PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 7, July 2013, Pages 2351–2359
S 0002-9939(2013)11504-2
Article electronically published on March 6, 2013

A CLASS OF DOMAINS WITH NONCOMPACT $\overline{\partial}$-NEUMANN OPERATOR

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(Communicated by Franc Forstneric)

Abstract. The $\overline{\partial}$-Neumann operator (the inverse of the complex Laplacian) is shown to be noncompact on certain domains in complex Euclidean space. These domains are either higher-dimensional analogs of the Hartogs triangle or have such a generalized Hartogs triangle imbedded appropriately in them.

1. Introduction

Let n_1, n_2 be positive integers, and let $n = n_1+n_2$. For a point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $'z$ for the point $(z_1, \dots, z_{n_1})$ in $\mathbb{C}^{n_1}$, and also write z' for the point $(z_{n_1+1}, \dots, z_n)$ in $\mathbb{C}^{n_2}$, and denote $z = ('z, z')$. Let $\alpha > 0$, and consider the bounded pseudoconvex domain H in $\mathbb{C}^n$ given by

$$H = \left\{ ('z, z') \in \mathbb{C}^n \colon \ \|'z\|_{(1)} < \left(\|z'\|_{(2)}\right)^{\alpha} < 1 \right\}, \tag{1}$$

where $\|\cdot\|_{(1)}$, $\|\cdot\|_{(2)}$ are arbitrary norms on the complex vector spaces $\mathbb{C}^{n_1}$, $\mathbb{C}^{n_2}$ respectively.

We may refer to the domain H of (1) as a *Hartogs triangle,* a term usually applied to the case $n_1 = n_2 = \alpha = 1$. This domain H belongs to the class of domains for which noncompactness of the $\overline{\partial}$-Neumann problem is established in this note. (See [11] for the relevant definitions, as well as a comprehensive discussion of compactness in the $\overline{\partial}$-Neumann problem. Other important texts dealing with the $\overline{\partial}$-Neumann problem include [4, 2].) To define precisely the class of domains we will be considering, note that the boundary $\mathsf{b}H$ of H contains the piece

$$\begin{aligned} S &= \left\{ ('z, z') \in \mathbb{C}^n \colon \ \|'z\|_{(1)} < 1, \|z'\|_{(2)} = 1 \right\} \\ &= \mathbb{B}_1 \times \mathsf{b}\mathbb{B}_2, \end{aligned} \tag{2}$$

where $\mathbb{B}_1 \subset \mathbb{C}^{n_1}$ and $\mathbb{B}_2 \subset \mathbb{C}^{n_2}$ are the unit balls in the norms $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ respectively. In this note, we prove noncompactness of the $\overline{\partial}$-Neumann operator on a bounded pseudoconvex domain Ω in $\mathbb{C}^n$, which has roughly speaking the following property: it is possible to embed the Hartogs triangle H of (1) into Ω in such a way that the subset S of the boundary of H given in (2) is mapped into the boundary of Ω. For example, it suffices to assume that there is a biholomorphic map from a neighborhood of $\overline{H}$ into $\mathbb{C}^n$ which maps H into Ω and S into $\mathsf{b}\Omega$.

Received by the editors July 13, 2011 and, in revised form, October 13, 2011.
2010 *Mathematics Subject Classification.* Primary 32W05.
Key words and phrases. $\overline{\partial}$-Neumann operator, Hartogs triangle.

More precisely, it is sufficient to assume the following hypotheses on Ω: (i) There exists a biholomorphic map

$$F : H \to F(H) \subset \Omega, \tag{3}$$

which extends to a $\mathcal{C}^\infty$-diffeomorphism of a neighborhood of $\overline{H}$ onto a neighborhood of $\overline{F(H)}$ in $\mathbb{C}^n$. (ii) Further, F itself extends biholomorphically to S in such a way that

$$F(S) \subset \mathsf{b}\Omega. \tag{4}$$

For such an Ω, we will prove the following:

