q_r$ generate $I$ up to radical.

We can apply Lemma 2.3 to have the following proposition.

Proposition 2.4. If a graph $G$ has a tree-like system of length $r$, then

$$\text{biara } I(G) \leq r.$$ 

A tree is defined to be a connected forest. Next, we introduce a notion of a primitive tree and define for a tree $T$ the number $b(T)$, which is a kind of combinatorial interpretation of the binomial arithmetical rank of an edge ideal.

Definition 2.5. We say that a tree $T$ is primitive if it is obtained by the following recursive procedure:

(1) The complete graph $K_2$ with two vertices is primitive.

(2) Let $T$ be a primitive tree and $\{u, v\} \in E(T)$. Take two new vertices $x, y \notin V(T)$. Then $T' := (V(T) \cup \{x, y\}, E(T) \cup \{\{u, x\}, \{v, y\}\})$ is primitive.
Note that \( \#E(T) \) is odd for a primitive tree \( T \).

Let \( T \) be a tree. A set \( \mathcal{D} = \{T_1, T_2, \ldots, T_m\} \) of subgraphs of \( T \) is called a decomposition of \( T \) with primitive trees if each \( T_i \) is a primitive tree and \( E(T) \) is the disjoint union of \( E(T_1), E(T_2), \ldots, E(T_m) \). Note that \( \frac{(\#E(T) + \#\mathcal{D})}{2} = \sum_{i=1}^{m}(\#E(T_i) + 1)/2 \) is an integer.

For \( e \in E(T) \), let \( T(e) \) denote the subgraph of \( T \) with the vertex set \( e \) and edge set \{\( e \)\}. Then \( T(e) \) is a primitive tree and \( \mathcal{D}_0 := \{T(e) : e \in E(T)\} \) is a decomposition of \( T \) with primitive trees. We call \( \mathcal{D}_0 \) the trivial decomposition of \( T \) with primitive trees.

We set
\[
m(T) := \min\{\#\mathcal{D} : \mathcal{D} \text{ is a decomposition of } T \text{ with primitive trees}\},
\]
\[
b(T) := \frac{1}{2}(\#E(T) + m(T)).
\]

We give a characterization of \( b(T) \) in terms of a tree-like system of \( T \).

**Proposition 2.6.** The integer \( b(T) \) of a tree \( T \) gives the minimum length among the tree-like systems of \( T \).

**Proof.** We first show that there is a tree-like system of \( T \) with length \( b(T) \). For this purpose it is sufficient to prove that a primitive tree \( T \) has a tree-like system of length \( b(T) = (\#E(T) + 1)/2 \). The proof proceeds by induction on \( \#E(T) \).

When \( \#E(T) = 1 \), the only edge \( e \in E(T) \) is a tree-like system of length \( b(T) = 1 \).

We assume that \( \#E(T) \geq 2 \). By the definition of the primitive tree, there exist a primitive tree \( T_0 \), an edge \( \{u, v\} \in E(T_0) \), and vertices \( x, y \in V(T) \setminus V(T_0) \) such that
\[
T = (V(T_0) \cup \{x, y\}, E(T_0) \cup \{(u, x), (v, y)\}).
\]
Since \( \#E(T) = \#E(T_0) + 2 \), we have that \( b(T) = b(T_0) + 1 \). Let \( T_1, T_2, \ldots, T_r \) be a tree-like system of \( T_0 \) of length \( r := b(T_0) \). We set \( T_{r+1} = \{(u, x), (v, y)\} \). Since \( \{u, v\} \) is an edge of \( T_0 \) and there exists some \( T_i \) which contains \( \{u, v\} \), the sequence \( T_1, T_2, \ldots, T_r, T_{r+1} \) forms a tree-like system of \( T \) with length \( r + 1 = b(T) \).

