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PARTIAL CROSSED PRODUCT DESCRIPTION OF THE C^*-ALGEBRAS ASSOCIATED WITH INTEGRAL DOMAINS

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Abstract. Recently, Cuntz and Li introduced the C^*-algebra $\mathfrak{A}[R]$ associated to an integral domain R with finite quotients. In this paper, we show that $\mathfrak{A}[R]$ is a partial group algebra of the group $K \rtimes K^\times$ with suitable relations, where K is the field of fractions of R. We identify the spectrum of these relations and we show that it is homeomorphic to the profinite completion of R. By using partial crossed product theory, we reconstruct some results proved by Cuntz and Li. Among them, we prove that $\mathfrak{A}[R]$ is simple by showing that the action is topologically free and minimal.

1. Introduction

Fifteen years ago, motivated by the work of Julia [14], Bost and Connes constructed a C^*-dynamical system having the Riemann ζ-function as a partition function [2]. The C^*-algebra of the Bost-Connes system, denoted by $C_{\mathbb{Q}}$, is a Hecke C^*-algebra obtained from the inclusion of the integers into the rational numbers. In [19], Laca and Raeburn showed that $C_{\mathbb{Q}}$ can be realized as a semigroup crossed product and, in [20], they characterized the primitive ideal space of $C_{\mathbb{Q}}$.

In [1], [4] and [15], by observing that the construction of $C_{\mathbb{Q}}$ is based on the inclusion of the integers into the rational numbers, Arledge, Cohen, Laca and Raeburn generalized the construction of Bost and Connes. They replaced the field $\mathbb{Q}$ by an algebraic number field K and $\mathbb{Z}$ by the ring of integers of K. Many of the results obtained for $C_{\mathbb{Q}}$ were generalized to arbitrary algebraic number fields (at least when the ideal class group of the field is $h = 1$) [16], [17].

Recently, a new construction appeared. In [5], Cuntz defined two new C^*-algebras: $\mathcal{Q}_{\mathbb{N}}$ and $\mathcal{Q}_{\mathbb{Z}}$. Both algebras are simple and purely infinite, and $\mathcal{Q}_{\mathbb{N}}$ can be seen as a C^*-subalgebra of $\mathcal{Q}_{\mathbb{Z}}$. These algebras encode the additive and multiplicative structure of the semiring $\mathbb{N}$ and of the ring $\mathbb{Z}$. Cuntz showed that the algebra $\mathcal{Q}_{\mathbb{N}}$ is, essentially, the algebra generated by $C_{\mathbb{Q}}$ and one unitary operator. In [25], Yamashita realized $\mathcal{Q}_{\mathbb{N}}$ as the C^*-algebra of a topological higher-rank graph.

The next step was given by Cuntz and Li. In [6], they generalized the construction of $\mathcal{Q}_{\mathbb{Z}}$ by replacing $\mathbb{Z}$ by a unital commutative ring R (which is an integral domain with finite quotients by principal ideals). This algebra was called $\mathfrak{A}[R]$.

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Cuntz and Li showed that $\mathfrak{A}[R]$ is simple and purely infinite (when R is not a field), and they related a C^*-subalgebra of it with the generalized Bost-Connes systems (when R is the ring of integers in an algebraic number field having $h = 1$ and, at most, one real place). In [23], Li extended the construction of $\mathfrak{A}[R]$ to an arbitrary unital ring.

The aim of this paper is to show that the algebra $\mathfrak{A}[R]$ can be seen as a partial crossed product (when R is an integral domain with finite quotients). We show that $\mathfrak{A}[R]$ is isomorphic to a partial group algebra of the group $K \rtimes K^\times$ with suitable relations, where K is the field of fractions of R. By using the relationship between partial group algebras and partial crossed products, we see that $\mathfrak{A}[R]$ is a partial crossed product by the group $K \rtimes K^\times$. We characterize the spectrum of the commutative algebra arising in the crossed product and show that this spectrum is homeomorphic to $\hat{R}$ (the profinite completion of R). Furthermore, we show that the partial action is topologically free and minimal. By using the fact that the group $K \rtimes K^\times$ is amenable, we conclude that $\mathfrak{A}[R]$ is simple.

Recently, some similar results appeared. In [21] and [3], Brownlowe, an Huef, Laca and Raeburn showed that $\mathcal{Q}_\mathbb{N}$ is a partial crossed product by using a boundary quotient of the Toeplitz (or Wiener-Hopf) algebra of the quasi-lattice ordered group $(\mathbb{Q} \rtimes \mathbb{Q}_+^\times, \mathbb{N} \rtimes \mathbb{N}^\times)$ (see [24] and [18] for Toeplitz algebras of quasi-lattice ordered groups). We observe that our techniques are different from theirs. We don't use Nica's construction [24] (indeed, our group $K \rtimes K^\times$ is not a quasi-lattice, in general). From our results, in the case $R = \mathbb{Z}$, we see that $\mathcal{Q}_\mathbb{Z}$ is a partial crossed product by the group $\mathbb{Q} \rtimes \mathbb{Q}^\times$. From this, it is immediate that $\mathcal{Q}_\mathbb{N}$ is a partial crossed product by $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ (as in [3]).

Before we go to the main result we give, in section 2, a brief review about the algebra $\mathfrak{A}[R]$ and the theories of partial crossed products and partial group algebras. In section 3, we state our main theorem: the algebra $\mathfrak{A}[R]$ is isomorphic to a partial group algebra. In section 4, we study $\mathfrak{A}[R]$ by using the techniques of partial crossed products. We recover the faithful conditional expectation constructed by Cuntz and Li in [6, Proposition 1] in a very natural way. Futhermore, we use the concepts of topological freeness and minimality of a partial action to show that $\mathfrak{A}[R]$ is simple.

2. Preliminaries

2.1. **The C^*-algebra $\mathfrak{A}[R]$ of an integral domain.** Throughout this text, R will be an integral domain (a unital commutative ring without zero divisors) with the property that the quotient $R/(m)$ is finite, for all $m \neq 0$ in R. We denote by $R^\times$ the set $R\backslash\{0\}$ and by R^* the set of units in R.

Definition 2.1 ([6, Definition 1]). The **regular C^*-algebra of** R, denoted by $\mathfrak{A}[R]$, is the universal C*-algebra generated by isometries $\{s_m \mid m \in R^\times\}$ and unitaries $\{u^n \mid n \in R\}$ subject to the relations

(CL1) $s_m s_{m'} = s_{mm'}$;
(CL2) $u^n u^{n'} = u^{n+n'}$;
(CL3) $s_m u^n = u^{mn} s_m$;
(CL4) $\displaystyle\sum_{l+(m)\in R/(m)} u^l s_m s_m^* u^{-l} = 1$;

for all $m, m' \in R^\times$ and $n, n' \in R$.

