EMBEDDING OF THE DUNCE HAT

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Abstract. In this note we show that the famous Borsuk contractible non-collapsible 2-polyhedron, generally known as the *dunce hat*, does not embed in any product of two curves but quasi-embeds in the “three-page book”.

All spaces discussed in this paper are assumed to be metrizable and all mappings (also called maps) are continuous. By a compactum we mean a compact (metric) space, by a continuum we mean a non-void connected compactum, and by a curve we mean a 1-dimensional continuum. All polyhedra are compact.

By $\mu$ we denote the Menger curve. It is well known that if a compactum $X$ quasi-embeds in $\mu$, then $X$ embeds in $\mu$. (A metric space $X$ is said to *quasi-embed* in $Y$ if for each $\varepsilon > 0$ there is an $\varepsilon$-mapping $f : X \to Y$.) In [3] the authors asked the following question about a possible extension of this result to finite products of copies of the Menger curve:

*Suppose $X$ quasi-embeds in $\mu^n$. Does $X$ embed in $\mu^n$?*

To our surprise, this question has been answered in the negative in a recent joint paper by S. A. Melikhov and J. Zajac [6]. Actually, they proved that the Sklyarenko absolute retract [8] quasi-embeds in a product of two dendrites but does not embed in any product of two curves. The problem was left open for cases where $X$ is less complicated, for instance, for $X$ being a polyhedron. (Sklyarenko’s example is not a polyhedron.)

The purpose of this note is to show that even for polyhedra the answer is negative. In fact, we are going to prove that the famous Borsuk contractible non-collapsible 2-dimensional polyhedron [1] (generally known as the *dunce hat* (cf. [9]); also called the *Borsuk tube* by Polish topologists (cf. [2])) is such a counterexample. In other words, we shall prove the following.

**Theorem.** The *dunce hat* does not embed in any product of two curves but quasi-embeds in the “three-page book” $T \times I$.

By $T$ we denote the simple triod, and by $I$ we denote the unit interval, $I = [0, 1]$.

First we recall a description of the Borsuk tube, slightly modifying the original description of Borsuk [1]. Let $B^2$ denote the unit disc in the complex plane $\mathbb{C}$, and let $S^1$ denote its boundary, the unit circle in $\mathbb{C}$. We define the Borsuk tube to be...
the quotient $B = \mathbb{B}^2/\sim$, where $(1 - t)\exp(2\pi it) \sim \exp(2\pi it)$ for each $t \in I$. (The interval $I$ is a subset of $\mathbb{B}^2$ because the reals are regarded to be a subset of $\mathbb{C}$.) Thus $B$ is the quotient of $\mathbb{B}^2$ obtained by identifying the arc $J = \{(1 - t)\exp(2\pi it) : t \in I\}$ with $S^1$; see Figure 1. (In the original description, Borsuk identifies $t \in I$ with $\exp(2\pi it) \in S^1$ for each $t \in I$.) By its basic property $B$ is a contractible but not collapsible 2-polyhedron (Figure 1; cf. [9]). One easily verifies that $B$ is a quasi-2-manifold.

Hence by the Structure Theorem in [4] we infer that $B$ does not embed in any product of two curves; see [4], Corollary 5.4. This proves the first assertion of our theorem. To complete the proof it remains to establish the second one.

From the definition it follows that $B$ is a compact metrizable space; let $d$ denote a fixed metric on $B$. Fix a number $\epsilon > 0$. Hence it remains to construct an $\epsilon$-mapping $f : B \to T \times I$. Let $q : \mathbb{B}^2 \to B$ denote the quotient mapping. Observe that there is a number $a > 0$ such that for all $z, z' \in \mathbb{B}^2$ we have

$$(1) \ |z - z'| < a \Rightarrow d(q(z), q(z')) < \epsilon.$$ We may also assume that $a < 1$. Then it is easy to define an equicontinuous family of mappings $g_t : (I, 0, 1) \to (I, 0, 1)$, $t \in I$, satisfying the conditions

