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BINOMIAL ARITHMETICAL RANK OF EDGE IDEALS OF FORESTS

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Abstract. We prove that the binomial arithmetical rank of the edge ideal of a forest coincides with its big height.

1. Introduction

Let $S = K[x_1, x_2, \dots, x_n]$ be a polynomial ring over a field K and I a squarefree monomial ideal of S. The *arithmetical rank* of I is defined by the minimum number r of elements $a_1, a_2, \dots, a_r \in S$ which *generate I up to radical*, that is, $\sqrt{(a_1, a_2, \dots, a_r)} = I$ holds. We denote it by $\operatorname{ara} I$. The *binomial arithmetical rank* of I is defined by the minimum number r of binomials or monomials $a_1, a_2, \dots, a_r \in S$ which *generate I up to radical*. We denote it by $\operatorname{biara} I$. By Lyubeznik [12] we know that

$$\operatorname{pd} S/I \leq \operatorname{ara} I \leq \operatorname{biara} I,$$

where $\operatorname{pd} S/I$ denotes the projective dimension of S/I. We are interested in the problem when the equality $\operatorname{ara} I = \operatorname{pd} S/I$ holds. It is proved for some classes of squarefree monomial ideals; see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16].

In this paper, a *graph* is assumed to be finite and simple. Denote by $G = (V(G), E(G))$ the graph with the vertex set $V(G)$ and the edge set $E(G)$. We consider the polynomial ring $K[V(G)]$ whose variables are x_v, $v \in V(G)$. The ideal of $K[V(G)]$ generated by the quadratic squarefree monomials $x_u x_v$, $\{u, v\} \in E(G)$ is called the *edge ideal* of G and denoted by $I(G)$. We work on the above problem for the edge ideals of forests, graphs with no cycle. This was first discussed by Barile [1] and settled for special forests G.

In this paper, we compare the binomial arithmetical rank with the big height instead of the projective dimension. Here the big height of a squarefree monomial ideal I, denoted by $\operatorname{bight} I$ is the maximum height of the minimal prime ideals of I. Note that the inequality $\operatorname{bight} I \leq \operatorname{pd} S/I$ always holds. Precisely, we prove the following theorem:

Theorem 1.1. *Let G be a forest. Then*

$$\operatorname{bight} I(G) = \operatorname{pd} K[V(G)]/I(G) = \operatorname{ara} I(G) = \operatorname{biara} I(G).$$

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To obtain Theorem 1.1 through combinatorial interpretations of the big height and binomial arithmetical rank of the edge ideal of a tree, we prove a purely graph-theoretic theorem, which shows an existence of a certain decomposition of a tree. See Theorem 3.2.

2. Tree-like systems and primitive trees

In this section we define some notions on a graph which have connections with the binomial arithmetical rank of an edge ideal.

First we introduce a notion of a tree-like system.

Definition 2.1. Let G be a graph. We call the sequence $\Sigma\colon \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_r$ of non-empty subsets of $E(G)$ with $\#\mathcal{T}_1 = 1$, $\#\mathcal{T}_i \leq 2$, $i = 2, 3, \ldots, r$ a *tree-like system* of G with length r if the following two conditions are satisfied:

1. The edge set $E(G)$ of G is the disjoint union of $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_r$.
2. For each i with $\#\mathcal{T}_i = 2$, there exist an integer $j < i$ and two vertices u, v each of which is contained in an edge in $\mathcal{T}_i$ such that $\{u, v\} \in \mathcal{T}_j$.

Remark 2.2. Barile [1, Definition 1, p. 4681] first introduced a tree-like system, which has little difference from ours.

A fundamental technique in the study of the arithmetical rank is the following lemma due to Schmitt and Vogel [15].

Lemma 2.3 (Schmitt and Vogel [15, Lemma, p. 249])**.** *Let S be a ring and $P_1, P_2, \ldots, P_r$ finite subsets of S. Assume that $P_1, P_2, \ldots, P_r$ satisfy the following two conditions:*

1. $\#P_1 = 1$.
2. *For all $i > 1$ and $a, a'' \in P_i$, $a \neq a''$, there exists $j < i$ and $a' \in P_j$ such that $aa'' \in (a')$.*

Let I be an ideal generated by $P_1, P_2, \ldots, P_r$. We set

$$q_i = \sum_{a \in P_i} a, \qquad i = 1, 2, \ldots, r.$$

Then $q_1, q_2, \ldots, q_r$ generate I up to radical.

