PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 7, July 2013, Pages 2225–2231
S 0002-9939(2013)11559-5
Article electronically published on February 20, 2013

LARGE FAMILIES OF STABLE BUNDLES ON ABELIAN VARIETIES

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(Communicated by Lev Borisov)

Abstract. A sequence of μ-stable bundles $\{E_m\}$ on a polarized variety (X, H) is said to be a large family if their ranks and the discriminants become arbitrarily large as m goes to infinity. We prove the existence of large families on a principally polarized abelian variety (X, Θ) such that the Neron-Severi group is generated by Θ.

1. Introduction

Let X be a smooth projective variety of dimensin $n \geq 2$ defined over $\mathbb{C}$ and let H be an ample line bundle on X. We fix an integer $r \geq 2$ and $c_i \in H^{2i}(X, \mathbb{Z})$ $(1 \leq i \leq n)$. One of the fundamental problems concerning μ-stable vector bundles on X is the *existence problem*. Namely we would like to determine the values r and c_i for which a μ-stable bundle E on X with $\mathrm{rk}(E) = r$ and Chern classes $c_i(E) = c_i$ exists. Although the complete answer to this problem is not known at present, some partial results have been obtained. For example, by the method of elementary transformations, Maruyama proved that for given $r \geq n$, c_1 and an integer s, there exists a μ-stable vector bundle E on X with $\mathrm{rk}(E) = r$, $c_1(E) = c_1$ and $c_2(E) \cdot H^{n-2} \geq s$ ([4]).

A consequence of Maruyama's theorem is that there exists a sequence of μ-stable bundles $\{E_m\}_{m=1}^{\infty}$ such that their ranks r_m and the discriminants $\Delta_m = (2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2) \cdot H^{n-2}$ become arbitrarily large as m goes to infinity. We call such sequence a *large family* of μ-stable bundles. More generally, for a given pair of positive real numbers (s, t), we consider the problem of finding a sequence $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles on X such that

$$r_m = O(m^s), \quad \Delta_m = O(m^t).$$

Such an *asymptotic existence problem* may be considered as a preliminary step to the solution of the original, much harder existence problem and is closely related to the *strong Bogomolov inequality* which has been discussed in the recent study of string theory ([1],[8]). In [7], we gave examples of Calabi-Yau manifolds (X, H) such that SBI does not hold by constructing certain large families of μ-stable bundles on them.

Received by the editors May 11, 2011 and, in revised form, October 9, 2011.

2010 *Mathematics Subject Classification.* Primary 14J60; Secondary 14K12.

Key words and phrases. Stable vector bundles, abelian varieties.

The author was supported in part by Grant-in-Aid for Scientific Research (C)(21540049).

The author is grateful to the referee for pointing out several mistakes in the original manuscript and for giving valuable comments.

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It seems that the asymptotic existence problem on abelian varieties has not been considered before. In this note we treat principally polarized abelian varieties (X, Θ) and give an explicit example of large families of μ-stable bundles on them. More precisely we prove the following.

Theorem 1.1. *Let (X, Θ) be a principally polarized abelian variety of dimension $n \geq 2$ such that the Neron-Severi group $NS(X)$ is generated by $\mathcal{O}_X(\Theta)$. Then there exist large families of μ-stable bundles $\{E_m\}_{m=1}^{\infty}$ on X with $r_m = O(m^s)$, $\Delta_m = O(m^t)$ for $(s,t) = (1,2)$ and $(1,3)$.*

The proof of the theorem is based on the construction of rank two μ-stable bundles on abelian threefolds in [2].

2. Large families of μ-stable bundles

A pair (X, H) of a smooth projective variety X and an ample line bundle H on X is said to be a polarized variety. A family $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles on a polarized variety (X, H) is a sequence of vector bundles E_m on X which are μ-stable with respect to H. Let r_m and Δ_m denote the rank and the discriminant of E_m with respect to H respectively. To investigate the asymptotic behavior of a given family, we introduce the following definition.

Definition 2.1. A family $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles is said to be *large* if $r_m \to \infty$ and $\Delta_m \to \infty$ as $m \to \infty$. A large family $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles is said to be *of order* (s,t) if there exist positive real numbers s, t such that

$$r_m = O(m^s) \quad \text{and} \quad \Delta_m = O(m^t).$$

Maruyama's theorem ([4]) yields immediately the following.

