TENSOR PRODUCTS OF LEAVITT PATH ALGEBRAS

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ABSTRACT. We compute the Hochschild homology of Leavitt path algebras over a field $k$. As an application, we show that $L_2$ and $L_2 \otimes L_2$ have different Hochschild homologies, and so they are not Morita equivalent; in particular, they are not isomorphic. Similarly, $L_\infty$ and $L_\infty \otimes L_\infty$ are distinguished by their Hochschild homologies, and so they are not Morita equivalent either. By contrast, we show that $K$-theory cannot distinguish these algebras; we have $K_\ast (L_2) = K_\ast (L_2 \otimes L_2) = 0$ and $K_\ast (L_\infty) = K_\ast (L_\infty \otimes L_\infty) = K_\ast (k)$.

1. INTRODUCTION

Elliott’s theorem [21] states that $O_2 \otimes O_2 \cong O_2$ plays an important role in the proof of the celebrated classification theorem of Kirchberg algebras in the UCT class, due to Kirchberg [13] and Phillips [19]. Recall that a Kirchberg algebra is a purely infinite, simple, nuclear and separable C*-algebra. The Kirchberg-Phillips theorem states that this class of simple C*-algebras is completely classified by its topological $K$-theory. The analogous question whether the algebras $L_2$ and $L_2 \otimes L_2$ are isomorphic has remained open for some time. Here $L_2$ is the Leavitt algebra of type $(1, 2)$ over a field $k$ (see [17]), that is, the $k$-algebra with generators $x_1, x_2, x_1^\ast, x_2^\ast$ and relations given by $x_i^\ast x_j = \delta_{i,j}$ and $\sum_{i=1}^2 x_i x_i^\ast = 1$.

In this paper we obtain a negative answer to this question. Indeed, we analyze a much larger class of algebras, namely the tensor products of Leavitt path algebras of finite quivers in terms of their Hochschild homology, and we prove that, for $1 \leq n < m \leq \infty$, the tensor products $E = \bigotimes_{i=1}^n L(E_i)$ and $F = \bigotimes_{j=1}^m L(F_j)$ of Leavitt path algebras of non-acyclic finite quivers $E_i, F_j$ are distinguished by their Hochschild homologies (Theorem 5.1). Because Hochschild homology is Morita invariant, we conclude that $E$ and $F$ are not Morita equivalent for $n < m$. Since $L_2$ is the Leavitt path algebra of the graph with one vertex and two arrows, we obtain that $L_2 \otimes L_2$ and $L_2$ are not Morita equivalent; in particular, they are not isomorphic.

Recall that, by a theorem of Kirchberg [15], a simple, nuclear and separable C*-algebra $A$ is purely infinite if and only if $A \otimes O_\infty \cong A$. We also show that the analogue of Kirchberg’s result is not true for Leavitt algebras. We prove in
Proposition 5.3 that if \( E \) is a non-acyclic quiver, then \( L_{\infty} \otimes L(E) \) and \( L(E) \) are not Morita equivalent, and also that \( L_{\infty} \otimes L_{\infty} \) and \( L_{\infty} \) are not Morita equivalent.

Using the results in [5] we prove that the algebras \( L_{2} \) and \( L_{2} \otimes L(F) \), for \( F \) an arbitrary finite quiver, have trivial \( K \)-theory: all algebraic \( K \)-theory groups \( K_{i} \), \( i \in \mathbb{Z} \), vanish on them (this follows from Lemma 6.1 and Proposition 6.2). We also compute \( K_{*}(L(F)) = K_{*}(L_{\infty} \otimes L(F)) \) and that \( K_{*}(L_{\infty}) = K_{*}(L_{\infty} \otimes L_{\infty}) = K_{*}(k) \) is the \( K \)-theory of the ground field (see Proposition 6.3 and Corollary 6.4). This implies in particular that, in contrast with the analytic situation, no classification result, in terms solely of \( K \)-theory, can be expected for a class of central, simple \( k \)-algebras, containing all purely infinite simple unital Leavitt path algebras and closed under tensor products. It is worth mentioning that an important step towards a \( K \)-theoretic classification of purely infinite simple Leavitt path algebras of finite quivers has been achieved in [2].

We refer the reader to [3], [7] and [20] for the basics on Leavitt algebras, Leavitt path algebras and graph \( C^{*} \)-algebras, and to [22] for a nice survey on the Kirchberg-Phillips Theorem.