We can apply Lemma 2.3 to have the following proposition.

Proposition 2.4. *If a graph G has a tree-like system of length r, then*

$$\operatorname{biara} I(G) \leq r.$$

A *tree* is defined to be a connected forest. Next, we introduce a notion of a primitive tree and define for a tree T the number $b(T)$, which is a kind of combinatorial interpretation of the binomial arithmetical rank of an edge ideal.

Definition 2.5. We say that a tree T is *primitive* if it is obtained by the following recursive procedure:

1. The complete graph K_2 with two vertices is primitive.
2. Let T be a primitive tree and $\{u, v\} \in E(T)$. Take two new vertices $x, y \notin V(T)$. Then $T' := (V(T) \cup \{x, y\}, E(T) \cup \{\{u, x\}, \{v, y\}\})$ is primitive.

Note that $\#E(T)$ is odd for a primitive tree T.

Let T be a tree. A set $\mathcal{D} = \{T_1, T_2, \dots, T_m\}$ of subgraphs of T is called a *decomposition of T with primitive trees* if each T_i is a primitive tree and $E(T)$ is the disjoint union of $E(T_1), E(T_2), \dots, E(T_m)$. Note that $(\#E(T) + \#\mathcal{D})/2 = \sum_{i=1}^{m}(\#E(T_i) + 1)/2$ is an integer.

For $e \in E(T)$, let $T(e)$ denote the subgraph of T with the vertex set e and edge set $\{e\}$. Then $T(e)$ is a primitive tree and $\mathcal{D}_0 := \{T(e) : e \in E(T)\}$ is a decomposition of T with primitive trees. We call $\mathcal{D}_0$ the trivial decomposition of T with primitive trees.

We set

$$m(T) := \min\{\#\mathcal{D} \ :\ \mathcal{D} \text{ is a decomposition of } T \text{ with primitive trees}\},$$

$$b(T) := \frac{1}{2}(\#E(T) + m(T)).$$

We give a characterization of $b(T)$ in terms of a tree-like system of T.

Proposition 2.6. *The integer $b(T)$ of a tree T gives the minimum length among the tree-like systems of T.*

Proof. We first show that there is a tree-like system of T with length $b(T)$. For this purpose it is sufficient to prove that a primitive tree T has a tree-like system of length $b(T) = (\#E(T) + 1)/2$. The proof proceeds by induction on $\#E(T)$.

When $\#E(T) = 1$, the only edge $e \in E(T)$ is a tree-like system of length $b(T) = 1$.

We assume that $\#E(T) \geq 2$. By the definition of the primitive tree, there exist a primitive tree T_0, an edge $\{u, v\} \in E(T_0)$, and vertices $x, y \in V(T) \setminus V(T_0)$ such that

$$T = (V(T_0) \cup \{x, y\},\ E(T_0) \cup \{\{u, x\}, \{v, y\}\}).$$

Since $\#E(T) = \#E(T_0) + 2$, we have that $b(T) = b(T_0) + 1$. Let $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r$ be a tree-like system of T_0 of length $r := b(T_0)$. We set $\mathcal{T}_{r+1} = \{\{u, x\}, \{v, y\}\}$. Since $\{u, v\}$ is an edge of T_0 and there exists some $\mathcal{T}_i$ which contains $\{u, v\}$, the sequence $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r, \mathcal{T}_{r+1}$ forms a tree-like system of T with length $r + 1 = b(T)$.