We denote by e_m the range projection of s_m, namely $e_m = s_m s_m^*$. It is easily seen that, under (CL2) and (CL3), $u^l e_m u^{-l} = u^{l'} e_m u^{-l'}$ if $l + (m) = l' + (m)$. From this, we see that the sum in (CL4) is independent of the choice of l.

Let $\{\xi_r \mid r \in R\}$ be the canonical basis of the Hilbert space $\ell^2(R)$ and consider the operators S_m and U^n on $\ell^2(R)$ given by $S_m(\xi_r) = \xi_{mr}$ and $U^n(\xi_r) = \xi^{n+r}$.

Definition 2.2 ([6, Section 2])**.** The **reduced regular C^*-algebra of** R, denoted by $\mathfrak{A}_r[R]$, is the C^*-subalgebra of $\mathcal{B}(\ell^2(R))$ generated by the operators $\{S_m \mid m \in R^\times\}$ and $\{U^n \mid n \in R\}$.

One can check that S_m is an isometry, U^n is a unitary and they satisfy (CL1)-(CL4). Hence, there exists a surjective $*$-homomorphism $\mathfrak{A}[R] \longrightarrow \mathfrak{A}_r[R]$.

In [6], Cuntz and Li showed that when R is not a field, $\mathfrak{A}[R]$ is simple; therefore the above $*$-homomorphism is a $*$-isomorphism. In section 4, we will show that $\mathfrak{A}[R]$ is simple (when R is not a field) by using the partial crossed product description of $\mathfrak{A}[R]$.

For future reference, we need the following lemma, proved by Cuntz and Li:

Lemma 2.3 ([6, Lemma 1])**.** *For all $n, n' \in R$ and $m, m' \in R^\times$, the projections (in $\mathfrak{A}[R]$) $u^n e_m u^{-n}$ and $u^{n'} e_{m'} u^{-n'}$ commute.*

More details about these algebras can be found in [5], [6], [7], [8], [22], [23] and [25].

2.2. **Partial crossed products.** Here, we review some basic facts about partial actions and partial crossed products.

Definition 2.4 ([9, Definition 1.1])**.** A **partial action** α of a (discrete) group G on a C^*-algebra $\mathcal{A}$ is a collection $(\mathcal{D}_g)_{g\in G}$ of ideals of $\mathcal{A}$ and $*$-isomorphisms $\alpha_g : \mathcal{D}_{g^{-1}} \longrightarrow \mathcal{D}_g$ such that

(PA1) $\mathcal{D}_e = \mathcal{A}$, where e represents the identity element of G;
(PA2) $\alpha_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}}) \subseteq \mathcal{D}_{(gh)^{-1}}$;
(PA3) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x), \quad \forall\, x \in \alpha_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}})$.

In the above definition, if we replace the C^*-algebra $\mathcal{A}$ by a locally compact space X, the ideals $\mathcal{D}_g$ by open sets X_g and the $*$-isomorphisms α_g by homeomorphisms $\theta_g : X_{g^{-1}} \longrightarrow X_g$, we obtain a **partial action** θ of the group G on the space X. A partial action θ on a space X naturally induces a partial action α on the C^*-algebra $C_0(X)$. The ideals D_g are $C_0(X_g)$ and $\alpha_g(f) = f \circ \theta_{g^{-1}}$.

We say that a partial action θ on a space X is **topologically free** if, for all $g \in G\backslash\{e\}$, the set $F_g = \{x \in X_{g^{-1}} \mid \theta_g(x) = x\}$ has empty interior. A subset V of X is **invariant** under the partial action θ if $\theta_g(V \cap X_{g^{-1}}) \subseteq V$, for every $g \in G$. The partial action θ is **minimal** if there are no invariant open subsets of X other than $\emptyset$ and X. It is easy to see that θ is minimal if, and only if, every $x \in X$ has dense orbit; namely $\mathcal{O}_x = \{\theta_g(x) \mid g \in G \text{ for which } x \in X_{g^{-1}}\}$ is dense in X.

Definition 2.5 ([9, Definition 6.1])**.** A **partial representation** π of a (discrete) group G into a unital C^*-algebra $\mathcal{B}$ is a map $\pi : G \longrightarrow \mathcal{B}$ such that, for all $g, h \in G$,

(PR1) $\pi(e) = 1$;
(PR2) $\pi(g^{-1}) = \pi(g)^*$;
(PR3) $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$.

From a partial action α, we can construct two **partial crossed products**: $\mathcal{A} \rtimes_\alpha G$ (full) and $\mathcal{A} \rtimes_{\alpha,r} G$ (reduced). We can define both as follows: let $\mathcal{L}$ be the normed $*$-algebra of the finite formal sums $\sum_{g\in G} a_g\delta_g$, where $a_g \in \mathcal{D}_g$. The operations and the norm in $\mathcal{L}$ are given by $(a_g\delta_g)(a_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)a_h)\delta_{gh}$, $(a_g\delta_g)^* = \alpha_{g^{-1}}(a_g^*)\delta_{g^{-1}}$ and $||\sum_{g\in G} a_g\delta_g|| = \sum_{g\in G} ||a_g||$. If we denote by B_g the vector subspace $\mathcal{D}_g\delta_g$ of $\mathcal{L}$, then the family $(B_g)_{g\in G}$ generates a Fell bundle. The full and the reduced crossed products are, respectively, the full and the reduced cross sectional algebra of $(B_g)_{g\in G}$. It is well known that $\mathcal{A} \rtimes_\alpha G$ is universal with respect to a covariant pair (φ, π), where $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a $*$-homomorphism ($\mathcal{B}$ is a unital C^*-algebra), $\pi : G \longrightarrow \mathcal{B}$ is a partial representation of G and the covariant equations are $\varphi(\alpha_g(x)) = \pi(g)\varphi(x)\pi(g^{-1})$ for $x \in \mathcal{D}_{g^{-1}}$ and $\varphi(x)\pi(g)\pi(g^{-1}) = \pi(g)\pi(g^{-1})\varphi(x)$ for $x \in \mathcal{A}$.

There exists a faithful conditional expectation $E : \mathcal{A} \rtimes_{\alpha,r} G \longrightarrow \mathcal{A}$ given by $E(a\delta_g) = a$ if $g = e$ and $E(a\delta_g) = 0$ if $g \neq e$. When the Fell bundle $(B_g)_{g\in G}$ is amenable (G amenable implies its), the full and reduced constructions are isomorphic and, in this case, there exists a faithful conditional expectation of $\mathcal{A} \rtimes_\alpha G$ onto $\mathcal{A}$.

