HIGHER DIOPHANTINE APPROXIMATION EXPONENTS
AND CONTINUED FRACTION SYMMETRIES
FOR FUNCTION FIELDS II

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Abstract. We construct families of non-quadratic algebraic laurent series
(over finite fields of any characteristic) which have only bad rational approx-
imations so that their rational approximation exponent is as near to 2 as we
wish and, at the same time, have very good quadratic approximations so that
the quadratic exponent is close to the Liouville bound and thus can be ar-
bitrarily large. In contrast, in the number field case, the Schmidt exponent
(analog of the Roth exponent of 2 for rational approximation) for approx-
imations by quadratics is 3. We do this by exploiting the symmetries of the
relevant continued fractions. We then generalize some of the aspects from the
degree 2 (= \(p^0 + 1\))-approximation to degree \(p^n + 1\)-approximation. We also
calculate the rational approximation exponent of an analog of \(\pi\).

1. Background

We recall [S80, Chapter 8] some basic definitions, facts and conjectures about
diophantine approximation of real numbers by rationals or (real) algebraic numbers.
(See also [B04, BG06] and [W] for a nice survey of recent developments.)

Definition 1 (Height and higher diophantine approximation exponents). For a
non-zero algebraic number \(\beta\), define \(H(\beta)\) to be the maximum of the absolute values
of the coefficients of a non-trivial irreducible polynomial with co-prime integral
coefficients that it satisfies.

For \(\alpha\) an irrational real number not algebraic of degree \(\leq d\), define \(E_d(\alpha)\) (\(E_{\leq d}(\alpha)\)
respectively) as \(\limsup(-\log|\alpha - \beta|/\log H(\beta))\), where \(\beta\) varies through all algebraic
real numbers of degree \(d\) (\(\leq d\) respectively).

Note that \(E_1(\alpha)\) is the usual exponent \(E(\alpha) := \limsup(-\log|\alpha - P/Q|/\log|Q|)\).

Then for irrational \(\alpha\), we have \(E(\alpha) \geq 2\) by Dirichlet’s theorem, whereas for
irrational algebraic \(\alpha\) of degree \(d\), we have \(E(\alpha) \leq d\) by Liouville’s theorem and
\(E(\alpha) = 2\) by Roth’s theorem, improving Liouville, Thue, Siegel, and Dyson bounds.

For real \(\alpha\) not algebraic of degree \(\leq d\), Wirsing (generalizing Dirichlet’s result)
conjectured \(E_{\leq d}(\alpha) \geq d + 1\) and proved a complicated lower bound (for this ex-
ponent) which is slightly better than \((d + 3)/2\), whereas Davenport and Schmidt
proved his conjecture for \(d = 2\). On the other hand, for \(\alpha\) of degree \(> d\), we have
the Liouville bound $E_{\leq d}(\alpha) \leq \deg \alpha$. Schmidt (generalizing Roth’s result) proved that for real algebraic $\alpha$ of degree greater than $d$, $E_{\leq d}(\alpha) \leq d + 1$.

From now on, unless stated otherwise, we only focus on the function field analogs (see, e.g., [T04] for general background and [T04] Ch. 9, [T09] for diophantine approximation, continued fractions background and references), where the role of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is played by $A = F[t], K = F(t), K_\infty = F((1/t))$ respectively, where $F$ is a finite field of characteristic $p$. With the usual absolute value coming from the degree in $t$ of polynomials or rational functions, we have exactly the same definitions of heights and exponents. Now by rationals, reals, and algebraic, we mean elements of $K, K_\infty$, and algebraic over $K$ respectively. (There should be no confusion between degrees in $t$ (as above) of rationals and algebraic degree over $K$ of irrational algebraics.)

Then analogs of Dirichlet and Liouville theorems hold, but a naive analog of Roth’s theorem fails, as shown by Mahler [M49]. For other results, see [aM70] [S00] [T04] and the references therein, and, for example, results [KT00] in the Wirsing direction. We also recall that for almost all real $\alpha$ (and for almost all $\alpha \in K_\infty$), $E_d(\alpha) = d + 1$. (Here ‘almost all’ means that the set of exceptions is of measure zero.)