Theorem. *For $1 \leq q \leq n_1$ the $\overline{\partial}$-Neumann operator of Ω acting on $L^2_{0,q}(\Omega)$ is noncompact.*

2. Some remarks

We note that for each $\zeta \in \mathsf{b}\mathbb{B}_2$, the subset $F(S) \subset b\Omega$ of (4) contains the n_1-dimensional analytic variety $F(\mathbb{B}_1 \times \{\zeta\})$, and this means absence of the classical compactness-entailing property (P_q), for $1 \leq q \leq n_1$. This strongly suggests the noncompactness of the $\overline{\partial}$-Neumann operator in degrees $(0, q)$ for $1 \leq q \leq n_1$. On the other hand, (P_q) is not known to be necessary for compactness, so this observation by itself does not show that the $\overline{\partial}$-Neumann operator is noncompact. When $n_2 = 1$, so that the boundary $\mathsf{b}\Omega$ contains $(n - 1 = n_1)$-dimensional complex manifolds, the noncompactness can be deduced from a result of Catlin, according to which the presence of such $(n-1)$-dimensional manifolds on the boundary of a weakly pseudoconvex domain in $\mathbb{C}^n$ implies noncompactness of the $\overline{\partial}$-Neumann operator on $(0, n-1)$ forms (see [5], where it is assumed that $n = 2$, but the boundary is allowed to be Lipschitz). In the analogs of this result for higher-codimensional manifolds in the boundary (see [11, Theorem 4.21]) one needs to assume that the boundary is strictly pseudoconvex in the directions transverse to the complex manifold. This is not true in general for the generalized Hartogs triangle of (1), since, e.g., we can take $\|w\|_{(2)} = \max_{1 \leq j \leq n_2} |w_2|$. In general, establishing compactness is a tricky business (see [11, Chapter 4]). Our interest in the rather special domains Ω stems from the fact that we are able to demonstrate noncompactness by an explicit and elementary counterexample in the spirit of [9, 10] and [8, Proposition 6.3] by exploiting the symmetry of Ω inherited from the rotational symmetry of H.

We also note here that the result can also be stated in the situation when Ω is a relatively compact Stein domain in a complex n-dimensional Hermitian manifold. Under the hypothesis of existence of a map F with the same properties as above, we can prove the noncompactness of the $\overline{\partial}$-Neumann operator on $L^2_{p,q}(\Omega)$, where $0 \leq p \leq n$ and $1 \leq q \leq n_1$. The changes required in the proof are purely formal, and for clarity of exposition we stick with the ambient manifold $\mathbb{C}^n$.

Let N be a positive integer, for $1 \leq j \leq N$, let n_j be a positive integer, and let $n = \sum_{j=1}^N n_j$. Suppose that we fix a norm $\|\cdot\|_{(j)}$ on the vector space $\mathbb{C}^{n_j}$ and denote a point $z \in \mathbb{C}^n$ by $(z^{(1)}, \ldots, z^{(N)})$, where $z^{(j)} \in \mathbb{C}^{n_j}$. In principle, it should be possible to modify the proof of the theorem to prove the noncompactness of the $\overline{\partial}$-Neumann operator on a domain of the form

$$\widetilde{H} = \left\{ z \in \mathbb{C}^n \colon \left\| z^{(1)} \right\|_{(1)} < \left\| z^{(2)} \right\|_{(2)}^{\alpha_2} < \left\| z^{(3)} \right\|_{(3)}^{\alpha_3} < \cdots < \left\| z^{(N)} \right\|_{(N)}^{\alpha_N} < 1 \right\}$$

or, more generally, on a domain in which $\widetilde{H}$ is appropriately embedded. The exposition in the general case will involve complicated notation arising from the more intricate geometry, and for clarity, we write the proof in a simple situation.