There is a close relation between topological freeness and minimality of the partial action and ideals of the reduced crossed product. If θ is a topologically free partial action on a space X, then θ is minimal if, and only if, $C_0(X) \rtimes_{\alpha,r} G$ is simple, where α is the action induced by θ. Under the amenability hypothesis, this result is valid for the full crossed product too.

For more details about partial crossed products, see [9], [10], [11], [12] and [13].

2.3. **Partial group algebras.** Let G be a discrete group, let $\mathcal{G}$ be the set G without the group operations and denote the elements in $\mathcal{G}$ by $[g]$ (namely, $\mathcal{G} = \{[g] \mid g \in G\}$). The **partial group algebra of** G, denoted by $C_\mathrm{p}^*(G)$, is defined to be the universal C^*-algebra generated by the set $\mathcal{G}$ with the relations

$$\mathcal{R}_\mathrm{p} = \{[e] = 1\} \cup \{[g^{-1}] = [g]^*\}_{g\in G} \cup \{[g][h][h^{-1}] = [gh][h^{-1}]\}_{g,h\in G}.$$

The algebra $C_\mathrm{p}^*(G)$ is universal with respect to a partial representation. Observe that the relations in $\mathcal{R}_\mathrm{p}$ correspond to the partial representation axioms (PR1), (PR2) and (PR3). Sometimes we will refer to a relation in $\mathcal{R}_\mathrm{p}$ by indicating the corresponding axiom.

Consider the natural bijection between $\mathcal{P}(G)$ and $\{0,1\}^G$, where $\mathcal{P}(G)$ is the power set of G. With the product topology, $\{0,1\}^G$ is a compact Hausdorff space. Give to $\mathcal{P}(G)$ the topology of $\{0,1\}^G$. Denote by X_G the subset of $\mathcal{P}(G)$ of the subsets ξ of G such that $e \in \xi$. Clearly, with the induced topology of $\mathcal{P}(G)$, X_G is a compact space. For each $g \in G$, let $X_g = \{\xi \in X_G \mid g \in \xi\}$. It is easy to see that $\theta_g : X_{g^{-1}} \longrightarrow X_g$ given by $\theta_g(\xi) = g\xi$ is a homeomorphism. The collection of open sets $(X_g)_{g\in G}$ of X_G with the homeomorphisms θ_g define a partial action θ of G on X_G. The partial crossed product $C(X_G) \rtimes_\alpha G$ is isomorphic to $C_\mathrm{p}^*(G)$ (where α is the partial action induced by θ).

For each $g \in G$, we abbreviate $[g][g^{-1}]$ by e_g. Let $\mathcal{R}$ be a set of relations on $\mathcal{G}$ such that every relation is of the form

$$\sum_i \lambda_i \prod_j e_{g_{ij}} = 0.$$

The **partial group algebra of G with relations $\mathcal{R}$**, denoted by $C^*_{\mathrm{p}}(G,\mathcal{R})$, is defined to be the universal C^*-algebra generated by the set $\mathcal{G}$ with the relations $\mathcal{R}_{\mathrm{p}} \cup \mathcal{R}$. Given a partial representation π of G, we can extend π naturally to sums of products of elements in $\mathcal{G}$. If this extension satisfies the relations $\mathcal{R}$, we say that π is a **partial representation that satisfies $\mathcal{R}$**. The algebra $C^*_{\mathrm{p}}(G,\mathcal{R})$ is universal with respect to a partial representation that satisfies the relations $\mathcal{R}$.

Denote by 1_g the function in $C(X_G)$ given by $1_g(\xi) = 1$ if $g \in \xi$ and $1_g(\xi) = 0$ otherwise. By an abuse of notation, we also denote by $\mathcal{R}$ the subset of $C(X_G)$ given by the functions $\sum_i \lambda_i \prod_j 1_{g_{ij}}$, where $\sum_i \lambda_i \prod_j e_{g_{ij}} = 0$ is a relation in (the original) $\mathcal{R}$. The **spectrum of the relations $\mathcal{R}$** is defined to be the compact Hausdorff space

$$\Omega_{\mathcal{R}} = \{\xi \in X_G \mid f(g^{-1}\xi) = 0,\ \forall\, f \in \mathcal{R},\ \forall\, g \in \xi\}.$$

Let $\Omega_g = \{\xi \in \Omega_{\mathcal{R}} \mid g \in \xi\}$. By restricting the above θ_g to $\Omega_{g^{-1}}$, we obtain a partial action (again denoted by θ) of G on $\Omega_{\mathcal{R}}$ (the open sets are the Ω_g's and the homeomorphisms are the restrictions of the θ_g's). The main result concerning $C^*_{\mathrm{p}}(G,\mathcal{R})$ says that this algebra is isomorphic to the partial crossed product $C(\Omega_{\mathcal{R}}) \rtimes_\alpha G$ (again, α is the partial action induced by θ).

The above results are proved in [12] and [13].

3. Partial group algebra description of $\mathfrak{A}[R]$

Let R be an integral domain satisfying the conditions stated in the previous section. Denote by K the field of fractions of R and consider the semidirect product $K \rtimes K^\times$. The elements of $K \rtimes K^\times$ will be denoted by a pair (u,w), where $u \in K$ and $w \in K^\times$. Recall that $(u,w)(u',w') = (u + u'w, ww')$ and $(u,w)^{-1} = (-u/w, 1/w)$. We denote by $[u,w]$ an element of the set $K \rtimes K^\times$ without the group operations (as the set $\mathcal{G}$ associated to G in the previous section).[1] Again, denote $[g][g^{-1}]$ by e_g. Consider the sets of relations

$$\mathcal{R}_1 = \left\{e_{(n,1)} = 1 \mid n \in R\right\}, \quad \mathcal{R}_2 = \left\{e_{\left(0,\frac{1}{m}\right)} = 1 \mid m \in R^\times\right\},$$

$$\mathcal{R}_3 = \left\{ \sum_{n+(m)\in R/(m)} e_{(n,m)} = 1 \;\middle|\; m \in R^\times \right\}$$

and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. We observe that, under the relations $\mathcal{R}_1$ and $\mathcal{R}_{\mathrm{p}}$ (relations stated in the previous section), the sum in $\mathcal{R}_3$ does not depend on the choice of n. Indeed, for $k \in R$,

$$\begin{aligned} e_{(n+km,m)} &= [n+km,m][(n+km,m)^{-1}] \overset{\mathcal{R}_1}{=} [(n,m)(k,1)]e_{(-k,1)}[(k,1)^{-1}(n,m)^{-1}] \\ &= [(n,m)(k,1)][(k,1)^{-1}][k,1][(k,1)^{-1}(n,m)^{-1}] \\ &\overset{\text{(PR3)}}{=} [n,m][k,1][(k,1)^{-1}][k,1][(k,1)^{-1}][(n,m)^{-1}] \\ &= [n,m]e_{(k,1)}e_{(k,1)}[(n,m)^{-1}] = e_{(n,m)}. \end{aligned}$$

Remark 3.1. The relations in $\mathcal{R}_1$ are unnecessary. They can be obtained from $\mathcal{R}_3$ with $m = 1$.