Now we show that \( b(T) \) gives the minimum length. Let \( \Sigma : T_1, T_2, \ldots, T_r \) be a tree-like system of \( T \). If
\[
\Sigma' : T_1, T_2, \ldots, T_{i-1}, T_j, T_i, T_{i+1}, \ldots, T_{j-1}, T_{j+1}, \ldots, T_r
\]
is also a tree-like system of \( T \) for a pair \( i < j \) with \( \#T_i = 1 \) and \( \#T_j = 2 \), then we consider \( \Sigma' \) instead of \( \Sigma \). If there is such a pair with respect to \( \Sigma' \), then we consider such a permutation for \( \Sigma' \), and we repeat this process as long as there is such a pair. Then from the beginning we may assume that \( \Sigma \) does not allow such a permutation, since the length does not change. We rename the elements of \( \Sigma \) as follows:
\[
\Sigma : T_{i,1}, T_{i,2}, \ldots, T_{i,r_i}, T_{j,1}, T_{j,2}, \ldots, T_{j,r_j}
\]
such that \( \#T_{i,1} = 1 \), \( \#T_{i,j} = 2 \) for \( i = 1, 2, \ldots, \ell \) and \( j > 1 \). We set
\[
\Sigma_i : T_{i,1}, T_{i,2}, \ldots, T_{i,r_i}
\]
for \( i = 1, 2, \ldots, \ell \). Let \( T_i \) be the graph whose edge set consists of all edges appearing in \( T_{i,r} \). Then \( \Sigma_i \) is a tree-like system of \( T_i \). It is easy to see that \( T_i \) is a tree. We claim that \( T_i \) is primitive. Let \( T_{i,k} (k = 1, 2, \ldots, r_i) \) be the tree whose edge set is the union of \( T_{i,1}, T_{i,2}, \ldots, T_{i,k} \). We prove that \( T_{i,k} \) is primitive for all \( k \). Since
$\#E(T_{i,1}) = \#T_{i,1} = 1$, the tree $T_{i,1}$ is primitive. We assume that $T_{i,k}$ is a primitive tree. Set $T_{i,k+1} = \{\{u, x\}, \{v, y\}\}$ and assume that $\{u, v\} \in E(T_{i,k})$. Since $T$ is a tree, $x$ and $y$ are distinct. Thus it is enough to prove that $x, y$ do not belong to $V(T_{i,k})$. Suppose that $x$ belongs to $V(T_{i,k})$. Then there exists a path in $T_{i,k}$ which connects $x$ and $u$. Since $\{x, u\} \in E(T) \setminus E(T_{i,k})$, it follows that $T$ contains a cycle. This is a contradiction. Therefore $D = \{T_1, T_2, \ldots, T_\ell\}$ is a decomposition of $T$ with primitive trees. Then the length of $\Sigma$ is equal to $r = (\#E(T) + \#D)/2$, which is greater than or equal to $b(T)$. 

**Example 2.7.** Let $T$ be a tree with the vertex set $V(T) = \{u, v, x, y\}$ and with the edge set $\{\{u, v\}, \{u, x\}, \{v, y\}\}$:

Then $T$ is a primitive tree. Indeed, it is obtained by one operation from a complete graph $T(\{u, v\}) \cong K_2$. In this case, $b(T) = 2$ and $T$ has the tree-like system $\{\{u, v\}\}, \{\{u, x\}, \{v, y\}\}$, which is of length 2.

### 3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In fact, we prove the following equality for a forest $G$:

$$\text{bight} I(G) = \text{biara} I(G),$$

since we know

$$\text{bight} I(G) \leq \text{pd} K[V(G)]/I(G) \leq \text{ara} I(G) \leq \text{biara} I(G).$$

First we reformulate it in terms of graph theory. Let $G = (V(G), E(G))$ be a graph. A subset $C$ of the vertex set $V(G)$ is called a vertex cover of $G$ if it intersects all edges of $G$. If it is minimal under inclusion, it is said to be a minimal vertex cover. Since the correspondence from a minimal vertex cover $C$ of $G$ to the minimal prime ideal $P_C = (x_v : v \in C)$ of $I(G)$ is one-to-one, we have the following lemma:

**Lemma 3.1.** For a graph $G$ the big height $\text{bight} I(G)$ of the edge ideal $I(G)$ is equal to the maximum cardinality among the minimal vertex covers of $G$.