3. Preliminaries

Pulling back the canonical metric of $\mathbb{C}^n$ via the map F^{-1} (with F as in (3)), followed by a partition of unity argument, there is a smooth Hermitian metric h on $\overline{\Omega}$, such that the map F is an isometry on H. We use the metric h to define the pointwise inner product on spaces of forms, and the volume form on $\overline{\Omega}$ in the standard way, and let $L^2_{p,q}(\Omega)$ be the L^2-space of (p,q)-forms defined by this pointwise inner product and volume form. Thanks to [1, Theorem 1], we know that the fact of compactness or noncompactness of the $\overline{\partial}$-Neumann operator on the bounded domain Ω is independent of the Hermitian metric on Ω, as long as the metric is smooth on $\overline{\Omega}$. Consequently, it will suffice to show that the $\overline{\partial}$-Neumann operator defined with respect to the metric h acting on $L^2_{0,q}(\Omega)$ (denoted by N_q) is noncompact, for $1 \leq q \leq n_1$. Since N_{q+1} is compact if N_q is compact for $1 \leq q \leq n-1$ (see [11, Proposition 4.5]), it follows that it is sufficient to show that the operator $\mathsf{N}_{n_1} : L^2_{0,n_1}(\Omega) \to L^2_{0,n_1}(\Omega)$ is noncompact.

Let $\overline{\partial}^*$ denote the Hilbert-space adjoint of $\overline{\partial}$ on $L^2_{0,n_1}(\Omega)$ with respect to the metric h, and denote by $\mathfrak{H}$ the space

$$\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \subset L^2_{0,n_1}(\Omega), \tag{5}$$

which is a Hilbert space with norm

$$\|f\|_{\mathfrak{H}} = \left(\left\|\overline{\partial} f\right\|^2_{L^2_{0,n_1+1}(\Omega)} + \left\|\overline{\partial}^* f\right\|^2_{L^2_{0,n_1-1}(\Omega)} \right)^{\frac{1}{2}}. \tag{6}$$

To prove noncompactness of N_{n_1}, it is sufficient to show that the inclusion map of $\mathfrak{H}$ into $L^2_{0,n_1}(\Omega)$ is noncompact ([11, Proposition 4.2]). Before we proceed to show this, we compute certain quantities which will be useful in the proof. We will denote the $(2n_2-1)$-dimensional Hausdorff measure induced on a hypersurface in $\mathbb{R}^{2n_2} = \mathbb{C}^{n_2}$ by $d\sigma$. For $\nu > 0$, we consider the function

$$\gamma(\nu) = \int_{\mathsf{b}\mathbb{B}_2} |t_{n_2}|^{2\nu}\, d\sigma(t), \tag{7}$$

where $t = (t_1, \ldots, t_{n_2}) \in \mathbb{C}^{n_2}$, and the integral extends over the boundary $\mathsf{b}\mathbb{B}_2 = \{\|t\|_{(2)} = 1\}$ of the unit ball $\mathbb{B}_2$ of the norm $\|\cdot\|_{(2)}$ on $\mathbb{C}^{n_2}$. We will denote by dV the Lebesgue measure on Euclidean spaces of arbitrary dimensions (the dimension being known from the context). We first prove a couple of lemmas.

Lemma 1. *For $\beta \geq 0$, we have* $\displaystyle\int_{\mathbb{B}_2} |w_{n_2}|^{2\nu} \|w\|^{2\beta}_{(2)}\, dV(w) = \frac{\gamma(\nu)}{2(\nu+\beta+n_2)}$.

Proof. Using the co-area formula, we can rewrite the integral on the left as

$$\int_{r=0}^{1} \left(\int_{\{\|w\|_{(2)}=r\}} |w_{n_2}|^{2\nu}\, r^{2\beta} d\sigma(w) \right) dr,$$

where the inner integral extends over the level set $\{\|w\|_{(2)} = r\} \subset \mathbb{C}^{n_2}$ and is with respect to the surface measure σ induced on this set by the ambient Lebesgue measure. To evaluate the inner integral, we make the substitution $w = rt$, where