Proposition 2.2. *For any polarized variety (X, H), there exists a large family $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles on (X, H).*

In general it is more difficult to show the existence of large families of μ-stable bundles with prescribed asymptotic growth of r_m and Δ_m. So we formulate the following.

Asymptotic Existence Problem. For given positive real numbers s and t, does there exist a large family $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles of order (s,t) on (X, H)?

We notice that there exists a relation between the asymptotic existence problem and the strong Bogomolov inequality.

Definition 2.3. For a given rational number $l \geq 0$, we say that the *strong Bogomolov inequality of type l* (*SBI_l* in short) holds on a polarized variety (X, H) if there exists a positive constant $\alpha(X, H)$ depending only on X, H such that every μ-stable bundle E satisfies the inequality

$$\Delta(E) \geq r^l \alpha(X, H).$$

SBI_l is a strengthening of the usual Bogomolov inequality $\Delta(E) \geq 0$. The case $l = 2$ has been discussed in the context of string theory ([1]). We have the following obvious proposition.

Proposition 2.4. *Assume that there exists a large family $\{E_m\}_{m=1}^{\infty}$ of μ-stable bundles of order (s,t) with $t < ls$. Then SBI_l does not hold on (X, H).*

Example 2.5. Let $\mathcal{O}_{\mathbb{P}^n}(1)$ be the tautological line bundle on $\mathbb{P}^n$. Then the evaluation map $\varphi_m : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(m)$ is surjective for all $m > 0$. The bundle $E_m = \operatorname{Ker}\varphi_m$, which is a special class of syzygy bundles on $\mathbb{P}^n$, is known to be μ-stable ([9]). Since we have $c_1(E_m) = \mathcal{O}_{\mathbb{P}^n}(-m)$ and $c_2(E_m) = m^2\mathcal{O}_{\mathbb{P}^n}(1)^2$, we obtain

$$r_m = \binom{n+m}{m} - 1, \quad \Delta_m = (\binom{n+m}{m} + 1)m^2.$$

Hence $\{E_m\}_{m=1}^{\infty}$ is a large family of order $(n, n+2)$.

Example 2.6. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension $n \geq 2$ such that $\operatorname{Pic}(X)$ is generated by $\mathcal{O}_X(1)$. We choose $D \in |\mathcal{O}_D(m)|$ for sufficiently large m so that $\mathcal{O}_D(m)$ is globally generated and $h^i(\mathcal{O}_D(m)) = 0$ for $i > 0$. Let E_m denote the dual of the elementary transformation of $H^0(D, \mathcal{O}_D(m)) \otimes \mathcal{O}_X$. Namely E_m fits in the exact sequence

$$0 \to E_m^{\vee} \to H^0(D, \mathcal{O}_D(m)) \otimes \mathcal{O}_X \to \mathcal{O}_D(m) \to 0.$$

Then $\{E_m\}_{m=1}^{\infty}$ is a family of μ-stable bundles on $(X, \mathcal{O}_X(1))$ as shown in [6]. By the Riemann-Roch formula, we have

$$\begin{aligned} r_m &= \chi(\mathcal{O}_D(m)) \\ &= \frac{\mathcal{O}_X(1)^n}{(n-1)!} m^{n-1} + O(m^{n-2}) \end{aligned}$$

and

$$\Delta_m = \frac{2(\mathcal{O}_X(1)^n)^2}{(n-1)!} m^n + O(m^{n-1}).$$

Thus $\{E_m\}_{m=1}^{\infty}$ is a large family of order $(n-1, n)$. In particular, SBI_l does not hold if $n < l(n-1)$ on (X, H).

Example 2.7. Let C be a smooth projective curve of genus $g \geq 2$. Let $J(C)$ be its Jacobian variety and Θ the theta divisor on $J(C)$. For a fixed point $c \in C$, let $\iota : C \hookrightarrow J(C)$ be the embedding defined by $x \mapsto x - c$. For $0 \leq i \leq g-1$, let $W_i = C + C + \cdots + C$ (i-times sum) be the distinguished subvariety of dimension i. Thus $W_{g-1} = \Theta - \kappa$, where κ is the theta characteristic.