$$(2) \ g_0 = g_1 = id_I,$$

$$(3) \ g_t \text{ is a relative homeomorphism } (I, [1 - t, 1]) \to (I, \{1\}) \text{ for each } 0 \leq t \leq a,$$

$$(4) \ g_t \text{ is a homeomorphism for each } a < t \leq 1.$$ These mappings define a mapping $g : \mathbb{B}^2 \to \mathbb{B}^2$ given by the formula

$$g(s \cdot \exp(2\pi it)) = g_t(s) \cdot \exp(2\pi it)$$

for each $s, t \in I$.

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2A 2-dimensional compactum is called a quasi-2-manifold if each point $x \in X$ admits an open neighborhood $U$ such that every closed set $F$ separating $X$ between $x$ and $X \setminus U$ admits an essential map into $S^1$.

3For instance, let us define: (i) if $t \in [0, a]$ put $g_t(s) = \frac{t}{1 - t}$ for $s \leq 1 - t$ and $g_t(s) = 1$ for $s \geq 1 - t$; (ii) if $t \in (a, 1]$ put $g_t(s) = \frac{1 - t}{(1 - a)^2} + \frac{1 - a}{1 - a} s$ for $s \leq 1 - a$ and $g_t(s) = \frac{1 - a}{1 - a} s + \frac{1 - t}{1 - a}$ for $s > 1 - a$. Notice that each $g_t$ is composed of two linear maps defined on adjacent (possibly degenerate) subintervals of $I$. 
Then one can define another quotient space $B'$ of $\mathbb{B}^2$ analogous to $B$, identifying the arc $g(J)$ with $S^1_1$; precisely, $B' = \mathbb{B}^2/\sim'$, where $g((1−t)\exp(2\pi it)) \sim' \exp(2\pi it)$ for each $t \in I$. Let $g' : \mathbb{B}^2 \to B'$ denote the quotient mapping. Since $g$ preserves the identifications, there exists a mapping $g' : B \to B'$ such that $g' \circ g = g' \circ q$; that is, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{B}^2 & \xrightarrow{g} & \mathbb{B}^2 \\
\downarrow & & \downarrow \\
B & \xrightarrow{g'} & B'.
\end{array}
\]

Then

(5) $g'$ is an $\epsilon$-mapping.

Indeed, suppose $g'$ sends two different points $g(z)$, $g(z')$ of $B$ to the same point of $B'$. Then $z$, $z'$ do not lie in the same fiber of $g$. Since $g(z)$, $g(z')$ lie in the same fiber of $q'$, it follows that both $z$, $z'$ lie in the same fiber of $g$, i.e. $z, z' \in [1−t, 1] \exp(2\pi it)$ for some $t \in [0, a]$. Therefore, $|z − z'| < a$; hence $d(g(z), g(z')) < \epsilon$ by (1). This proves (5).

Notice that $J$ is the union of two subarcs $J_0 = \{(1−t)\exp(2\pi it) : t \in [0, a]\}$ and $J_1 = \{(1−t)\exp(2\pi it) : t \in [a, 1]\}$. Likewise, $S^1_1$ is the union of two arcs $S_0 = \{\exp(2\pi it) : t \in [0, a]\}$ and $S_1 = \{\exp(2\pi it) : t \in [a, 1]\}$. Moreover, we have: $g(J)$ is the union of two subarcs $g(J_0)$ and $g(J_1)$, $g(J_0) = S_0$ and $g(J_1)$ meets $S_1$ at point $\exp(2\pi i\alpha)$ which is a common endpoint of these arcs, and off that point $g(J_1)$ lies in the interior of $\mathbb{B}^2$. It follows that $B'$ is the quotient of $\mathbb{B}^2$ obtained by identifying the arcs $g(J_1)$ and $S_1$. Therefore, $B'$ is a collapsible 2-polyhedron and $q'(S_0)$ is a free face of $B'$ (i.e. each interior point $x$ of $q'(S_0)$ admits a neighborhood that is a closed 2-disc with $x$ lying on its boundary). To complete the proof it suffices to show that $B'$ embeds in $T \times I$.