[1] Sometimes we work with the element $(u,w)^{-1}$ or the element $(u_1,w_1)(u_2,w_2)$. For these elements, our corresponding notation will be $[(u,w)^{-1}]$ and $[(u_1,w_1)(u_2,w_2)]$.

Consider the partial group algebra $C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$. We will show that this algebra is isomorphic to $\mathfrak{A}[R]$.

Proposition 3.2. *There exists a $*$-homomorphism $\Psi : \mathfrak{A}[R] \longrightarrow C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$ such that $\Psi(u^n) = [n, 1]$ and $\Psi(s_m) = [0, m]$.*

Proof. We need to show that $[n, 1]$ is a unitary (for $n \in R$), that $[0, m]$ is an isometry (for $m \in R^\times$) and that the relations (CL1)-(CL4) are satisfied. From $\mathcal{R}_1$ and (PR2), we have $[n, 1][n, 1]^* = e_{(n,1)} = 1$ and $[n, 1]^*[n, 1] = e_{(-n,1)} = 1$; i.e., $[n, 1]$ is a unitary. Similarly, from $\mathcal{R}_2$ and (PR2) we see that $[0, m]$ is an isometry. By using this fact,

$$\Psi(s_m s_{m'}) = [0, m][0, m'] = [0, m][0, m'][0, m']^*[0, m']$$

$$\overset{\text{(PR3)}}{=} [0, mm'][0, m']^*[0, m'] = [0, mm'] = \Psi(s_{mm'});$$

hence (CL1) is satisfied. We can prove (CL2) in the same way. To show (CL3), note that

$$\Psi(s_m u^n) = [0, m][n, 1] = [0, m][n, 1][n, 1]^*[n, 1] \overset{\text{(PR3)}}{=} [mn, m][n, 1]^*[n, 1] = [mn, m],$$

because $[n, 1]$ is a unitary. On the other hand,

$$\Psi(u^{mn} s_m) = [mn, 1][0, m] = [mn, 1][mn, 1]^*[mn, 1][0, m]$$

$$\overset{\text{(PR3)}}{=} [mn, 1][mn, 1]^*[mn, m] = [mn, m].$$

Finally, (CL4) follows from $\mathcal{R}_3$ and[2]

$$\Psi(u^n e_m u^{-n}) = [n, 1][0, m][0, m]^*[-n, 1] = [n, m][0, 1/m][-n, 1][-n, 1]^*[-n, 1]$$

$$\overset{\text{(PR3)}}{=} [n, m][(n, m)^{-1}][-n, 1]^*[-n, 1] = [n, m][(n, m)^{-1}] = e_{(n,m)}.$$

□

Now, we will construct an inverse for Ψ. In the next claim, note that every element in $K \rtimes K^\times$ can be written under the form $\left(\frac{n}{m'}, \frac{m}{m'}\right)$, where $n \in R$ and $m, m' \in R^\times$.

Claim 3.3. The map $\pi : K \rtimes K^\times \longrightarrow \mathfrak{A}[R]$ given by $\pi\left(\left(\frac{n}{m'}, \frac{m}{m'}\right)\right) = s^*_{m'} u^n s_m$ is independent of the representation of $\left(\frac{n}{m'}, \frac{m}{m'}\right)$.

Proof. Let $\left(\frac{n}{m'}, \frac{m}{m'}\right) = \left(\frac{q}{p'}, \frac{p}{p'}\right)$, i.e., $pm' = p'm$ and $m'q = p'n$. Hence,

$$s^*_{p'} u^q s_p = s^*_{p'} s^*_{m'} s_{m'} u^q s_p \overset{\text{(CL3)}}{=} s^*_{p'} s^*_{m'} u^{m'q} s_{m'} s_p \overset{\text{(CL1)}}{=} s^*_{p'm'} u^{m'q} s_{m'p}$$

$$\overset{\text{(CL1)}}{=} s^*_{m'} s^*_{p'} u^{np'} s_{p'} s_m \overset{\text{(CL3)}}{=} s^*_{m'} s^*_{p'} s_{p'} u^n s_m = s^*_{m'} u^n s_m.$$

□

Proposition 3.4. *The map π defined above is a partial representation of $K \rtimes K^\times$ that satisfies $\mathcal{R}$.*

[2]Be careful with the e's! The notation e_m represents $s_m s^*_m$ in $\mathfrak{A}[R]$, and $e_{(n,m)}$ represents $[n, m][n, m]^*$ in $C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$.

Proof. First, we will show that π is a partial representation. Since $\pi((0,1)) = s_1^* u^0 s_1 = 1$, we have (PR1). Observe that

$$\pi\left(\left(\frac{n}{m'},\frac{m}{m'}\right)^{-1}\right) = \pi\left(\left(\frac{-n}{m},\frac{m'}{m}\right)\right) = s_m^* u^{-n} s_{m'} = \pi\left(\left(\frac{n}{m'},\frac{m}{m'}\right)\right)^*,$$

which shows (PR2). To see (PR3), let $g = \left(\frac{q}{p'},\frac{p}{p'}\right)$ and $h = \left(\frac{n}{m'},\frac{m}{m'}\right)$. We have $gh = \left(\frac{m'q+pn}{p'm'},\frac{pm}{p'm'}\right)$ and, therefore,

$$\begin{aligned}
\pi(gh)\pi(h^{-1}) &= \pi(gh)\pi(h)^* = (s_{p'm'}^* u^{m'q+pn} s_{pm})(s_m^* u^{-n} s_{m'}) \\
&\overset{\text{(CL1),(CL2),(CL3)}}{=} s_{p'}^* u^q s_{m'}^* s_p u^n s_m s_m^* u^{-n} s_{m'} \\
&= s_{p'}^* u^q s_{m'}^* s_p \underbrace{u^n s_m s_m^* u^{-n}}\,\underbrace{s_{m'} s_{m'}^*}\, s_{m'} \\
&\overset{\text{Lemma 2.3}}{=} s_{p'}^* u^q s_{m'}^* s_p s_{m'} s_{m'}^* u^n s_m s_m^* u^{-n} s_{m'} \\
&\overset{\text{(CL1)}}{=} (s_{p'}^* u^q s_p)(s_{m'}^* u^n s_m)(s_m^* u^{-n} s_{m'}) = \pi(g)\pi(h)\pi(h^{-1}).
\end{aligned}$$