Now we show that $b(T)$ gives the minimum length. Let $\Sigma\colon \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r$ be a tree-like system of T. If

$$\Sigma'\colon \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{i-1}, \mathcal{T}_j, \mathcal{T}_i, \mathcal{T}_{i+1}, \dots, \mathcal{T}_{j-1}, \mathcal{T}_{j+1}, \dots, \mathcal{T}_r$$

is also a tree-like system of T for a pair $i < j$ with $\#\mathcal{T}_i = 1$ and $\#\mathcal{T}_j = 2$, then we consider Σ' instead of Σ. If there is such a pair with respect to Σ', then we consider such a permutation for Σ', and we repeat this process as long as there is such a pair. Then from the beginning we may assume that Σ does not allow such a permutation, since the length does not change. We rename the elements of Σ as follows:

$$\Sigma\colon \mathcal{T}_{1,1}, \mathcal{T}_{1,2}, \dots, \mathcal{T}_{1,r_1}, \mathcal{T}_{2,1}, \mathcal{T}_{2,2}, \dots, \mathcal{T}_{2,r_2}, \dots, \mathcal{T}_{\ell,1}, \mathcal{T}_{\ell,2}, \dots, \mathcal{T}_{\ell,r_\ell}$$

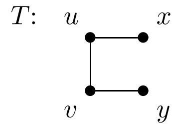
such that $\#\mathcal{T}_{i,1} = 1$, $\#\mathcal{T}_{i,j} = 2$ for $i = 1, 2, \dots, \ell$ and $j > 1$. We set

$$\Sigma_i\colon \mathcal{T}_{i,1}, \mathcal{T}_{i,2}, \dots, \mathcal{T}_{i,r_i}$$

for $i = 1, 2, \dots, \ell$. Let T_i be the graph whose edge set consists of all edges appearing in Σ_i. Then Σ_i is a tree-like system of T_i. It is easy to see that T_i is a tree. We claim that T_i is primitive. Let $T_{i,k}$ $(k = 1, 2, \dots, r_i)$ be the tree whose edge set is the union of $\mathcal{T}_{i,1}, \mathcal{T}_{i,2}, \dots, \mathcal{T}_{i,k}$. We prove that $T_{i,k}$ is primitive for all k. Since

$\#E(T_{i,1}) = \#\mathcal{T}_{i,1} = 1$, the tree $T_{i,1}$ is primitive. We assume that $T_{i,k}$ is a primitive tree. Set $\mathcal{T}_{i,k+1} = \{\{u,x\},\{v,y\}\}$ and assume that $\{u,v\} \in E(T_{i,k})$. Since T is a tree, x and y are distinct. Thus it is enough to prove that x, y do not belong to $V(T_{i,k})$. Suppose that x belongs to $V(T_{i,k})$. Then there exists a path in $T_{i,k}$ which connects x and u. Since $\{x,u\} \in E(T) \setminus E(T_{i,k})$, it follows that T contains a cycle. This is a contradiction. Therefore $\mathcal{D} = \{T_1, T_2, \dots, T_\ell\}$ is a decomposition of T with primitive trees. Then the length of Σ is equal to $r = (\#E(T) + \#\mathcal{D})/2$, which is greater than or equal to $b(T)$. ☐

Example 2.7. Let T be a tree with the vertex set $V(T) = \{u,v,x,y\}$ and with the edge set $\{\{u,v\},\{u,x\},\{v,y\}\}$:



Then T is a primitive tree. Indeed, it is obtained by one operation from a complete graph $T(\{u,v\}) \cong K_2$. In this case, $b(T) = 2$ and T has the tree-like system $\{\{u,v\}\}, \{\{u,x\},\{v,y\}\}$, which is of length 2.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In fact, we prove the following equality for a forest G:

$$\operatorname{bight} I(G) = \operatorname{biara} I(G),$$

since we know

$$\operatorname{bight} I(G) \le \operatorname{pd} K[V(G)]/I(G) \le \operatorname{ara} I(G) \le \operatorname{biara} I(G).$$

First we reformulate it in terms of graph theory. Let $G = (V(G), E(G))$ be a graph. A subset C of the vertex set $V(G)$ is called a *vertex cover* of G if it intersects all edges of G. If it is minimal under inclusion, it is said to be a *minimal vertex cover*. Since the correspondence from a minimal vertex cover C of G to the minimal prime ideal $P_C = (x_v : v \in C)$ of $I(G)$ is one-to-one, we have the following lemma:

Lemma 3.1. *For a graph G the big height* $\operatorname{bight} I(G)$ *of the edge ideal $I(G)$ is equal to the maximum cardinality among the minimal vertex covers of G.*

For Theorem 1.1 it is enough to consider the case of a tree. By Propositions 2.4 and 2.6 we have

$$\operatorname{bight} I(T) \le \operatorname{biara} I(T) \le b(T). \tag{3.1}$$

Hence to show $\operatorname{bight} I(T) = \operatorname{biara} I(T)$, it is enough to prove $\operatorname{bight} I(T) = b(T)$. By Lemma 3.1 the equality $\operatorname{bight} I(T) = b(T)$ is nothing but the following characterization of $b(T)$ in terms of graph theory.