We now deal with the similar phenomena for quadratic approximations. In [T11], for $p = 2$ and any integer $m > 1$, we constructed algebraic elements $\alpha$ of degree at most $2^m$ having continued fractions with folding pattern symmetries and a bounded sequence of partial quotients, so that $E(\alpha) = 2$, but with $E_2(\alpha) \geq 2^m > 3$. In this paper, with a different construction, based on some ideas of [T99] [T11], we prove:

**Theorem 1.** Let $p$ be a prime, $q$ be a power of $p$ and $\epsilon > 0$ be given. Then we can construct infinitely many algebraic $\alpha$, with explicit equations and continued fractions, such that

$$q \leq \deg(\alpha) \leq q + 1, \quad E(\alpha) < 2 + \epsilon, \quad E_2(\alpha) > q - \epsilon,$$

with an explicit sequence of quadratic approximations realizing the last bound.

**Theorem 2.** Let $p$ be a prime, $q$ be a power of $p$ and $m, n > 1$ be given. Then we can construct infinitely many algebraic $\alpha_{m,n}$, with explicit equations and continued fractions, such that

$$\deg(\alpha_{m,n}) \leq q^m + 1, \quad \lim_{n \to \infty} E(\alpha_{m,n}) = 2, \quad \lim_{n \to \infty} E_{q+1}(\alpha_{m,n}) \geq q^{m-1} + \frac{q - 1}{(q + 1)q},$$

with an explicit sequence of degree $q + 1$-approximations realizing the last bound.

A natural question raised by these considerations is whether there are algebraic $\alpha$’s of each degree $d$, with $E(\alpha) = 2$ (or even with bounded partial quotients) and for which the Liouville bound for the lower degree approximations is attained, or whether some of these requirements need to be relaxed.

## 2. Continued fractions

Continued fractions are natural tools of the theory of diophantine approximation. See [aM70] [BS76] [S00] [T04] for the basics in the function field case.

Let us review some standard notation. We write $\alpha = a_0 + 1/(a_1 + 1/(a_2 + \cdots))$ in the short-form $[a_0, a_1, \cdots]$. We write $a_n = [a_n, a_{n+1}, \cdots]$ so that $\alpha = \alpha_0$. Let us
define $p_n$ and $q_n$ as usual in terms of the partial quotients $a_i$ so that $p_n/q_n$ is the
$n$-th convergent $[a_0, \cdots, a_n]$ to $\alpha$. Hence $\deg q_n = \sum_{i=1}^n \deg a_i$.

Following the basic analogies mentioned above, we use the absolute value coming
from the degree in $t$ to generate the continued fraction in the function field case, and
we use the ‘polynomial part’ in place of the ‘integral part’ of the ‘real’ number $\alpha \in
K_\infty$. In the function field case, for $i > 0$, $a_i$ can be any non-constant polynomial,
and so the degree of $q_i$ increases with $i$, but $a_i$ or $q_i$ need not be monic. As usual,
we have

\begin{equation}
(1) \quad p_nq_{n-1} - q_np_{n-1} = (-1)^{n-1}, \quad \alpha = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}},
\end{equation}

implying the usual basic approximation formula

\begin{equation}
(2) \quad \alpha - \frac{p_n}{q_n} = (-1)^n/(\langle \alpha_{n+1} + q_{n-1}/q_n \rangle q_n^2),
\end{equation}

which because of the non-archimedean nature of the absolute value, now implies

\begin{equation}
(3) \quad |\alpha - \frac{p_n}{q_n}| = 1/(|a_{n+1}|q_n^2).
\end{equation}

If we know the continued fraction for $\alpha$, the equation allows us to calculate the
exponent, using $\deg q_n = \sum_{i=1}^n \deg a_i$, as

\begin{equation}
(4) \quad E(\alpha) = 2 + \limsup \frac{\deg a_{n+1}}{\sum_{i=1}^n \deg a_i}.
\end{equation}

3. Proof of Theorem 1

First, we note that if $\gamma = [a_0, \cdots, a_m, b, \cdots]$ (with arbitrary entries after $b$),
then, by (1),

\begin{equation}
\alpha - \gamma = \frac{\alpha_{m+1}p_{m-1} + p_m - \gamma_{m+1}p_{m-1} + p_m}{(\alpha_{m+1}q_m + q_{m-1})} = \pm \frac{\alpha_{m+1} - \gamma_{m+1}}{(\cdots)(\cdots)}.\end{equation}

Hence the non-archimedean nature of the absolute values implies that

\begin{equation}
(5) \quad |\alpha - \gamma| = \frac{1}{|bq_{m+1}|} \quad \text{if } \deg(a_{m+1} - b) = \deg a_{m+1}.
\end{equation}