This shows that π is a partial representation. It remains to show that the extension of π satisfies the relations in $\mathcal{R}$. By Remark 3.1, it suffices to show that the relations in $\mathcal{R}_2$ and $\mathcal{R}_3$ are satisfied. This follows from

$$\pi(e_{(0,1/m)}) = \pi([0,1/m][0,m]) = s_m^* u^0 s_1 s_1^* u^0 s_m = 1$$

and

$$\pi\left(\sum_{n+(m)\in R/(m)} e_{(n,m)}\right) = \sum_{n+(m)\in R/(m)} s_1^* u^n s_m s_m^* u^{-n} s_1 = 1.$$

□

Remark 3.5. We can define π for a general representation of an element in $K \rtimes K^\times$ by $\pi\left(\left(\frac{n}{m''},\frac{m}{m'}\right)\right) = s_{m''}^* u^n s_{m'}^* s_{m''} s_m$.

Theorem 3.6. *The $*$-homomorphism Ψ defined above is a $*$-isomorphism. Its inverse $\Phi : C_{\mathrm{p}}^*(K \rtimes K^\times, \mathcal{R}) \longrightarrow \mathfrak{A}[R]$ is given by $\Phi\left(\left[\frac{n}{m'},\frac{m}{m'}\right]\right) = s_{m'}^* u^n s_m$.*

Proof. The existence of Φ follows from π and the universal property of $C_{\mathrm{p}}^*(K \rtimes K^\times, \mathcal{R})$. It remains to show that Ψ and Φ are inverses of each other. Indeed, $\Phi(\Psi(u^n)) = \Phi([n,1]) = s_1^* u^n s_1 = u^n$, $\Phi(\Psi(s_m)) = \Phi([0,m]) = s_1^* u^0 s_m = s_m$ and

$$\Psi\left(\Phi\left(\left[\frac{n}{m'},\frac{m}{m'}\right]\right)\right) = \Psi(s_{m'}^* u^n s_m) = [0,1/m']\,[n,1]\,[0,m]$$

$$= [0,1/m']\,[0,1/m']^*\,[0,1/m']\,[n,1]\,[n,1]^*\,[n,1]\,[0,m] = \left[\frac{n}{m'},\frac{m}{m'}\right].$$

□

4. Partial crossed product description of $\mathfrak{A}[R]$

Before characterizing $\mathfrak{A}[R]$ as a partial crossed product, note that the group $K \rtimes K^\times$ is solvable and, hence, amenable. Therefore, there exists a faithful conditional expectation (imported from the partial crossed product realization) $E : C_{\mathrm{p}}^*(K \rtimes K^\times, \mathcal{R}) \longrightarrow C^*(\{e_g\}_{g\in K\rtimes K^\times})$ given by

$$E([g_1][g_2]\cdots[g_k]) = \delta_{g_1g_2\cdots g_k,e}[g_1][g_2]\cdots[g_k].$$

In [6, Proposition 1], Cuntz and Li constructed a faithful conditional expectation Θ on $\mathfrak{A}[R]$ given by $\Theta(s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m'}) = \delta_{m',m''}\delta_{n,n'} s_{m'}^* u^n s_m s_m^* u^{-n} s_{m'}$. The next proposition shows that, under the $*$-isomorphism Ψ, E and Θ are the same conditional expectation.

Proposition 4.1. $E \circ \Psi = \Psi \circ \Theta$.

Proof. First of all, observe that $\left(\frac{n}{m''}, \frac{m}{m''}\right)\left(\frac{-n'}{m}, \frac{m'}{m}\right) = (0,1)$ if, and only if, $m' = m''$ and $n = n'$. Hence,

$$E \circ \Psi(s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m'}) = E\left(\left[\frac{n}{m''}, \frac{m}{m''}\right]\left[\frac{-n'}{m}, \frac{m'}{m}\right]\right)$$

$$= \delta_{m',m''}\delta_{n,n'}\left[\frac{n}{m'}, \frac{m}{m'}\right]\left[\frac{-n}{m}, \frac{m'}{m}\right].$$

On the other hand,

$$\Psi \circ \Theta(s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m'}) = \Psi(\delta_{m',m''}\delta_{n,n'} s_{m'}^* u^n s_m s_m^* u^{-n} s_{m'})$$

$$= \delta_{m',m''}\delta_{n,n'}\left[\frac{n}{m'}, \frac{m}{m'}\right]\left[\frac{-n}{m}, \frac{m'}{m}\right].$$

□

We already know that $\mathfrak{A}[R]$ is a partial crossed product. Indeed, every partial group algebra is a partial crossed product (see section 2.3). From now on, our goal is to study $\mathfrak{A}[R]$ in this manner.

There exists a natural partial order on $R^\times$ given by the divisibility: we say that $m \leq m'$ if there exists $r \in R$ such that $m' = mr$. Whenever $m \leq m'$, we can consider the canonical projection $p_{m,m'} : R/(m') \longrightarrow R/(m)$. Since $(R^\times, \leq)$ is a directed set, we can consider the inverse limit

$$\hat{R} = \varprojlim\{R/(m),\ p_{m,m'}\},$$

which is the **profinite completion of** R. In this text, we shall use the following concrete description of $\hat{R}$:

$$\hat{R} = \left\{ (r_m + (m))_m \in \prod_{m \in R^\times} R/(m) \;\middle|\; p_{m,m'}(r_{m'} + (m')) = r_m + (m),\ \text{if } m \leq m' \right\}.$$

Give to $R/(m)$ the discrete topology, to $\prod_{m\in R^\times} R/(m)$ the product topology and to $\hat{R}$ the induced topology of $\prod_{m\in R^\times} R/(m)$. With the operations defined componentwise, $\hat{R}$ is a compact topological ring. There exists a canonical inclusion of R into $\hat{R}$ given by $r \longmapsto (r + (m))_m$ (to see injectivity, take $r \neq 0$, m non-invertible and note that $r \notin (rm)$).