Theorem 3.2. *Let T be a tree. Then $b(T)$ is equal to the maximum cardinality among the minimal vertex covers of T.*

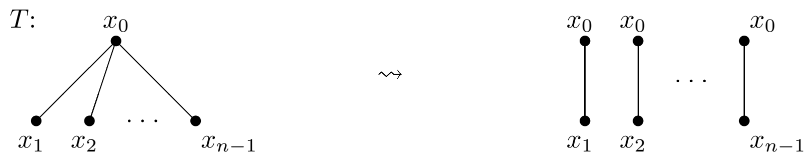
By (3.1) we know that $\operatorname{bight} I(T) \le b(T)$. Hence, by Lemma 3.1, to show the above theorem it is enough to prove the following lemma:

Lemma 3.3. *Let T be a tree. Then there exists a minimal vertex cover W of T with $\#W \geq b(T)$.*

Proof. The proof proceeds by induction on $n = \#V(T)$. When $n = 2$, the tree T is a complete graph with two vertices and we have $\#E(T) = m(T) = 1$. Thus $b(T) = 1$ and the assertion is trivially true.

We assume that the assertion is true for trees T' with $\#V(T') < n$. Let T be a tree with $\#V(T) = n$. We denote by $l(T)$ the maximum length of paths in T.

If $l(T) = 2$, then T is a complete bipartite graph of type $(1, n-1)$: $E(T) = \{\{x_0, x_1\}, \{x_0, x_2\}, \ldots, \{x_0, x_{n-1}\}\}$.

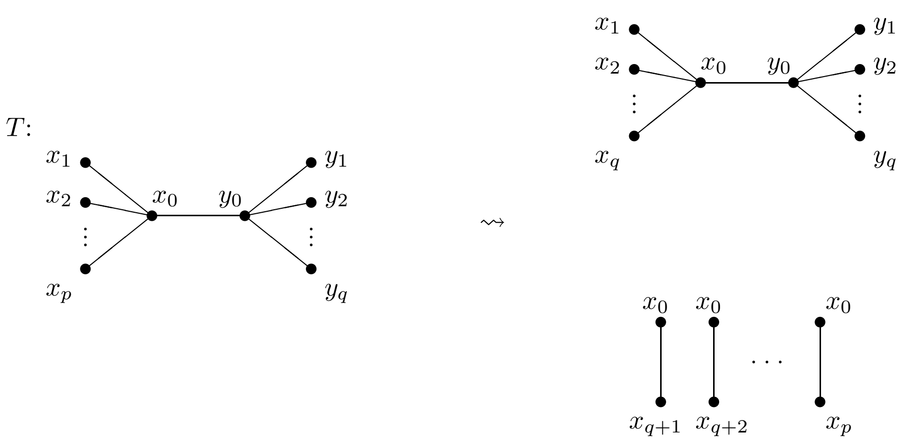


In this case T has only the trivial decomposition of T with primitive trees. Thus $\#E(T) = m(T) = n-1$ and $b(T) = n-1$. Then $W = \{x_1, x_2, \ldots, x_{n-1}\}$ is a minimal vertex cover of T with $\#W = n - 1 = b(T)$.

When $l(T) = 3$, the edge set of T is

$$E(T) = \{\{x_0, y_0\}, \{x_0, x_1\}, \{x_0, x_2\}, \ldots, \{x_0, x_p\}, \{y_0, y_1\}, \{y_0, y_2\}, \ldots, \{y_0, y_q\}\},$$

where $p, q \geq 1$. We may assume that $p \geq q$. Then $\#E(T) = p + q + 1$. Moreover $m(T) = 1 + p - q$.