$\|t\|_{(2)} = 1$; i.e., t lies on the boundary $\mathsf{b}\mathbb{B}_2$ of the unit ball of the $\|\cdot\|_{(2)}$ norm. Taking advantage of the fact that the level sets of the norm are related by dilations, we can compute this integral as

$$\begin{aligned}
&\int_{r=0}^{1}\left(\int_{\mathsf{b}\mathbb{B}_2} r^{2\nu}\,|t_{n_2}|^{2\nu}\,r^{2\beta}r^{2n_2-1}d\sigma(t)\right)dr\\
&=\left(\int_0^1 r^{2\nu+2\beta+2n_2-1}dr\right)\left(\int_{\mathsf{b}\mathbb{B}_2}|t_{n_2}|^{2\nu}\,d\sigma(t)\right)\\
&=\frac{1}{2(\nu+\beta+n_2)}\gamma(\nu).
\end{aligned}$$

□

We will need the following fact regarding removable singularities (cf. [7]):

Lemma 2. *Let $N \geq 3$, and let P be a first-order linear partial differential operator on a domain $U \subset \mathbb{R}^N$. Let $q \in U$, and suppose that $u, v \in L^2(U)$ are such that on $U \setminus \{q\}$ we have in the sense of distributions $Pu = v$. Then, as distributions, $Pu = v$ on U.*

Proof. For $\epsilon > 0$, let $\chi_\epsilon \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be a smooth cutoff such that $0 \leq \chi_\epsilon \leq 1$, χ_ϵ is equal to 1 near q, χ_ϵ vanishes outside the ball $B(q,\epsilon)$, and we have $|\nabla\chi_\epsilon| = O(\epsilon^{-1})$. Let $\phi \in \mathcal{C}_0^\infty(U)$. Writing $\phi = (1-\chi_\epsilon)\phi + \chi_\epsilon\phi$ and denoting the adjoint of P by P^* we have

$$\begin{aligned}
|\langle Pu - v, \phi\rangle| &= |\langle Pu - v, \chi_\epsilon\phi\rangle|\\
&\leq |\langle u, P^*(\chi_\epsilon\phi)\rangle| + |\langle v, \chi_\epsilon\phi\rangle|\\
&\leq C\left(\epsilon^{-1}+1\right)\sqrt{\mathrm{Vol}(B(q,\epsilon))}\\
&\to 0 \text{ as } \epsilon \to 0^+,
\end{aligned}$$

which proves the lemma. □

We note here that although Lemma 2 is stated for operators acting on functions, the result, as well as the proof (after formal changes), continues to hold for differential operators such as $\overline{\partial}$ that act on sections of vector bundles.

4. Counterexample to compactness of N_{n_1}

We now construct a sequence $\{u_\nu\}$ bounded in $\mathfrak{H}$ which has no convergent subsequence when viewed as a sequence in $L^2_{0,n_1}(\Omega)$. Let χ be a real-valued nonvanishing smooth function on the interval $[0,1]$ which vanishes in a neighborhood of 1. Denote by $(z_1, z_2, \ldots, z_n) = ('z, z') \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ the coordinates on $F(H) \subset \Omega$ given by the map $F^{-1} : F(H) \to \mathbb{C}^n$. For each positive integer ν we consider the $(0, n_1)$-form

$$u_\nu = \begin{cases} \sqrt{\dfrac{\nu}{\gamma(\nu)}}\cdot\chi\left(\dfrac{\|'z\|_{(1)}}{\|z'\|^\alpha_{(2)}}\right) z_n^\nu \cdot \displaystyle\bigwedge_{j=1}^{n_1} d\overline{z_j}, & \text{on } F(H),\\ 0, \text{ elsewhere.}\end{cases} \tag{8}$$

(Here α is as in (1), and γ is as in (7)). Then u_ν has support in $F(H)$, is bounded on $\overline{\Omega}$, and is smooth everywhere on $\overline{\Omega}$ except at $F(0)$. In particular, each $u_\nu \in L^2_{0,n_1}(\Omega)$. We claim that the sequence $\{u_\nu\}$ is bounded in $\mathfrak{H}$ but has no convergent subsequence as a sequence in $L^2_{0,n_1}(\Omega)$.