Let $\widehat{J(C)}$ denote the dual abelian variety and let $\mathcal{L}$ be the normalized Poincaré bundle on $J(C) \times \widehat{J(C)}$. Then the Picard bundle E_m is defined to be the Fourier-Mukai transform $R^1\mathcal{S}(\iota_*\mathcal{O}_C(-mc))$ with respect to $\mathcal{L}$ ([5]). It is well known that E_m are μ-stable bundles and $c_i(E_m) = [W_{g-i}]$. By the formula

$$[W_{g-i}] = \frac{\Theta^i}{i!}, \quad \Theta^g = g!,$$

we obtain

$$r_m = m + g - 1, \quad \Delta_m = g!.$$

Thus we see that the family $\{E_m\}_{m=1}^{\infty}$ is not a large family.

It is not clear to us whether it is possible to construct large families of μ-stable bundles on $J(C)$ by means of Fourier-Mukai transforms. In the next section we construct large families on principally polarized abelian varieties using a different method.

Definition 2.8. For vector bundles E_1, E_2 on a polarized variety (X, H) and a non-trivial subspace $i : U \subset \mathrm{Ext}^1(E_2, E_1)$, the *universal extension* is the extension

$$0 \to U^\vee \otimes E_1 \to E \to E_2 \to 0,$$

which corresponds to the inclusion map $i : U \hookrightarrow \mathrm{Ext}^1(E_2, E_1)$, regarded as an element in $\mathrm{Hom}(U, \mathrm{Ext}^1(E_2, E_1))$ under the isomorphism

$$\mathrm{Hom}(U, \mathrm{Ext}^1(E_2, E_1)) \cong \mathrm{Ext}^1(E_2, U^\vee \otimes E_1).$$

Concerning the stability of the universal extension, the following result has been proved in [6, Lemma 1.4].

Lemma 2.9. *Let (X, H) be a polarized variety such that the Neron-Severi group $NS(X)$ is generated by H. Let E_i $(i = 1, 2)$ be μ-stable bundles of rank r_i on X. Let $U \subset \mathrm{Ext}^1(E_2, E_1)$ be a non-trivial subspace and let*

$$0 \to U^\vee \otimes E_1 \to E \to E_2 \to 0$$

be the universal extension. If $\delta(E_1, E_2) := r_1 c_1(E_2) - r_2 c_1(E_1) = H$, then E is μ-stable.

Let (X, H) be a polarized variety and E a vector bundle of rank two on X with $c_1(E) = L$. We have

$$\mathrm{Ext}^1(E, \mathcal{O}_X) \cong H^1(X, E^\vee) \cong H^1(X, E(-L)) \cong \mathrm{Ext}^1(L, E)$$

since $E^\vee \cong E(-L)$.

Definition 2.10. For any non-trivial subspace $U \subset \mathrm{Ext}^1(E, \mathcal{O}_X)$ of dimension s, the universal extension

$$0 \to U^\vee \otimes \mathcal{O}_X \to E_U \to E \to 0$$

is a bundle of rank $s + 2$, $c_1(E_U) = L$ and $c_2(E_U) = c_2(E)$. E_U is said to be an extension *of type I*. Similarly, for any non-trivial subspace $V \subset \mathrm{Ext}^1(L, E)$ of dimension s, the universal extension

$$0 \to V^\vee \otimes E \to E_V \to L \to 0$$

is a vector bundle of rank $2s + 1$ and

$$\begin{aligned} c_1(E_V) &= c_1(L) + s c_1(E), \\ c_2(E_V) &= \frac{s(s-1)}{2} c_1(E)^2 + s c_1(L) \cdot c_1(E) + s c_2(E). \end{aligned}$$

E_V is said to be an extension *of type II*.

To prove Theorem 1.1. we need the following result, which allows us to construct large families of μ-stable bundles from a family of rank two bundles.

Proposition 2.11. *Let (X, H) be a polarized variety such that $NS(X)$ is generated by H. Let E be a μ-stable bundle of rank two on X with $c_1(E) = H$ and $\dim \mathrm{Ext}^1(E, \mathcal{O}_X) > 0$. Then the extensions of type I and type II are both μ-stable.*

Proof. Since $\delta(E, \mathcal{O}_X) = \delta(H, E) = H$, the claim follows from Lemma 2.9. $\square$