**Notation.** We fix a field \( k \); all vector spaces, tensor products and algebras are over \( k \). If \( R \) and \( S \) are unital \( k \)-algebras, then by an \((R,S)\)-bimodule we understand a left module over \( R \otimes S^{op} \). By an \( R \)-bimodule we shall mean an \((R,R)\) bimodule, that is, a left module over the enveloping algebra \( R^{e} = R \otimes R^{op} \). Hochschild homology of \( k \)-algebras is always taken over \( k \). If \( M \) is an \( R \)-bimodule, we write

\[
HH_{n}(R, M) = \text{Tor}_{n}^{R^{e}}(R, M)
\]

for the Hochschild homology of \( R \) with coefficients in \( M \) and we abbreviate \( HH_{n}(R) = HH_{n}(R,R) \).

### 2. Hochschild homology

Let \( k \) be a field, \( R \) a \( k \)-algebra and \( M \) an \( R \)-bimodule. The Hochschild homology \( HH_{*}(R, M) \) of \( R \) with coefficients in \( M \) was defined in the introduction. It is computed by the Hochschild complex \( HH(R, M) \), which is given in degree \( n \) by

\[
HH(R, M)_{n} = M \otimes R^{\otimes n}.
\]

It is equipped with the Hochschild boundary map \( b \) defined by

\[
b(a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n}) = \sum_{i=0}^{n-1} (-1)^{i} a_{0} \otimes \cdots \otimes a_{i} a_{i+1} \otimes \cdots \otimes a_{n} + (-1)^{n} a_{n} a_{0} \otimes \cdots \otimes a_{n-1}.
\]

If \( R \) and \( M \) happen to be \( \mathbb{Z} \)-graded, then \( HH(R, M) \) splits into a direct sum of subcomplexes

\[
HH(R, M) = \bigoplus_{m \in \mathbb{Z}} mHH(R, M).
\]

The homogeneous component of degree \( m \) of \( HH(R, M)_{n} \) is the linear subspace of \( HH(R, M)_{n} \) generated by all elementary tensors \( a_{0} \otimes \cdots \otimes a_{n} \) with \( a_{i} \) homogeneous and \( \sum_{i} |a_{i}| = m \). One of the first basic properties of the Hochschild complex is that it commutes with filtering colimits. Thus we have

**Lemma 2.1.** Let \( I \) be a filtered ordered set and let \( \{(R_{i}, M_{i}) : i \in I\} \) be a directed system of pairs \((R_{i}, M_{i})\) consisting of an algebra \( R_{i} \) and an \( R_{i} \)-bimodule \( M_{i} \), with algebra maps \( R_{i} \to R_{j} \) and \( R_{i} \)-bimodule maps \( M_{i} \to M_{j} \) for each \( i \leq j \). Let \((R, M) = \text{colim}_{i}(R_{i}, M_{i})\). Then \( HH_{n}(R, M) = \text{colim}_{i} HH_{n}(R_{i}, M_{i}) \) \((n \geq 0)\).
Let $R_i$ be a $k$-algebra and $M_i$ an $R_i$-bimodule ($i = 1, 2$). The Künneth formula establishes a natural isomorphism \([23 9.4.1]\)

$$HH_n(R_1 \otimes R_2, M_1 \otimes M_2) \cong \bigoplus_{p=0}^{n} HH_p(R_1, M_1) \otimes HH_{n-p}(R_2, M_2).$$

Another fundamental fact about Hochschild homology which we shall need is Morita invariance. Let $R$ and $S$ be Morita equivalent algebras, and let $P \in R \otimes S^{op}$-mod and $Q \in S \otimes R^{op}$-mod implement the Morita equivalence. Then \([23 \text{ Thm. 9.5.6}]\)

\[(2.2)\]

$$HH_n(R, M) = HH_n(S, Q \otimes_R M \otimes_R P).$$

**Lemma 2.3.** Let $R_1, \ldots, R_n$ and $S_1, \ldots, S_m, \ldots$ be a finite and an infinite sequence of algebras, and let $R = \bigotimes_{i=1}^{n} R_i$, $S_{\leq m} = \bigotimes_{j=1}^{m} S_j$ and $S = \bigotimes_{j=1}^{\infty} S_j$. Assume:

1. $HH_q(R_i) \neq 0 \neq HH_q(S_j)$ $(q = 0, 1)$, $(1 \leq i \leq n)$, $(1 \leq j)$.
2. $HH_p(R_i) = HH_p(S_j) = 0$ for $p > 2$, $1 \leq i \leq n$, $1 \leq j$.
3. $n \neq m$.

Then no two of $R, S_{\leq m}$ and $S$ are Morita equivalent.