Before we begin the proof of the claim, we collect a couple of simple computations. Note that a norm $\|\cdot\|$ on the vector space $\mathbb{C}^k$ is Lipschitz, and hence differentiable almost everywhere, and it is easy to see that its gradient is bounded; i.e., there is a constant $K_1 > 0$, depending only on the norm $\|\cdot\|$ such that

$$|\nabla(\|z\|)| \leq K_1, \text{ for almost every } z \in \mathbb{C}^k,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^k$. Now, a computation shows that

$$\nabla\left(\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|_{(2)}^{\alpha}}\right)\right) = \chi'\left(\frac{\|'z\|_{(1)}}{\|z'\|_{(2)}^{\alpha}}\right)\left(\frac{1}{\|z'\|_{(2)}^{\alpha}}\nabla\left(\|'z\|_{(1)}\right) - \alpha\frac{\|'z\|_{(1)}}{\|z'\|_{(2)}^{\alpha+1}}\nabla\left(\|z'\|_{(2)}\right)\right),$$

where the gradient is taken in $\mathbb{C}^n$. Therefore there is a constant $K_2 > 0$ such that for any $z \in H$, we have

$$\left|\nabla\left(\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|_{(2)}^{\alpha}}\right)\right)\right| < \frac{K_2}{\|z'\|_{(2)}^{\alpha}} \text{ a.e.} \tag{9}$$

We now estimate $\left\|\overline{\partial}u_\nu\right\|_{L^2_{0,n_1+1}(\Omega)}$. Since $u_\nu \in L^2_{0,n_1}(\Omega)$, by Lemma 2, the singularity at $F(0)$ of the form u_ν can be ignored while computing $\overline{\partial}u_\nu$, provided $\overline{\partial}u_\nu \in L^2_{0,n_1+1}(\Omega)$. Since z_n^ν is holomorphic, we have on $F(H)$:

$$\overline{\partial}u_\nu = \sqrt{\frac{\nu}{\gamma(\nu)}} \cdot z_n^\nu \cdot \overline{\partial}\left(\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|_{(2)}^{\alpha}}\right)\right) \wedge \left(\bigwedge_{j=1}^{n_1} d\overline{z_j}\right).$$

From now on, C denotes a constant independent of ν, which may be different at different occurrences of the symbol. Using (9), we have

$$\left\|\overline{\partial}u_\nu\right\|_{L^2_{0,n_1+1}(\Omega)} \leq \sqrt{\frac{\nu}{\gamma(\nu)}} \cdot \left\|\left|\nabla\left(\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|_{(2)}^{\alpha}}\right)\right)\right| z_n^\nu\right\|_{L^2(F(H))} \tag{10}$$

$$\leq C\sqrt{\frac{\nu}{\gamma(\nu)}} \cdot \left\|\frac{z_n^\nu}{\|z'\|_{(2)}^{\alpha}}\right\|_{L^2(H)}. \tag{11}$$

To compute integrals over the Hartogs triangle, we will use the new coordinates (v, w) induced by the homeomorphism Φ from $\mathbb{B}_1 \times (\mathbb{B}_2 \setminus \{0\}) \subset \mathbb{C}^n$ to $H \subset \mathbb{C}^n$ given by

$$\Phi(v, w) = (\|w\|_{(2)}^{\alpha}\, v, w), \tag{12}$$

which is differentiable a.e. and which has Jacobian determinant $\|w\|_{(2)}^{2\alpha n_1}$ a.e. Recalling that dV denotes the Lebesgue measure on Euclidean space, we have

$$\begin{aligned}
\left\|\frac{z_n^\nu}{\|z'\|_{(2)}^{\alpha}}\right\|_{L^2(H)}^2 &= \int_H \frac{|z_n|^{2\nu}}{\|z'\|_{(2)}^{2\alpha}} dV(z) \\
&= \int_{\mathbb{B}_1\times(\mathbb{B}_2\setminus\{0\})} \frac{|w_{n_2}|^{2\nu}}{\|w\|_{(2)}^{2\alpha}} \|w\|_{(2)}^{2\alpha n_1}\, dV(v, w) \\
&= \mathrm{Vol}(\mathbb{B}_1) \int_{\mathbb{B}_2} |w_{n_2}|^{2\nu} \|w\|_{(2)}^{2\alpha(n_1-1)}\, dV(w) \\
&= \mathrm{Vol}(\mathbb{B}_1) \frac{\gamma(\nu)}{2(\nu + \alpha(n_1 - 1) + n_2)},
\end{aligned} \tag{13}$$