3. Proof of Theorem 1.1

Let (X, Θ) be a principally polarized abelian variety of dimension $g \geq 3$ defined over $\mathbb{C}$. We assume that the Neron-Severi group $NS(X)$ is generated by $\mathcal{O}_X(\Theta)$. For a point $x \in X$, let $T_x : X \to X$ be the translation map by x. For $x \in X$, let $\mathcal{P}_x = \mathcal{O}_X(\Theta_x - \Theta)$. We let Θ_x denote the divisor $\Theta + x$. Thus $\mathcal{O}_X(\Theta_x) = T^*_{-x}\mathcal{O}_X(\Theta)$. We have

$$\Theta_x + \Theta_{-x} \sim 2\Theta.$$

By the theorem of the cube, we have

$$T^*_{2x}\mathcal{O}_X(\Theta) = T^*_x\mathcal{O}_X(\Theta) \otimes T^*_x\mathcal{O}_X(\Theta) \otimes \mathcal{O}_X(-\Theta).$$

Hence

$$\Theta_{-2x} \sim 2\Theta_{-x} - \Theta.$$

Further, we have

$$\begin{aligned} 2\Theta_x + \Theta_{-2x} &\sim 2\Theta_x + 2\Theta_{-x} - \Theta \\ &\sim 2(\Theta_x + \Theta_{-x}) - \Theta \\ &\sim 4\Theta - \Theta = 3\Theta. \end{aligned}$$

Proposition 3.1. *Let (X, Θ) be as above. Let Y be a locally complete intersection subscheme of codimension two. Let E be a rank two bundle which fits in the extension*

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_Y(3\Theta) \to 0.$$

Then E is μ-stable with respect to $\mathcal{O}_X(\Theta)$ if Y is not contained in any translate of Θ.

Proof. If E is not μ-stable, then there exists a subline bundle $L \subset E$, which has the form $L = \mathcal{P}_x(m\Theta)$ for some integer $m \geq 2$ and a point $x \in X$. Let $f : \mathcal{P}_x(m\Theta) \to \mathcal{I}_Y \otimes \mathcal{O}_X(3\Theta)$ be the composite map. If $f = 0$, then we would obtain a non-trivial map $\mathcal{P}_x(m\Theta) \to \mathcal{O}_X$, which is impossible. It follows that $f \neq 0$; hence there exists a non-trivial section s of $H^0(\mathcal{I}_Y \otimes \mathcal{P}_x^\vee((3-m)\Theta))$. Since $m \geq 2$, we must have $m = 2$. Thus $H^0(\mathcal{I}_Y \otimes \mathcal{P}_x^\vee(\Theta)) \cong H^0(\mathcal{I}_Y \otimes \mathcal{P}_{-x}(\Theta)) \cong H^0(\mathcal{I}_Y \otimes T^*_x\mathcal{O}_X(\Theta)) \neq 0$, which contradicts the assumption. Hence E is μ-stable. $\square$

Let $a_1, a_2, \ldots, a_N$ be N distinct points of X. Let

$$Y_i = D_i \cap \Theta_{-a_i}, \quad Y = \bigcup_{i=1}^{N} Y_i,$$

where $D_i \in |2\Theta_{a_i}|$.

Lemma 3.2. *Let Y_i and Y be as above. If a_i and D_i are chosen to be general, then Y_i are smooth and mutually disjoint and the canonical bundle of Y is given by $\omega_Y = \mathcal{O}_Y(3\Theta)$. Further we have $h^0(\mathcal{O}_{Y_i}(\Theta)) = 1$ and $h^0(\mathcal{O}_Y(\Theta)) = N$.*

Proof. For $i \neq j$, let $V = 2\Theta_{a_i} \cap 2\Theta_{a_j}$ and $W = \Theta_{-2a_i} \cap \Theta_{-2a_j}$. Then V and W have codimension two, and we have $Y_i \cap Y_j = V \cap W$. By the Moving Lemma ([3]), for general $x \in X$, $V + x$ and W have proper intersection; hence it is empty. It follows that

$$\left(V + \frac{x}{2}\right) \cap \left(W - \frac{x}{2}\right) = \emptyset.$$

We have

$$V + \frac{x}{2} = 2\Theta_{a_i + \frac{x}{2}} \cap 2\Theta_{a_j + \frac{x}{2}}$$

and

$$W - \frac{x}{2} = \Theta_{-2(a_i+\frac{x}{4})} \cap \Theta_{-2(a_j+\frac{x}{4})}.$$

Hence, replacing a_i, a_j by $a_i + \frac{x}{2}$and $a_j + \frac{x}{4}$ respectively, we have $Y_i \cap Y_j = \emptyset$.