Next, we normalize the absolute value and the logarithm so that $\log |a|$ is a
degree of $a$ in $t$ and denote by $h := \log H$ the resulting logarithmic height and give
a simple bound on the height of a quadratic irrational $\theta$ in terms of the degrees
of the partial quotients of its continued fraction, which is eventually periodic by
analogy (see, e.g., [SU]) of Lagrange’s theorem. Thus consider $\theta = [b_0, \cdots, b_j, \mu]$,\nwhere $\mu = [X]$ is purely periodic continued fraction obtained by repeating period
(tuple) $X = (a_0, \cdots, a_n)$. Thus $\mu = (\mu p_n + p_n-1)/(\mu q_n + q_n-1)$ implies
$h(\mu) \leq \deg(p_n) = \sum \deg a_i$, where the sum runs over $i$ from 0 to $n$. For $a \in A$, comparing
the polynomials satisfied by quadratic $\gamma$ and $a + 1/\gamma$, we see that $h(a + 1/\gamma) \leq
h(\gamma) + 2\deg(a)$, whose repeated application gives

\begin{equation}
(6) \quad h(\theta) \leq 2 \sum_{i=0}^j \deg b_i + \sum_{i=0}^n \deg a_i.
\end{equation}

Next, we proceed to the construction of the $\alpha$’s and their quadratic approxima-
tions $\beta_i$. By capital letters $X$, $Y$, etc., we will denote tuples of partial quotients,
and we will denote by $X^m$ the tuple resulting from $X$ by raising each of its entries
to the $m$-th power. Let $n + 1 = r \ell$, let $a_0, \ldots, a_{\ell - 1} \in A$ be non-constant polynomials, and let $Y = (a_0, \ldots, a_{\ell - 1})$ and $X = (Y, \ldots, Y) = (a_0, \ldots, a_n)$, obtained by repeating $Y$ $r$ times. Then

$$\alpha := [X, X^q, X^{q^2}, \cdots] = [X, \alpha^q] = \frac{\alpha^q b_n + p_{n-1}}{\alpha^q q_n + q_{n-1}}$$

so that $\alpha$ is algebraic of degree at most $q + 1$. For any $s > 1$, let

$$\beta_s := [X, X^q, \cdots, X^{q^{s-1}}, \sqrt[q^s]{\alpha^{s}}].$$

Let $L = \sum_{i=0}^{\ell-1} \deg a_i$ so that $\sum_{i=0}^{n} \deg a_i = L r$. We see using (6) that $h(\beta_s) \leq 2Lr(q^s - 1)/(q - 1) + q^s L$, while (5) implies (since $b = a_0^q$) that $-\log|\alpha - \beta_s| = q^s \deg a_0 + 2(Lr(q^{s+1} - 1)/(q - 1) - \deg a_0)$. Hence, letting $s$ tend to infinity, we see that $E_2(\alpha) \geq (2Lr/q(q - 1) \deg a_0)/(2Lr + L(q - 1))$ (when is this an inequality?), which is at most $q$ and tends to $q$ if $r$ tends to infinity. On the other hand, by (4), for some $i$, $E(\alpha) - 2 \leq q^{s+1} \deg(a_i)/(r q^m \deg(a_i))$, which tends to 0, as $r$ tends to infinity. Hence, given $\epsilon > 0$, choosing $r$ appropriately large, we satisfy the claims of the theorem, with the Liouville bound implying that $\deg(\alpha) \geq q$, since without loss of generality we can assume that $\epsilon < 1$. This completes the proof.

Finally, we remark that if our equation for $\alpha$ is reducible, then we reach the Liouville bound (of $q$ in that case), at least as $\epsilon$ tends to zero. It might also be possible to tighten the inequalities to get a better lower bound for $E_2$.

4. Proof of Theorem 2

We follow the strategy of the previous section, but now we define

$$\alpha := \alpha_{m,n} := [X, X^{q^m}, X^{q^{2m}}, \cdots, X^{q^{(s-1)m}}, X^{q^m}, \cdots]$$

with $X = (B, B^q, \cdots, B^{q^{n-1}}, C)$, where $B, C \in \mathbb{F}[t], \text{with } b := \deg B > 0, \deg C = q^n b$, and $\deg(B^{q^n} - C) = q^n b$, which is clearly possible if $\mathbb{F} \neq \mathbb{F}_2$. (We will deal with the case $\mathbb{F} = \mathbb{F}_2$ at the end.) Next we let

$$\beta_s := [X, X^{q^m}, X^{q^{2m}}, \cdots, X^{q^{(s-1)m}}, B^{q^m}, B^{q^{m+1}}, B^{q^{m+2}}, \cdots].$$

The mobius transformation expression for $\alpha$ and $\beta_s$, as in the proof of the last theorem, shows that $\deg \alpha \leq q^m + 1$ and that $\deg(\beta_s) \leq q + 1$. The last inequality is in fact equality by the Liouville theorem by calculating its exponent by (4).