The above partial order can be extended to $K^\times$. For $w, w' \in K^\times$, we say that $w \leq w'$ if there exists $r \in R$ such that $w' = wr$. Denote by (w) the fractional ideal generated by w, namely $(w) = wR \subseteq K$. As before, if $w \leq w'$, we can consider the canonical projection[3] $p_{w,w'} : (R + (w'))/(w') \longrightarrow (R + (w))/(w)$. As before,

[3]By the second isomorphism theorem, it could be $p_{w,w'} : R/(R \cap (w')) \longrightarrow R/(R \cap (w))$.

we consider the inverse limit

$$\hat{R}_K = \varprojlim\{(R+(w))/(w),\ p_{w,w'}\}$$
$$\cong \left\{(u_w+(w))_w \in \prod_{w\in K^\times}(R+(w))/(w) \;\middle|\; p_{w,w'}(u_{w'}+(w'))=u_w+(w),\ \text{if } w\le w'\right\}.$$

It is a compact topological ring too. In fact, $\hat{R}_K$ is naturally isomorphic to $\hat{R}$ as a topological ring. In this text, we use $\hat{R}_K$ instead of $\hat{R}$ to simplify our proofs.

It is easy to see that when R is a field, then $\hat{R} \cong \hat{R}_K \cong \{0\}$.

Let Ω be the spectrum of the relations $\mathcal{R}$ (see section 2.3). We will show that Ω is homeomorphic to $\hat{R}_K$ (hence, homeomorphic to $\hat{R}$). Define

$$\begin{aligned}\rho : \hat{R}_K &\longrightarrow \mathcal{P}(K \rtimes K^\times),\\ (u_w+(w))_w &\longmapsto \{(u_w+rw, w) \mid w\in K^\times,\ r\in R\}.\end{aligned}$$

Note that the definition is independent of the choice of u_w in $u_w+(w)$.

Claim 4.2. $\rho(\hat{R}_K) \subseteq \Omega$.

Proof. Let $(u_w+(w))_w \in \hat{R}_K$. By the definition of $\hat{R}_K$, if $w\le w'$, then $u_{w'} = u_w + kw$ for some $k\in R$. Denote $\rho((u_w+(w)))$ by ξ. Clearly, $(0,1)\in\xi$. We need to show that $f(g^{-1}\xi)=0$, for all $f\in\mathcal{R}$ and $g\in\xi$. Fix $g=(u_w+rw,w)\in\xi$. Let $f = 1_{(n,1)}-1$ in $\mathcal{R}_1$ and note that $f(g^{-1}\xi)=0$ is equivalent to $g(n,1)\in\xi$. Since $g(n,1)=(u_w+rw,w)(n,1)=(u_w+(r+n)w,w)$, we have $g(n,1)\in\xi$. Now, let $f=1_{(0,1/m)}-1$ in $\mathcal{R}_2$. Similarly, we must show that $g(0,1/m)\in\xi$. Observe that $g(0,1/m)=(u_w+rw,w)(0,1/m)=(u_w+rw,w/m)$. Since $w/m\le w$, then $g(0,1/m)=(u_{w/m}+k(w/m)+rw,w/m)=(u_{w/m}+(k+rm)(w/m))\in\xi$. To finish, fix $m\in R^\times$ and let $f=\sum_{n+(m)}1_{(n,m)}-1$ in $\mathcal{R}_3$. We must show that there exists one, and only one, class $n+(m)$ such that $g(n,m)\in\xi$. Indeed, $g(n,m)=(u_w+rw,w)(n,m)=(u_w+(n+r)w,wm)=(u_{wm}+(n+r-k)w,wm)$ and, because it belongs to ξ, we must have $(n+r-k)w\in(wm)$. Hence, $n\equiv k-r \mod m$; in other words, there exists only one class $n+(m)$ such that $g(n,m)\in\xi$. Since $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3$, the proof is completed. □

Proposition 4.3. *$\rho : \hat{R}_K \longrightarrow \Omega$ is a homeomorphism.*

Proof.

Injectivity. Let $(u_w+(w))_w, (v_w+(w))_w \in \hat{R}_K$ such that $\rho((u_w+(w))) = \rho((v_w+(w)))$. By the definition of ρ, the elements in $\rho((u_w+(w)))$ whose second component equals w are of the form (u_w+rw,w). Since $(v_w,w)\in\rho((v_w+(w)))$ and, therefore, $(v_w,w)\in\rho((u_w+(w)))$, we must have $v_w=u_w+rw$ for some $r\in R$. This shows that $(u_w+(w))_w=(v_w+(w))_w$.

Surjectivity. Let $\xi\in\Omega$. The relations in $\mathcal{R}_1$ and $\mathcal{R}_2$ together imply that if $g\in\xi$, then $g(q/p,1/p)\in\xi$ for all $q\in R$ and $p\in R^\times$ (fix g and apply $f(g^{-1}\xi)=0$ for various f). For each $m\in R^\times$, let $f=\sum_{n+(m)}1_{(n,m)}-1$ in $\mathcal{R}_3$ and apply $f(g^{-1}\xi)=0$ with $g=(0,1)$ to see that there exists only one class $n+(m)$ such that $(n,m)\in\xi$. Denote this class by $u_m+(m)$. Since $g(0,1/p)\in\xi$ if $g\in\xi$, then $p_{m,mp}(u_{mp}+(mp))=(u_m+(m))$. From this, we can unambiguously define $u_w+(w)=u_m+(w)$ for $w=m/m'\in K^\times$. One can see that the classes $u_w+(w)$ are compatible with the projections $p_{w,w'}$ by using that $g(q/p,1/p)\in\xi$ if $g\in\xi$.

Hence, we have constructed $(u_w + (w))_w \in \hat{R}_K$. We claim that $\rho((u_w + (w))) = \xi$. Since $(u_w, w) \in \xi$, $(u_w, w)(q, 1) = (u_w + qw, w)$ must belong to ξ. This shows that $\rho((u_w + (w))) \subseteq \xi$. Suppose, by contradiction, that $\rho((u_w + (w))) \neq \xi$. Hence, there exists $h \in \xi$ such that $h \notin \rho((u_w + (w)))$. If we write $h = (n'/m', m/m')$, then $h \notin \rho((u_w + (w)))$ is equivalent to $n' - m'u_m \notin (m)$. Let $g = (u_m, 1/m')$, $h' = (u_m, m/m')$ and note that both belong to $\rho((u_w + (w)))$ (hence, belong to ξ). Since $g^{-1}h = (-m'u_m, m')(n'/m', m/m') = (n' - m'u_m, m)$, $g^{-1}h' = (0, m)$ and $n' - m'u_m \notin (m)$, then $f(g^{-1}\xi) \neq 0$ if $f = \sum_{n+(m)} 1_{(n,m)} - 1$, which contradicts the fact that $\xi \in \Omega$. Hence, $\rho((u_w + (w))) = \xi$.