Next we notice that the normal bundle $N_{Y_i/X}$ of Y_i is given by

$$N_{Y_i/X} = \mathcal{O}_{Y_i}(2\Theta_{a_i}) \oplus \mathcal{O}_{Y_i}(\Theta_{-2a_i}).$$

Thus, the canonical bundle ω_Y is given by $\det N_{Y_i/X} = \mathcal{O}_{Y_i}(3\Theta)$ for each i; hence $\omega_Y = \mathcal{O}_Y(3\Theta)$.

Finally, we consider the Koszul resolution of Y_i tensored with $\mathcal{O}_X(\Theta)$:

$$0 \to \mathcal{O}_X(\Theta - 2\Theta_{a_i} - \Theta_{-2a_i}) \to \mathcal{O}_X(\Theta - 2\Theta_{a_i}) \oplus \mathcal{O}_X(\Theta - \Theta_{-2a_i}) \to \mathcal{I}_{Y_i}(\Theta) \to 0.$$

Since a_i is chosen to be general, we have $H^k(X, \mathcal{O}_X(\Theta - 2\Theta_{a_i} - \Theta_{-2a_i})) = 0$ for $k \leq 2$ and $H^k(\mathcal{O}_X(\Theta - 2\Theta_{a_i}) \oplus \mathcal{O}_X(\Theta - \Theta_{-2a_i})) = 0$ for $k \leq 1$. Hence we obtain $H^k(X, \mathcal{I}_{Y_i}(\Theta)) = 0$ for $k \leq 1$. Then, considering the exact sequence

$$0 \to \mathcal{I}_{Y_i}(\Theta) \to \mathcal{O}_X(\Theta) \to \mathcal{O}_{Y_i}(\Theta) \to 0,$$

we obtain $H^0(\mathcal{O}_{Y_i}(\Theta)) \cong H^0(\mathcal{O}_X(\Theta)) \cong \mathbb{C}$ and $H^0(\mathcal{O}_Y(\Theta)) \cong \bigoplus_{i=1}^N H^0(\mathcal{O}_{Y_i}(\Theta)) \cong \mathbb{C}^N$. □

By the lemma above, Y is a locally complete intersection subscheme of codimension two which is subcanonical. Hence, by the Serre correspondence there exists a rank two bundle E' on X which fits in the extension

$$0 \to \mathcal{O}_X \to E' \to \mathcal{I}_Y(3\Theta) \to 0.$$

Lemma 3.3. *Let $E := E'(-\Theta)$. Then E is a μ-stable bundle with $c_1(E) = \mathcal{O}_X(\Theta)$, $c_2(E) = 2(N-1)\Theta^2$ and $\dim \operatorname{Ext}^1(E, \mathcal{O}_X) = \dim \operatorname{Ext}^1(\mathcal{O}_X(\Theta), E) = N - 1$.*

Proof. For a point $x \in X$, we consider the Koszul resolution of Y_i tensored with $\mathcal{O}_X(\Theta_x)$:

$$0 \to \mathcal{O}_X(\Theta_x - 2\Theta_{a_i} - \Theta_{-2a_i}) \to \mathcal{O}_X(\Theta_x - 2\Theta_{a_i}) \oplus \mathcal{O}_X(\Theta_x - \Theta_{-2a_i}) \to \mathcal{I}_{Y_i}(\Theta_x) \to 0.$$

Since $H^k(X, \mathcal{O}_X(\Theta_x - 2\Theta_{a_i} - \Theta_{-2a_i})) = 0$ for $k \leq 1$ and $H^0(\mathcal{O}_X(\Theta_x - 2\Theta_{a_i})) = 0$, we have

$$H^0(\mathcal{I}_{Y_i}(\Theta_x)) \cong H^0(\mathcal{O}_X(\Theta_x - \Theta_{-2a_i})).$$

Thus, if a divisor $D \in |T_x^* \mathcal{O}_X(\Theta)|$ contains Y_i, then $D = \Theta_{-2a_i}$. It follows that for $N \geq 2$, Y cannot be contained in any $D \in |T_x^* \mathcal{O}_X(\Theta)|$ for all $x \in X$. Hence $E = E'(-\Theta)$ is μ-stable by Proposition 3.1 and $c_1(E) = \mathcal{O}_X(\Theta)$ and $c_2(E) = 2(N-1)\Theta^2$.