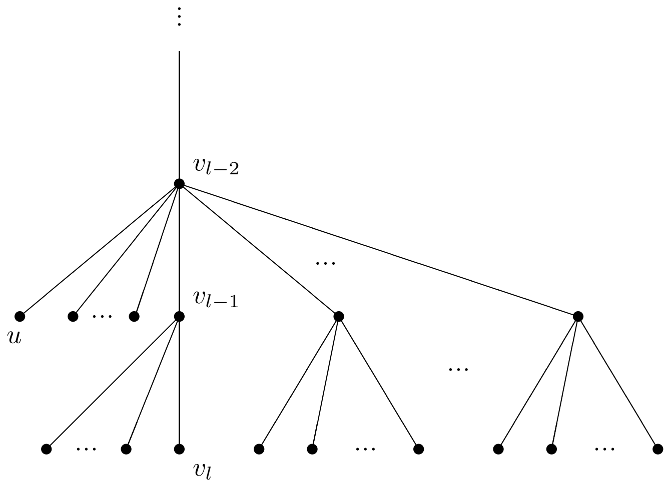
Hence

$$b(T) = \frac{1}{2}((p + q + 1) + (1 + p - q)) = p + 1$$

and $W = \{y_0, x_1, x_2, \ldots, x_p\}$ is a desired vertex cover of T.

Now we assume that $l := l(T) \geq 4$. Let $\{v_0, v_1, \ldots, v_l\}$ be a path of T with the maximum length. We divide the proof into two parts.

Case 1. First we assume that there exists a leaf u of T which is a neighbourhood of v_{l-2}.



We consider the induced subgraph $T' := T_{V(T)\setminus\{u,v_l\}}$. Let $\mathcal{D} = \{T_1, T_2, \ldots, T_m\}$ be a decomposition of T' with primitive trees. We may assume that $\{v_{l-2}, v_{l-1}\} \in E(T_1)$. Then the tree $\tilde{T}_1$ with the vertex set $V(T_1) \cup \{u, v_l\}$ and the edge set $E(T_1) \cup \{\{v_{l-2}, u\}, \{v_{l-1}, v_l\}\}$ is primitive. Thus $\tilde{\mathcal{D}} = \{\tilde{T}_1, T_2, \ldots, T_m\}$ is a decomposition of T with primitive trees. Hence we have $b(T) \le b(T') + 1$.

By the inductive hypothesis, there exists a minimal vertex cover W' of T' with $\#W' \ge b(T')$. If $v_{l-2} \notin W'$, then $v_{l-1} \in W'$. In this case, $W := W' \cup \{u\}$ is a minimal vertex cover of T with

$$b(T) \le b(T') + 1 \le \#W' + 1 = \#W.$$

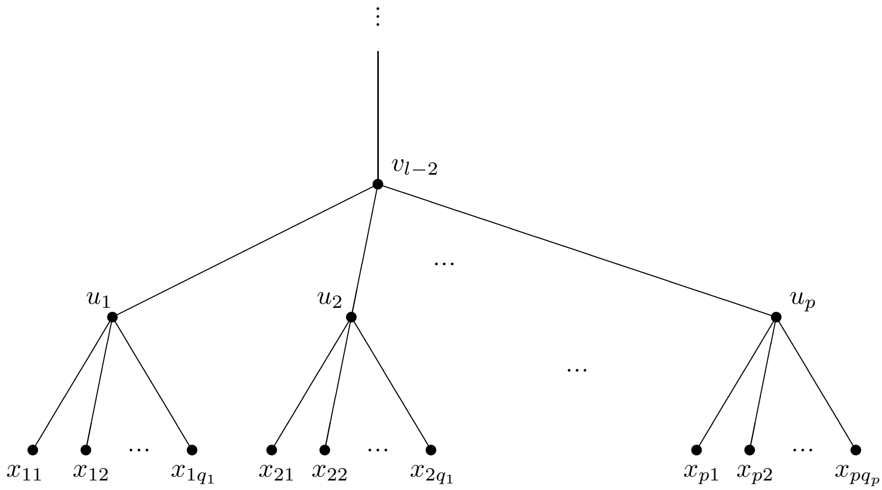
Next we assume that $v_{l-2} \in W'$. If there is no leaf which is adjacent to v_{l-1} in T', then $v_{l-1} \notin W'$. Thus $W := W' \cup \{v_l\}$ is a minimal vertex cover of T with