**Proof.** By the Künneth formula, we have

$$HH_n(R) = \bigotimes_{i=1}^{n} HH_1(R_i) \neq 0, \quad HH_p(R) = 0, \quad p > n.$$ 

By the same argument, $HH_p(S_{\leq m})$ is non-zero for $p = m$ and zero for $p > m$. Hence if $n \neq m$, $R$ and $S_{\leq m}$ do not have the same Hochschild homology, and therefore they cannot be Morita equivalent, by \([22]\). Similarly, by Lemma \([21]\) we have

$$HH_n(S) = \bigoplus_{J \subset [n], |J| = m} \bigotimes_{j \in J} HH_1(S_j) \otimes \bigotimes_{j \not\in J} HH_0(S_j)$$

so that $HH_n(S)$ is non-zero for all $n \geq 1$, and thus it cannot be Morita equivalent to either $R$ or $S_{\leq m}$. \(\square\)

3. **Hochschild homology of crossed products**

Let $R$ be a unital algebra and $G$ a group acting on $R$ by algebra automorphisms. Form the crossed-product algebra $S = R \rtimes G$, and consider the Hochschild complex $HH(S)$. For each conjugacy class $\xi$ of $G$, the graded submodule $HH^\xi(S) \subset HH(S)$ generated in degree $n$ by the elementary tensors $a_0 \otimes g_0 \otimes \cdots \otimes a_n \otimes g_n$ with $g_0 \cdots g_n \in \xi$ is a subcomplex, and we have a direct sum decomposition $HH(S) = \bigoplus \xi HH^\xi(S)$. The following theorem of Lorenz describes the complex $HH^\xi(S)$ corresponding to the conjugacy class $\xi = [g]$ of an element $g \in G$ as hyperhomology over the centralizer subgroup $Z_g \subset G$.

**Theorem 3.1 \([16]\).** Let $R$ be a unital $k$-algebra, $G$ a group acting on $R$ by automorphisms, $g \in G$ and $Z_g \subset G$ the centralizer subgroup. Let $S = R \rtimes G$ be the crossed product algebra and $HH^{(g)}(S) \subset HH(S)$ be the subcomplex described above. Consider the $S$-submodule $S_g = R \rtimes g \subset S$. Then there is a quasi-isomorphism

$$HH^{(g)}(S) \xrightarrow{\cong} HH(Z_g, HH(R, S_g)).$$

In particular, we have a spectral sequence

$$E^2_{p,q} = HH_p(Z_g, HH_q(R, S_g)) \Rightarrow HH^{|g|}_{p+q}(S).$$
Remark 3.2. Lorenz formulates his result in terms of the spectral sequence alone, but his proof shows that there is a quasi-isomorphism as stated above. An explicit formula is given for example in the proof of [11, Lemma 7.2].

Let $A$ be a not necessarily unital $k$-algebra and write $\tilde{A}$ for its unitalization. Recall from [24] that $A$ is called $H$-unital if the groups $\text{Tor}^A_{\mathbb{Z}}(k, A)$ vanish for all $n \geq 0$. Wodzicki proved in [24] that $A$ is $H$-unital if and only if for every embedding $A \triangleleft R$ of $A$ as a two-sided ideal of a unital ring $R$, the map

$$HH(A) \to HH(R : A) = \ker(HH(R) \to HH(R/A))$$

is a quasi-isomorphism.

Lemma 3.3. Theorem 3.1 still holds if the condition that $R$ be unital is replaced by the condition that it be $H$-unital.

Proof. Follows from Theorem 3.1 and the fact, proved in [11, Prop. A.6.5], that $R \rtimes G$ is $H$-unital if $R$ is as well.

Let $R$ be a unital algebra and $\phi : R \to pRp$ a corner isomorphism. As in [6], we consider the skew Laurent polynomial algebra $R[t_+, t_-, \phi]$. This is the $R$-algebra generated by elements $t_+$ and $t_-$ subject to the following relations:

$$t_+ a = \phi(a)t_+$$
$$a t_- = t_- \phi(a)$$
$$t_- t_+ = 1$$
$$t_+ t_- = p$$

Observe that the algebra $S = R[t_+, t_-, \phi]$ is $\mathbb{Z}$-graded by $\deg(r) = 0$, $\deg(t_+) = 1$, $\deg(t_-) = -1$. The homogeneous component of degree $n$ is given by

$$R[t_+, t_-, \phi]_n = \begin{cases} t_-^n R & n < 0 \\ R & n = 0 \\ R t_+^n & n > 0 \end{cases}$$