To prove $\dim \operatorname{Ext}^1(E, \mathcal{O}_X) = N - 1$, we consider the sequence

$$0 \to \mathcal{O}_X(-\Theta) \to E \to \mathcal{I}_Y(2\Theta) \to 0,$$

which induces the sequence of cohomology

$$\to H^{g-1}(\mathcal{O}_X(-\Theta)) \to H^{g-1}(E) \to H^{g-1}(\mathcal{I}_Y(2\Theta)) \to H^g(\mathcal{O}_X(-\Theta)) \to H^g(E).$$

We have $H^{g-1}(\mathcal{O}_X(-\Theta)) = 0$ by the Kodaira vanishing theorem and $H^g(\mathcal{O}_X(-\Theta)) \cong H^0(\mathcal{O}_X(\Theta)) \cong \mathbb{C}$. Further, $H^g(E) \cong H^0(E^\vee) = 0$ by stability. It follows that $h^{g-1}(E) = h^{g-1}(\mathcal{I}_Y(2\Theta)) - 1$. We consider the exact sequence

$$0 \to \mathcal{I}_Y(2\Theta) \to \mathcal{O}_X(2\Theta) \to \mathcal{O}_Y(2\Theta) \to 0$$

and the induced sequence of cohomology

$$\to H^{g-2}(\mathcal{O}_X(2\Theta)) \to H^{g-2}(\mathcal{O}_Y(2\Theta)) \to H^{g-1}(\mathcal{I}_Y(2\Theta)) \to H^{g-1}(\mathcal{O}_X(2\Theta)).$$

Since $H^i(\mathcal{O}_X(-\Theta)) = 0$ for $i = g-1,\, g$, we have

$$H^{g-1}(\mathcal{I}_Y(2\Theta)) \cong H^{g-2}(\mathcal{O}_Y(2\Theta)) \cong \bigoplus_{i=1}^{N} H^0(\mathcal{O}_{Y_i}(\Theta))^\vee \cong \mathbb{C}^N.$$

Hence $h^{g-1}(E) = h^{g-1}(\mathcal{I}_Y(2\Theta)) - 1 = N-1$, which yields $\dim \operatorname{Ext}^1(E, \mathcal{O}_X) = \dim \operatorname{Ext}^1(\mathcal{O}_X(\Theta), E) = N-1$. □

We shall construct μ-stable bundles of rank > 2 by means of extension.

Proposition 3.4. *Let (X, Θ) be as in Theorem* 1.1. *For any integer $N > 1$ and $0 < s \leq N-1$, there exist μ-stable bundles $E_{N,s}$ and $E'_{N,s}$ on X such that*

(1) $\operatorname{rank} E_{N,s} = s+2$, $c_1(E_{N,s}) = \Theta$, $c_2(E_{N,s}) = 2(N-1)\Theta^2$;

(2) $\operatorname{rank} E'_{N,s} = 2s+1$, $c_1(E'_{N,s}) = (s+1)\Theta$, $c_2(E'_{N,s}) = (\frac{s(s+1)}{2} + 2(N-1)s)\Theta^2$.

Proof. Given N, there exists a μ-stable bundle E_N of rank two with $c_1(E_N) = \mathcal{O}_X(\Theta)$ and $c_2(E_N) = 2(N-1)\Theta^2$ by Lemma 3.3. For a subspace $U \subset \operatorname{Ext}^1(E_N, \mathcal{O}_X)$ of dimension $s > 0$, let $E_{N,s}$ denote the universal extension of type I. Similarly, for a subspace $V \subset \operatorname{Ext}^1(\mathcal{O}_X(\Theta), E_N)$ of dimension $s > 0$, let $E'_{N,s}$ denote the universal extension of type II. By Proposition 2.11, they are μ-stable bundles. □

Given integers $m > 1$, the ranks and the discriminants of $E_m := E_{m,m-1}$ are given by

$$r_m = m+1, \quad \Delta_m = (4m^2 - m - 4)\Theta^g.$$

Similarly, we have

$$r'_m = 2m-1, \quad \Delta'_m = (m-1)(8m^2 - 13m + 4)\Theta^g$$

for $E'_m := E'_{m,m-1}$. Therefore, $\{E_m\}_{m=1}^{\infty}$ (resp. $\{E'_m\}_{m=1}^{\infty}$) is a family of μ-stable bundles of type $(1,2)$ (resp.$(1,3)$) on (X,Θ). Thus Theorem 1.1 is proved.

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