Write

$$D := b q^{n+1} - 1 \frac{q^{sm} - 1}{q - 1}, \quad F := q^{m b} q^{n+1} - 1 \frac{q^{sm} - 1}{q - 1},$$

which are the sums of the degrees of entries in $(X, X^{q^m}, \cdots, X^{q^{(s-1)m}})$ and $X^{q^m}$, respectively. Then a straight calculation using (5) shows that

$$-\log|\alpha - \beta_s| = 2(D - b + F) - q^{ms + n} b.$$

Now the height of $\theta := [B^{q^m}, B^{q^{m+1}}, \cdots] = B^{q^m} + 1/\theta^q$ is clearly $q^{sm} b$. For $a \in A$, by comparing heights of $\gamma$ and $\alpha + 1/\gamma$ for a $\gamma$ satisfying an equation of the form $\gamma = (P \gamma^q + Q)/(R \gamma^q + S)$, we see that $h(\alpha + 1/\gamma) \leq h(\gamma) + (q + 1) \deg(a)$, whose repeated application gives

$$h(\beta_s) \leq (q + 1) D + q^{ms} b.$$
Taking the limit of the ratio as \( s \) tends to infinity, we get
\[
E_{q+1}(\alpha) \geq (q^{n+1} - 1)\left(\frac{q^m}{q^m - 1}\right) - q^n)/((q + 1)(\frac{q^{n+1} - 1}{q - 1}) - \frac{1}{q^m - 1}) + 1),
\]
and taking the limit of the right side, as \( n \) tends to infinity, we get the lower bound claimed in the theorem.

Finally, using (4), we see that
\[
E(\alpha) = 2 + \lim_{s \to \infty} \frac{q^{m+b}}{D - b} = 2 + \frac{(q - 1)(q^m - 1)}{q^{n+1} - 1},
\]
implying the claim on the limit as \( n \) tends to infinity.

Finally, we consider the case \( F = F_2 \). We can choose \( B, C \in F[t] \), with \( \deg B = b \), \( \deg(C) = q^nb \) and \( \deg(B^{q^n} - C) = q^nb - 1 \). The whole asymptotic analysis is the same, as \( b \) tends to infinity, leading to the same bounds. This finishes the proof.

5. Exponent for an analog of \( \pi \)

In \( \text{T11, Sec. 7} \), we calculated the exponent of an analog of \( \pi \) for \( F_q[t] \). (For this section, we take \( F = F_q \).) But as discussed in \( \text{T04, pp. 47-48} \), there are a couple of good candidates for analogs of \( \pi \) (up to rational multiples, which do not change exponents). We now consider \( \pi_1 := \prod (1 - [j]/[j + 1]) \in K_\infty \), where \([j] := t^{q^j} - t\) and the product is over \( j \) from 1 to \( \infty \). As explained in the reference above, the Carlitz period (good analog of \( 2\pi i \)) is then \([-1]^{1/(q-1)}\pi_1 \).

Theorem 3. For \( \pi_1 \) as above, \( E(\pi_1) \geq (q - 1)^2/q \), with equality when \( q > 5 \).

Proof. Note that \([j + 1] - [j] = [1]q^j \) and \([1] \) divides \([n]\) with the quotient co-prime to \([1]\). Thus the truncation of the product at \( j = N - 1 \), which equals \([1]q^2 + \cdots + q^{N-1}/([2][3] \cdots [N]) \), has a denominator of degree \( q^2 + q^3 + \cdots + q^N - (N - 1)q \), whereas it approximates \( \pi_1 \) with error of degree \( q^N - q^{N+1} \) (resulting from \( 1 - (1 - [N]/[N + 1]) \)), showing that the inequality claimed, as the limit of the ratio of these two quantities as \( N \) tends to \( \infty \), tends to \((q - 1)^2/q \). When \( q > 5 \), we have \((q - 1)^2/q > \sqrt{q} + 1 \), and this implies by a proposition of Voloch (see \[V88, Prop. 5\] or \[T04, Lemma 9.3.3\]) the equality claimed.

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References


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