$$b(T) \le b(T') + 1 \le \#W' + 1 = \#W.$$

Otherwise, let $w_1, \ldots, w_p$ be all the leaves adjoining the v_{l-1} in T'. Then $W'' := (W' \setminus \{v_{l-1}\}) \cup \{w_1, \ldots, w_p\}$ is a minimal vertex cover of T' with $\#W'' \ge \#W'$. We set $W := W'' \cup \{v_l\}$. Then W is a minimal vertex cover of T with

$$b(T) \le b(T') + 1 \le \#W'' + 1 = \#W.$$

Case 2. Next we assume that there is no leaf adjoining v_{l-2} in T. For $v \in V(T)$, we denote by $N(v)$ the set of neighbours of v. We set $U := N(v_{l-2}) \setminus \{v_{l-3}\} := \{u_1, \ldots, u_p\}$ $(p \ge 1)$, $X_i := N(u_i) \setminus \{v_{l-2}\} := \{x_{i1}, \ldots, x_{iq_i}\}$ $(q_i \ge 1)$, and $X = X_1 \cup \cdots \cup X_p$. Note that $v_{l-1} \in U$ and $v_l \in X$.

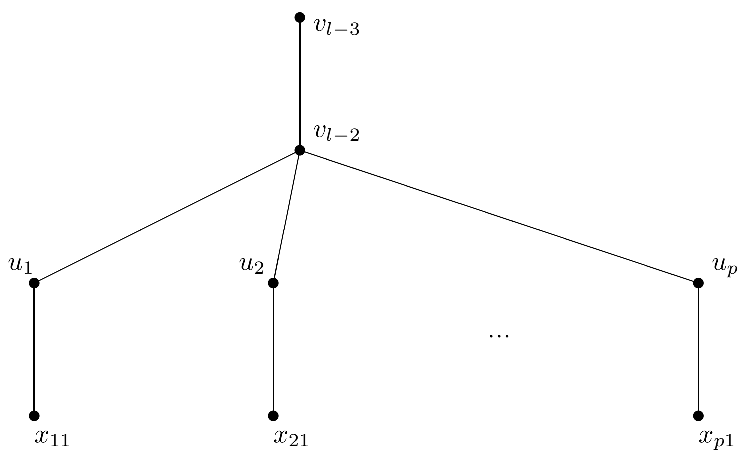


Also we set

$$T' := T_{V(T)\setminus(\{v_{l-2}\}\cup U\cup X)}, \qquad T'' := T_{\{v_{l-3},v_{l-2}\}\cup U\cup X}.$$

Then $T_0 := T''_{\{v_{l-3},v_{l-2}\}\cup U\cup\{x_{11},x_{21},\ldots,x_{p1}\}}$ is a primitive tree.

T_0:



Indeed, we can construct T_0 inductively by adding edges with the following order:

$$\begin{aligned}
&\{v_{l-2}, u_1\},\\
&\{v_{l-2}, u_2\}, \{u_1, x_{11}\},\\
&\{v_{l-2}, u_3\}, \{u_2, x_{21}\},\\
&\cdots,\\
&\{v_{l-2}, u_p\}, \{u_{p-1}, x_{p-11}\},\\
&\{v_{l-2}, v_{l-3}\}, \{u_p, x_{p1}\}.
\end{aligned}$$

Thus $m(T'') \leq 1 + \sum_{i=1}^{p}(q_i - 1)$ and we have

$$b(T'') \leq \frac{1}{2}\left\{\left(1 + p + \sum_{i=1}^{p} q_i\right) + \left(1 + \sum_{i=1}^{p}(q_i - 1)\right)\right\} = 1 + \sum_{i=1}^{p} q_i.$$

We set $W'' = \{v_{l-2}\} \cup X$. Then $\#W'' = 1 + \sum_{i=1}^{p} q_i \geq b(T'')$. By the inductive hypothesis on T', there exists a minimal vertex cover W' of T' with $\#W' \geq b(T')$. Then $W' \cap W'' = \emptyset$ and $W := W' \cup W''$ is a minimal vertex cover of T with

$$\#W = \#W' + \#W'' \geq b(T') + b(T'') \geq b(T),$$

as desired. □

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