For Theorem 1.1 it is enough to consider the case of a tree. By Propositions 2.4 and 2.6 we have

(3.1) $$\text{bight} I(T) \leq \text{biara} I(T) \leq b(T).$$

Hence to show $\text{bight} I(T) = \text{biara} I(T)$, it is enough to prove $\text{bight} I(T) = b(T)$. By Lemma 3.1 the equality $\text{bight} I(T) = b(T)$ is nothing but the following characterization of $b(T)$ in terms of graph theory.

**Theorem 3.2.** Let $T$ be a tree. Then $b(T)$ is equal to the maximum cardinality among the minimal vertex covers of $T$.

By (3.1) we know that $\text{bight} I(T) \leq b(T)$. Hence, by Lemma 3.1 to show the above theorem it is enough to prove the following lemma:
Lemma 3.3. Let $T$ be a tree. Then there exists a minimal vertex cover $W$ of $T$ with $\#W \geq b(T)$.

Proof. The proof proceeds by induction on $n = \#V(T)$. When $n = 2$, the tree $T$ is a complete graph with two vertices and we have $\#E(T) = m(T) = 1$. Thus $b(T) = 1$ and the assertion is trivially true.

We assume that the assertion is true for trees $T'$ with $\#V(T') < n$. Let $T$ be a tree with $\#V(T) = n$. We denote by $l(T)$ the maximum length of paths in $T$.

If $l(T) = 2$, then $T$ is a complete bipartite graph of type $(1,n-1)$: $E(T) = \{\{x_0,x_1\},\{x_0,x_2\},\ldots,\{x_0,x_{n-1}\}\}$.

In this case $T$ has only the trivial decomposition of $T$ with primitive trees. Thus $\#E(T) = m(T) = n-1$ and $b(T) = n-1$. Then $W = \{x_1,x_2,\ldots,x_{n-1}\}$ is a minimal vertex cover of $T$ with $\#W = n-1 = b(T)$.

When $l(T) = 3$, the edge set of $T$ is

$$E(T) = \{\{x_0,y_0\},\{x_0,x_1\},\{x_0,x_2\},\ldots,\{x_0,x_p\},\{y_0,y_1\},\{y_0,y_2\},\ldots,\{y_0,y_q\}\},$$

where $p,q \geq 1$. We may assume that $p \geq q$. Then $\#E(T) = p + q + 1$. Moreover $m(T) = 1 + p - q$.

Hence

$$b(T) = \frac{1}{2}((p + q + 1) + (1 + p - q)) = p + 1$$

and $W = \{y_0,x_1,x_2,\ldots,x_p\}$ is a desired vertex cover of $T$.

Now we assume that $l := l(T) \geq 4$. Let $\{v_0,v_1,\ldots,v_l\}$ be a path of $T$ with the maximum length. We divide the proof into two parts.

Case 1. First we assume that there exists a leaf $u$ of $T$ which is a neighbourhood of $v_{l-2}$. 
We consider the induced subgraph $T' := T_{V(T) \setminus \{u, v_l\}}$. Let $\mathcal{D} = \{T_1, T_2, \ldots, T_m\}$ be a decomposition of $T'$ with primitive trees. We may assume that $\{v_{l-2}, v_{l-1}\} \subseteq E(T_1)$. Then the tree $T_1$, with the vertex set $V(T_1) \cup \{u, v_l\}$ and the edge set $E(T_1) \cup \{\{v_{l-2}, u\}, \{v_{l-1}, v_l\}\}$ is primitive. Thus $\mathcal{D} = \{T_1, T_2, \ldots, T_m\}$ is a decomposition of $T$ with primitive trees. Hence we have $b(T) \leq b(T') + 1$.