To finish the proof, observe that $\hat{R}_K$ and Ω are compact Hausdorff. Therefore it suffices to show that ρ (or ρ^{-1}) is continuous to conclude that ρ is a homeomorphism. We will prove that ρ^{-1} is continuous by showing that $\pi_w \circ \rho^{-1}$ is continuous for all $w \in K^\times$, where $\pi_w : \hat{R}_K \longrightarrow (R + (w))/(w)$ is the canonical projection. Since $(R + (w))/(w)$ is discrete, it suffices to show that $\rho \circ \pi_w^{-1}(\{u_w + (w)\})$ is an open set of Ω, for all $u_w + (w) \in (R + (w))/(w)$. To see this, note that

$$\rho \circ \pi_w^{-1}(\{u_w + (w)\}) = \{\xi \in \Omega \mid (u_w, w) \in \xi\},$$

which is an open set of Ω (recall that the topology on Ω is induced by the product topology of $\{0,1\}^{K \rtimes K^\times}$). □

Following section 2.3, there exists a partial action of $K \rtimes K^\times$ on Ω. By the above proposition, we can define this partial action on $\hat{R}_K$. Let $\hat{R}_g = \rho^{-1}(\Omega_g)$, where $\Omega_g = \{\xi \in \Omega \mid g \in \xi\}$, and let θ_g be the homeomorphism between $\hat{R}_{g^{-1}}$ and $\hat{R}_g$. It is easy to see that

$$\hat{R}_{(u,w)} = \{(u_{w'} + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) = u + (w)\}$$

and

$$\theta_{(u,w)}((u_{w'} + (w'))_{w'}) = (u + wu_{w'} + (ww'))_{ww'} = (u + wu_{w^{-1}w'} + (w'))_{w'};$$

i.e., $\theta_{(u,w)}$ acts on $\hat{R}_{(u,w)^{-1}}$ by the affine transformation corresponding to (u, w). The next proposition, whose proof is trivial, will be useful later.

Proposition 4.4. *We have that*

(i) $\hat{R}_{(u,w)} = \emptyset \iff u \notin R + (w)$;

(ii) $\hat{R}_{(u,w)} = \hat{R}_K \iff R \subseteq u + (w)$.

Now, we describe the topology on $\hat{R}_K$. Since $\hat{R}_K$ is a singleton set when R is a field, we shall assume that R is not a field in this paragraph. For $w \in K^\times$ and $C_w \subseteq (R + (w))/(w)$, we define the open set

$$V_w^{C_w} = \{(u_{w'} + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) \in C_w\}.$$

Clearly, if $w \leq w'$, then $V_w^{C_w} = V_{w'}^{C_{w'}}$, where $C_{w'} = \{u + (w') \in (R + (w'))/(w') \mid u + (w) \in C_w\}$. From the product topology, we know that the finite intersections of open sets $V_w^{C_w}$ form a basis for the topology on $\hat{R}_K$. By taking a common multiple of the w's in the intersection, we see that every basic open set is of the form $V_w^{C_w}$ (since $V_w^{C_1} \cap V_w^{C_2} = V_w^{C_1 \cap C_2}$). Furthermore, if $C_w \neq \emptyset$, r is a non-invertible element in R and $V_w^{C_w} = V_{wr}^{C_{wr}}$, then C_{wr} has at least two elements. Indeed, let $u + (w) \in C_w$ and $r_1, r_2 \in R$ be such that $r_1 + (r) \neq r_2 + (r)$. It is easy to see that $u + wr_1 + (wr)$ and $u + wr_2 + (wr)$ are in C_{wr} and that $u + wr_1 + (wr) \neq u + wr_2 + (wr)$. This says that if $V_w^{C_w}$ is non-empty, we can suppose that C_w has more than one element.

Proposition 4.5. *The partial action θ on $\hat{R}_K$ is topologically free if, and only if, R is not a field.*

Proof. If R is a field, then $\hat{R}_K = \{0\}$ and, hence, θ is not topologically free. Conversely, suppose that R is not a field. We need to show that $F_g = \{x \in \hat{R}_{g^{-1}} \mid \theta_g(x) = x\}$ has empty interior, for all $g \in K \rtimes K^\times \backslash \{(0,1)\}$. We shall consider two cases: $g = (u,1)$ and $g = (u,w)$, $w \neq 1$.

Case 1. If $u \notin R$, then Proposition 4.4 says that $\hat{R}_{g^{-1}} = \emptyset$. So, we can suppose $u \in R$. If $F_g \neq \emptyset$, then the equation $\theta_g(x) = x$ implies that $u \in (m)$ for every $m \in R^\times$. Since R is not a field, then $u = 0$. This shows that $F_g = \emptyset$ if $g = (u,1)$ and $u \neq 0$.

Case 2. Let $g = (u,w)$ be such that $w \neq 1$ and $u \in R + (w)$ (if $u \notin R + (w)$, then $\hat{R}_{g^{-1}} = \emptyset$). Let V be a non-empty open set contained in $\hat{R}_{g^{-1}}$. We will show that there exists $x \in V$ such that $\theta_g(x) \neq x$. By shrinking V if necessary, we can suppose that $V = V_{w'}^{C_{w'}}$. Furthermore, we can assume that $C_{w'}$ has more than one element. Let $u_1 + (w')$ and $u_2 + (w')$ be distinct elements of $C_{w'}$; hence $u_1 - u_2 \notin (w')$. Suppose, by contradiction, that $\theta_g(x) = x$ for all $x \in V$. Since $(u_i + (w''))_{w''} \in V$, $i = 1, 2$, then

$$\theta_{(u,w)}((u_i + (w''))_{w''}) = (u_i + (w''))_{w''} \implies (u + wu_i + (w''))_{w''} = (u_i + (w''))_{w''}.$$

By choosing $w'' = (w-1)w'$ (note that $w \neq 1$), we see that $u + (w-1)u_i \in ((w-1)w')$, for $i = 1, 2$. By subtracting the equations (for different i's), we have $(w-1)(u_1 - u_2) \in ((w-1)w')$, and therefore $u_1 - u_2 \in (w')$, which is a contradiction! This shows that F_g has empty interior. □

Proposition 4.6. *The partial action θ is minimal.*

Proof. If R is a field, then the result is trivial. Now, suppose that R is not a field. We will prove that every $x \in \hat{R}_K$ has dense orbit (see section 2.2) by showing that if V is a non-empty open set, then there exists $g \in K \rtimes K^\times$ such that $x \in \hat{R}_{g^{-1}}$ and $\theta_g(x) \in V$. Let $x = (u_w + (w))_w \in \hat{R}_K$ and $V = V_{w'}^{C_{w'}}$ be non-empty. Take $u' + (w') \in C_{w'}$ and observe that we can suppose, without loss of generality, that $u' \in R$ and $u_{w'} \in R$. Let $g = (u' - u_{w'}, 1)$. By Proposition 4.4, $\hat{R}_{g^{-1}} = \hat{R}_K$ and, hence, $x \in \hat{R}_{g^{-1}}$. To finish, note that $\theta_g(x) = \theta_{(u'-u_{w'},1)}((u_w + (w))_w) = (u' - u_{w'} + u_w + (w))_w \in V$. □

In the following, we summarize the results of this section.