where we have used Lemma 1, proved in Section 3. Combining (11) and (13) gives the estimate

$$\left\|\overline{\partial} u_\nu\right\|_{L^2_{0,n_1+1}(\Omega)} \leq C\sqrt{\frac{\nu}{\nu+\alpha(n_1-1)+n_2}};$$

hence combined with the fact that $u_\nu \in L^2_{0,n_1}(\Omega)$, each u_ν is in the domain $\mathrm{Dom}(\overline{\partial})$ of the Hilbert space operator $\overline{\partial}$, and there is a C independent of ν such that

$$\left\|\overline{\partial} u_\nu\right\|_{L^2_{0,n_1+1}(\Omega)} \leq C. \tag{14}$$

Let ϑ denote the formal adjoint of $\overline{\partial}$ on $L^2_{0,n_1}(\Omega)$ with respect to the metric h defined above. Then on $F(H)$, in the coordinates $(z_1,\dots,z_n)$, the formal expression for ϑ coincides with the usual one in Euclidean space. If

$$U_\nu(z) = \sqrt{\frac{\nu}{\gamma(\nu)}}\cdot\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|^\alpha_{(2)}}\right)z_n^\nu, \tag{15}$$

so that on $F(H)$ we have $u_\nu = U_\nu d\overline{z_1}\wedge\cdots\wedge d\overline{z_{n_1}}$, we have on $F(H)$:

$$\begin{aligned}\vartheta u_\nu &= -\sum_{j=1}^{n_1}\frac{\partial U_\nu}{\partial z_j}d\overline{z_1}\wedge\cdots\wedge\widehat{d\overline{z_j}}\wedge\cdots\wedge d\overline{z_{n_1}}\\ &= -\sqrt{\frac{\nu}{\gamma(\nu)}}\cdot z_n^\nu\cdot\sum_{j=1}^{n_1}\frac{\partial}{\partial z_j}\left(\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|^\alpha_{(2)}}\right)\right)d\overline{z_1}\wedge\cdots\wedge\widehat{d\overline{z_j}}\wedge\cdots\wedge d\overline{z_{n_1}},\end{aligned}$$

where the hat on $\widehat{d\overline{z_j}}$ denotes the omission of this factor from the wedge product. Therefore, using (9) we compute

$$\begin{aligned}\|\vartheta u_\nu\|_{L^2_{0,n_1-1}(\Omega)} &= \|\vartheta u_\nu\|_{L^2_{0,n_1-1}(H)}\\ &\leq C\sqrt{\frac{\nu}{\gamma(\nu)}}\left\|\left|\nabla\left(\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|^\alpha_{(2)}}\right)\right)\right|z_n^\nu\right\|_{L^2(F(H))},\end{aligned}$$

which is the same quantity as in (10). Therefore, the same arguments as used before for $\overline{\partial}u_\nu$ show that $\vartheta u_\nu \in L^2_{0,n_1-1}(\Omega)$ and indeed is uniformly bounded in $L^2_{0,n_1-1}(\Omega)$ independently of ν.