To this end, present $B'$ as a union of two sets $q'(D_0)$ and $q'(D_1)$, where $D_0$ is the disc bounded by the arcs $I$, $S_0$ and $g(J_0)$, and $D_1$ is the disc bounded by $I$, $S_1$ and $g(J_1)$. Clearly, $q'(D_1)$ is a disc and the circle $q'(I)$ is its boundary. On the other hand, $q'(D_0)$ meets $q'(D_1)$ along the union $q'(g(J_1) \cup I)$. Since the arc $q'(g(J_1))$ off its endpoint $q'(0)$ lies entirely in the interior of the disc $q'(D_1)$ and $q'(I)$ is the boundary of that disc, one easily sees that $B'$ embeds in $T \times I$.

**Lemma.** The suspension of a starlike compact set $X$ lying in $\mathbb{R}^n$ embeds in the product $X \times I$.

To prove this lemma we may assume that $X$ is starlike with respect to the point $0 \in \mathbb{R}^n$ and that $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Identify $X$ with $X \times \{0\}$. Then the suspension is the union of two cones over $X$ with vertices $(0, −1)$ and $(0, 1)$. In this setting, the suspension is a subset of $X \times [−1, 1]$.

Applying our theorem and this lemma we get the following.

**Corollary.** For each $n \geq 2$ there is an $n$-dimensional contractible polyhedron $P$ not embeddable in any product of $n$ curves but quasi-embeddable in a product of $n$ triods. Moreover, if $n = 2k$, then $P$ quasi-embeds in $T^k \times I^k$, and if $n = 2k + 1$, then $P$ quasi-embeds in $T^k \times I^{k+1}$. 
Proof. Actually, we shall show that for $P$ we can take either the product $B^k = B \times \ldots \times B$ if $n = 2k$, or the suspension $\sum B^k$ if $n = 2k + 1$. In fact, the product $B^k$ of $k$ copies of the Borsuk tube is a quasi-$2k$-manifold that is a product of quasi-2-manifolds; see [4], Corollary 2.3. Since $B^k$ is contractible, by the Structure Theorem 5.1 of [4], it does not embed in any product of $2k$ curves. Since $B$ quasi-embeds in $T \times I$, $B^k$ quasi-embeds in $(T \times I)^k = T^k \times I^k$. On the other hand, by Corollary 2.3 of [4], $\sum B^k$ is a quasi-$(2k + 1)$-manifold off two points (cf. the proof of Theorem 1.3 in [5]). Notice that $\sum B^k$ is contractible. Hence, by the Second Structure Theorem 6.1 of [5], $\sum B^k$ does not embed in any product of $2k + 1$ curves. Moreover, $\sum B^k$ quasi-embeds in $\sum (T \times I)^k$, hence quasi-embeds in $(T \times I)^k \times I = T^k \times I^{k+1}$, by the above lemma and the starlikeness of $(T \times I)^k$ in $\mathbb{R}^{3k}$.

□

Problems


Problem 2. Does the “Bing house” quasi-embed in a product of two curves?

The “Bing house” is another example of a contractible non-collapsible 2-polyhedron [7]. By the same argument as in the case of the Borsuk tube, it does not embed in any product of two curves.

Problem 3. Does the suspension $\sum^n B$, $n > 1$, embed in a product of $n + 2$ curves?

Notice that by the Structure Lemma 4.8 in [4], $\sum^n B$ does not embed in any product of $n + 2$ graphs. Therefore, a positive solution of the problem from [5] implies a negative solution to Problem 3 above. Also notice that by our lemma, $\sum^n B$ quasi-embeds in $T \times I^{n+1}$ because $B$ quasi-embeds in $T \times I$.

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References


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