Theorem 4.7. *The algebra $\mathfrak{A}[R]$ is $*$-isomorphic to the partial crossed product $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times$, where α is the partial action induced by θ. The $*$-isomorphism is given by $u^n \longmapsto 1\delta_{(n,1)}$ and $s_m \longmapsto 1_{(0,m)}\delta_{(0,m)}$, where $1_{(0,m)}$ is the characteristic function of $\hat{R}_g$.*

Theorem 4.8. $\mathfrak{A}[R]$ *is simple.*

Proof. By Propositions 4.5 and 4.6, the reduced crossed product $C(\hat{R}_K) \rtimes_{\alpha,r} K \rtimes K^\times$ is simple. Since $K \rtimes K^\times$ is amenable, then $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times \cong C(\hat{R}_K) \rtimes_{\alpha,r} K \rtimes K^\times$ and, therefore, $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times$ is simple. The result follows from the previous theorem. □

Corollary 4.9. $\mathfrak{A}[R] \cong \mathfrak{A}_r[R]$.

When $R = \mathbb{Z}$, we can restrict our partial action to the subgroup $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ of $\mathbb{Q} \rtimes \mathbb{Q}^*$. The corresponding partial crossed product is the algebra $\mathcal{Q}_{\mathbb{N}}$ introduced by Cuntz in [5] and realized as a partial crossed product in [3] by Brownlowe, an Huef, Laca and Raeburn.

References

[1] J. Arledge, M. Laca and I. Raeburn, *Semigroup crossed products and Hecke algebras arising from number fields*, Doc. Math. **2** (1997), 115–138. MR1451963 (98k:46111)

[2] J. B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math., New Series, **1** (1995), no. 3, 411-457. MR1366621 (96m:46112)

[3] N. Brownlowe, A. an Huef, M. Laca and I. Raeburn, *Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers*, Ergodic Theory Dynam. Systems **32** (2012), no. 1, 35–62. MR2873157 (2012j:46101)

[4] P. B. Cohen, *A C^*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking*, Journées Arithmétiques de Limoges, 1997. MR1730430 (2001f:46104)

[5] J. Cuntz, *C^*-algebras associated with the $ax+b$-semigroup over $\mathbb{N}$*, K-Theory and Noncommutative Geometry (Valladolid, 2006), European Math. Soc., 2008, 201–215. MR2513338 (2010i:46086)

[6] J. Cuntz and X. Li, *The regular C^*-algebra of an integral domain*, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010. MR2732050 (2012c:46173)

[7] J. Cuntz and X. Li, *C^*-algebras associated with integral domains and crossed products by actions on adele spaces*, J. Noncomm. Geom. **5** (2011), no. 1, 1–37. MR2746649 (2011k:46093)

[8] J. Cuntz and X. Li, *K-theory for ring C^*-algebras attached to function fields*, arXiv:0911.5023v1, 2009.

[9] M. Dokuchaev and R. Exel, *Associativity of crossed products by partial actions, enveloping actions and partial representations*, Trans. Amer. Math. Soc. **357** (2005), 1931–1952. MR2115083 (2005i:16066)

[10] R. Exel, *Circle actions on C^*-algebras, partial automorphisms and a generalized Pimsner-Voiculescu exact sequence*, J. Funct. Analysis **122** (1994), 361–401. MR1276163 (95g:46122)

[11] R. Exel, *Amenability for Fell bundles*, J. reine angew. Math. **492** (1997), 41–73. MR1488064 (99a:46131)

[12] R. Exel, *Partial actions of groups and actions of inverse semigroups*, Proc. Amer. Math. Soc. **126** (1998), 3481–3494. MR1469405 (99b:46102)

[13] R. Exel, M. Laca and J. Quigg, *Partial dynamical systems and C^*-algebras generated by partial isometries*, J. Operator Theory **47** (2002), 169–186. MR1905819 (2003f:46108)

[14] B. Julia, *Statistical theory of numbers*, Number Theory and Physics, Les Houches Winter School, J.-M. Luck, P. Moussa and M. Waldschmidt, eds., Springer-Verlag, 1990. MR1058473 (91h:11088)

[15] M. Laca, *Semigroups of $*$-endomorphisms, Dirichlet series and phase transitions*, J. Funct. Anal. **152** (1998), 330–378. MR1608003 (99f:46097)

[16] M. Laca and M. van Frankenhuijsen, *Phase transitions on Hecke C^*-algebras and class-field theory over $\mathbb{Q}$*, J. reine angew. Math. **595** (2006), 25–53. MR2244797 (2007e:11135)

[17] M. Laca, N. S. Larsen and S. Neshveyev, *On Bost-Connes type systems for number fields*, J. Number Theory **129** (2009), no. 2, 325–338. MR2473881 (2010f:11156)

[18] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal. **139** (1996), 415–440. MR1402771 (97h:46109)

[19] M. Laca and I. Raeburn, *A semigroup crossed product arising in number theory*, J. London Math. Soc. (2) **59** (1999), 330–344. MR1688505 (2000g:46097)

[20] M. Laca and I. Raeburn, *The ideal structure of the Hecke C^*-algebra of Bost and Connes*, Math. Ann. **318** (2000), 433–451. MR1800765 (2002a:46095)

[21] M. Laca and I. Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. **225** (2010), 643–688. MR2671177

[22] N. S. Larsen and X. Li, *Dilations of semigroup crossed products as crossed products of dilations*, arXiv:1009.5842v1, 2010, to appear in Proc. Amer. Math. Soc.
[23] X. Li, *Ring C^*-algebras*, Math. Ann. **348** (2010), no. 4, 859–898. MR2721644 (2012a:46100)
[24] A. Nica, *C^*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory **27** (1992), 17–52. MR1241114 (94m:46094)
[25] S. Yamashita, *Cuntz's $ax + b$-semigroup C^*-algebra over $\mathbb{N}$ and product system C^*-algebras*, J. Ramanujan Math. Soc. **24** (2009), 299–322. MR2568059 (2010i:46089)

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