Further, the form u_ν vanishes everywhere on the boundary $b\Omega$ except in the patch $\mathsf{b}\Omega\cap\overline{F(H)}$, which consists of the disjoint union of the sets $F(S)$ and $\{F(0)\}\cap b\Omega$, with F as in (3). (Note that $\{F(0)\}\cap b\Omega$ is empty if $F(0)\in\Omega$.) Near $F(S)$ the boundary is represented in the local coordinates $('z,z')\in\mathbb{C}^{n_1+n_2}$ as $\{('z,z')\in\mathbb{C}^n\colon \|z'\|_{(2)}=1\}$. From formula (8) we see that the complex-normal component of u_ν vanishes on $\mathsf{b}\Omega\cap F(S)$. For $\epsilon>0$, let χ_ϵ be a cutoff of the type used in the proof of Lemma 2: $0\leq\chi_\epsilon\leq 1$, $\chi_\epsilon\equiv 1$ near $F(0)$, χ_ϵ vanishes outside $B(F(0),\epsilon)$ and $|\nabla\chi_\epsilon| = O(\epsilon^{-1})$. Note from the definition that u_ν has bounded coefficients. Writing $\psi_\epsilon = 1-\chi_\epsilon$ we have

$$\vartheta(\psi_\epsilon u_\nu) = U_\nu(z)\cdot\left(-\sum_{j=1}^{n_1}\frac{\partial\psi_\epsilon}{\partial z_j}d\overline{z_1}\wedge\cdots\wedge\widehat{d\overline{z_j}}\wedge\cdots\wedge d\overline{z_{n_1}}\right)+\psi_\epsilon\vartheta u_\nu,$$

where U_ν is as in (15) on $F(H)$ and extended as zero elsewhere. The second term approaches ϑu_ν in the L^2-topology as $\epsilon\to 0^+$, and the L^2-norm of the first term is

bounded by

$$C\,\|U_\nu\|_{L^\infty(\Omega)}\,\epsilon^{-1}\sqrt{\mathrm{Vol}(B(F(0),\epsilon))},$$

which goes to 0 as $\epsilon \to 0^+$. Note that $\psi_\epsilon u_\nu$ is in $\mathrm{Dom}(\overline{\partial}^*)$, since it is a smooth $(0,n_1)$-form on $\overline{\Omega}$ whose complex-normal component vanishes along $F(S) \subset \mathsf{b}\Omega$, and $\psi_\epsilon u_\nu$ itself vanishes elsewhere on the boundary. Further, $\psi_\epsilon u_\nu \to u_\nu$ in the graph norm of ϑ. It follows that $u_\nu \in \mathrm{Dom}(\overline{\partial}^*)$, and from the computation above, $\left\|\overline{\partial}^* u_\nu\right\|_{L^2_{0,n_1-1}(\Omega)} \le C$, where C does not depend on ν. Combining this with (14), it follows that each u_ν lies in the space $\mathfrak{H}$ of (5), and we have

$$\|u_\nu\|_{\mathfrak{H}} \le C.$$

We now compute $\|u_\nu\|_{L^2_{0,n_1}(\Omega)}$. We again use the change of coordinates given by (12):

$$\begin{aligned}
\|u_\nu\|^2_{L^2_{0,n_1}(\Omega)} &= \|u_\nu\|^2_{L^2_{0,n_1}(F(H))} \\
&= \frac{\nu}{\gamma(\nu)}\int_H \left|\chi\left(\frac{\|'z\|_{(1)}}{\|z'\|^\alpha_{(2)}}\right) z_n^\nu\right|^2 dV(z) \\
&= \frac{\nu}{\gamma(\nu)}\int_{\mathbb{B}_1\times(\mathbb{B}_2\setminus\{0\})} \chi^2(\|v\|_{(1)})\,|w_{n_2}|^{2\nu}\,\|w\|^{2\alpha n_1}_{(2)}\,dV(v,w) \\
&= \frac{\nu}{\gamma(\nu)}\int_{\mathbb{B}_1} \chi^2(\|v\|_{(1)})dV(v) \times \int_{\mathbb{B}_2} |w_{n_2}|^{2\nu}\,\|w\|^{2\alpha n_1}_{(2)}\,dV(w) \\
&= C\frac{\nu}{\gamma(\nu)} \times \frac{\gamma(\nu)}{2(\nu+\alpha n_1+n_2)} \\
&= C\frac{\nu}{\nu+\alpha n_1+n_2},
\end{aligned}$$

where in the line before the last, we have made use of Lemma 1. Therefore we see that there is a constant $\lambda > 0$ independent of ν such that $\|u_\nu\|_{L^2_{0,n_1}(\Omega)} \ge \lambda$.