By the inductive hypothesis, there exists a minimal vertex cover $W'$ of $T'$ with $\#W' \geq b(T')$. If $v_{l-2} \notin W'$, then $v_{l-1} \in W'$. In this case, $W := W' \cup \{u\}$ is a minimal vertex cover of $T$ with

$$b(T) \leq b(T') + 1 \leq \#W' + 1 = \#W.$$

Next we assume that $v_{l-2} \in W'$. If there is no leaf which is adjacent to $v_{l-1}$ in $T'$, then $v_{l-1} \notin W'$. Thus $W := W' \cup \{v_l\}$ is a minimal vertex cover of $T$ with

$$b(T) \leq b(T') + 1 \leq \#W' + 1 = \#W.$$ 

Otherwise, let $w_1, \ldots, w_p$ be all the leaves adjoining the $v_{l-1}$ in $T'$. Then $W'' := (W' \setminus \{v_{l-1}\}) \cup \{w_1, \ldots, w_p\}$ is a minimal vertex cover of $T'$ with $\#W'' \geq \#W'$. We set $W := W'' \cup \{v_l\}$. Then $W$ is a minimal vertex cover of $T$ with

$$b(T) \leq b(T') + 1 \leq \#W'' + 1 = \#W.$$ 

Case 2. Next we assume that there is no leaf adjoining $v_{l-2}$ in $T$. For $v \in V(T)$, we denote by $N(v)$ the set of neighbours of $v$. We set $U := N(v_{l-2}) \setminus \{v_{l-3}\} = \{u_1, \ldots, u_p\}$ ($p \geq 1$), $X_i := N(u_i) \setminus \{v_{l-2}\} = \{x_{i1}, \ldots, x_{i,q_i}\}$ ($q_i \geq 1$), and $X = X_1 \cup \cdots \cup X_p$. Note that $v_{l-1} \in U$ and $v_l \in X$. 

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Also we set
\[ T' := T_v(T) \setminus \{v_{l-2}\} \cup U \cup X, \quad T'' := T_{\{v_{l-3}, v_{l-2}\}} \cup U \cup X. \]
Then \( T_0 := T''_{\{v_{l-3}, v_{l-2}\}} \cup U \cup \{x_{11}, x_{21}, \ldots, x_{p1}\} \) is a primitive tree.

\[ T_0: \]

Indeed, we can construct \( T_0 \) inductively by adding edges with the following order:

\[ \{v_{l-2}, u_1\}, \]
\[ \{v_{l-2}, u_2\}, \{u_1, x_{11}\}, \]
\[ \{v_{l-2}, u_3\}, \{u_2, x_{21}\}, \]
\[ \ldots, \]
\[ \{v_{l-2}, u_p\}, \{u_{p-1}, x_{p-1}\}, \]
\[ \{v_{l-2}, v_{l-3}\}, \{u_p, x_{p1}\}. \]
Thus \(m(T'') \leq 1 + \sum_{i=1}^{p} (q_i - 1)\) and we have
\[
b(T'') \leq \frac{1}{2} \left( 1 + p + \sum_{i=1}^{p} q_i \right) + \left( 1 + \sum_{i=1}^{p} (q_i - 1) \right) = 1 + \sum_{i=1}^{p} q_i.
\]
We set \(W'' = \{v_{n-2}\} \cup X\). Then \(#W'' = 1 + \sum_{i=1}^{p} q_i \geq b(T'')\). By the inductive hypothesis on \(T'\), there exists a minimal vertex cover \(W'\) of \(T'\) with \(#W' \geq b(T')\). Then \(W' \cap W'' = \emptyset\) and \(W := W' \cup W''\) is a minimal vertex cover of \(T\) with
\[#W = #W' + #W'' \geq b(T') + b(T'') \geq b(T),\]
as desired. \(\square\)

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