Now let μ and ν be distinct positive integers. We have, using the same change of variables (12):

$$\begin{aligned}
(u_\mu,u_\nu)_{L^2_{0,n_1}(\Omega)} &= (u_\mu,u_\nu)_{L^2_{0,n_1}(H)} \\
&= \frac{\nu}{\gamma(\nu)}\int_H \chi^2\left(\frac{\|'z\|_{(1)}}{\|z'\|^\alpha_{(2)}}\right) z_n^\mu \overline{z_n}^\nu dV(z) \\
&= \frac{\nu}{\gamma(\nu)}\int_{\mathbb{B}_1\times(\mathbb{B}_2\setminus\{0\})} \chi^2\left(\|v\|_{(1)}\right) w_{n_2}^\mu\,\overline{w_{n_2}}^\nu\,\|w\|^{2\alpha n_1}_{(2)}\,dV(v,w) \\
&= \frac{\nu}{\gamma(\nu)}\int_{\mathbb{B}_1} \chi^2(\|v\|_{(1)})dV(v) \times \int_{\mathbb{B}_2} w_{n_2}^\mu\,\overline{w_{n_2}}^\nu\,\|w\|^{2\alpha n_1}_{(2)}\,dV(w).
\end{aligned}$$

Denote the integral over $\mathbb{B}_2$ in this product by K. We claim that $\mathsf{K} = 0$. Assuming this for a moment it follows that $(u_\mu,u_\nu)_{L^2_{0,n_1}(\Omega)} = 0$, and hence $\|u_\mu - u_\nu\|_{L^2_{0,n_1}(\Omega)} \ge \sqrt{2}\lambda$ for each pair of distinct indices μ and ν, and hence $\{u_\nu\}$ has no convergent subsequence in $L^2_{0,n_1}(\Omega)$. The proof of the theorem is complete, provided we can show that $\mathsf{K} = 0$.

Let θ be a real number which is not a rational multiple of π, and in the integral representing K, we make a change of variables $w = e^{i\theta}t$, where $t \in \mathbb{C}^{n_2}$. Then, since this transformation maps $\mathbb{B}_2$ to itself and its Jacobian is identically 1, we have

$$\begin{aligned}\mathsf{K} &= \int_{\mathbb{B}_2} w_{n_2}^{\mu}\,\overline{w_{n_2}}^{\nu}\,\|w\|_{(2)}^{2\alpha n_1}\,dV(w)\\ &= \int_{\mathbb{B}_2} e^{i(\mu-\nu)\theta} t_{n_2}^{\mu}\,\overline{t_{n_2}}^{\nu}\,\|t\|_{(2)}^{2\alpha n_1}\,dV(t)\\ &= e^{i(\mu-\nu)\theta}\mathsf{K},\end{aligned}$$

so that $\mathsf{K} = 0$. This completes the proof.

5. Concluding remarks

The noncompactness of the $\overline{\partial}$-Neumann operator implies that there is a point in the spectrum of the complex Laplacian $\Box_q$ acting on $L^2_{0,q}(\Omega)$ which is not an eigenvalue of finite multiplicity. It is of interest to determine the nature of this essential spectrum. In the case of the polydisc or product domains, the spectrum can be determined explicitly (see [6, 3]), and while the spectrum of a polydisc consists of eigenvalues only, there are infinitely many eigenvalues in the essential spectrum, each of infinite multiplicity (this is true, in particular, for the smallest nonzero eigenvalue). It would be of interest to know whether the same is true for the domain H, for example in the classical situation $n_1 = n_2 = \alpha = 1$.

Acknowledgments

The author thanks Professors Mei-Chi Shaw and Emil Straube for their help in the preparation of this article, the referee for helpful suggestions for improvements in it, and Prof. Giuseppe Zampieri for pointing out reference [8].

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