Proposition 3.4. Let $R$ be a unital ring, $\phi : R \to pRp$ a corner isomorphism, and $S = R[t_+, t_-, \phi]$. Consider the weight decomposition $HH(S) = \bigoplus_{m \in \mathbb{Z}} mHH(S)$. There is a quasi-isomorphism

$$(3.5) \quad mHH(S) \sim \text{Cone}(1 - \phi : HH(R, S_m) \to HH(R, S_m)).$$

Proof. If $\phi$ is an automorphism, then $S = R \rtimes \mathbb{Z}$, the right hand side of (3.5) computes $H(Z, HH(R, S_m))$, and the proposition becomes the particular case $G = Z$ of Theorem 3.1. In the general case, let $A$ be the colimit of the inductive system

$$R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \ldots .$$

Note that $\phi$ induces an automorphism $\tilde{\phi} : A \to A$. Now $A$ is $H$-unital, since it is a filtering colimit of unital algebras, and thus the assertion of the proposition is true for the pair $(A, \tilde{\phi})$, by Lemma 3.3. Hence it suffices to show that for $B = A \rtimes \tilde{\phi} \mathbb{Z}$ the maps $HH(S) \to HH(B)$ and $\text{Cone}(1 - \phi : HH(R, S_m) \to HH(R, S_m))$ induce quasi-isomorphisms. The analogous property for $K$-theory is shown in the course of the third step of the proof of [5, Thm. 3.6]. Since the proof in [5] uses only that $K$-theory commutes with filtering colimits and is matrix invariant on those rings for which it satisfies excision, it applies verbatim to Hochschild homology. This concludes the proof. □
4. Hochschild homology of the Leavitt path algebra

Let \( E = (E_0, E_1, r, s) \) be a finite quiver and let \( \hat{E} = (E_0, E_1 \sqcup E_1^*, r, s) \) be the double of \( E \), which is the quiver obtained from \( E \) by adding an arrow \( \alpha^* \) for each arrow \( \alpha \in E_1 \), going in the opposite direction. The Leavitt path algebra of \( E \) is the algebra \( L(E) \) with one generator for each arrow \( \alpha \in E_1 \) and one generator \( p_i \) for each vertex \( i \in E_0 \), subject to the following relations:

\[
p_{i}p_{j} = \delta_{i,j}p_{i} \quad (i, j \in E_0)
\]

\[
p_{s(\alpha)}\alpha = \alpha = \alpha p_{r(\alpha)} \quad (\alpha \in \hat{E}_1)
\]

\[
\alpha^*\beta = \delta_{\alpha, \beta}p_{r(\alpha)} \quad (\alpha, \beta \in E_1)
\]

\[
p_{i} = \sum_{\alpha \in E_1, s(\alpha) = i} \alpha \alpha^* \quad (i \in E_0 \setminus \text{Sink}(E))
\]

The algebra \( L = L(E) \) is equipped with a \( \mathbb{Z} \)-grading. The grading is determined by \( |\alpha| = 1 \), \( |\alpha^*| = -1 \), for \( \alpha \in \hat{E}_1 \). Let \( L_{0,n} \) be the linear span of all elements of the form \( \gamma\nu^* \), where \( \gamma \) and \( \nu \) are paths with \( r(\gamma) = r(\nu) \) and \( |\gamma| = |\nu| = n \). By [7, proof of Theorem 5.3], we have \( L_{0} = \bigcup_{n=0}^{\infty} L_{0,n} \). For each \( i \in E_0 \) and each \( n \in \mathbb{Z}^+ \), let us denote by \( P(n, i) \) the set of paths \( \gamma \) in \( E \) such that \( |\gamma| = n \) and \( r(\gamma) = i \). The algebra \( L_{0,0} \) is isomorphic to \( \prod_{i \in E_0} k \). In general, the algebra \( L_{0,n} \) is isomorphic to

\[
(4.1) \quad \left[ \prod_{m=0}^{n-1} \left( \prod_{i \in \text{Sink}(E)} M_{|P(m,i)|}(k) \right) \right] \times \left[ \prod_{i \in E_0} M_{|P(n,i)|}(k) \right].
\]

The transition homomorphism \( L_{0,n} \rightarrow L_{0,n+1} \) is the identity on the factors

\[
\prod_{i \in \text{Sink}(E)} M_{|P(m,i)|}(k),
\]

for \( 0 \leq m \leq n - 1 \), and also on the factor

\[
\prod_{i \in \text{Sink}(E)} M_{|P(n,i)|}(k)
\]

of the last term of the displayed formula. The transition homomorphism

\[
\prod_{i \in E_0 \setminus \text{Sink}(E)} M_{|P(n,i)|}(k) \rightarrow \prod_{i \in E_0} M_{|P(n+1,i)|}(k)
\]

is a block diagonal map induced by the following identification in \( L(E)_0 \): A matrix unit in a factor \( M_{|P(n,i)|}(k) \), where \( i \in E_0 \setminus \text{Sink}(E) \), is a monomial of the form \( \gamma\nu^* \), where \( \gamma \) and \( \nu \) are paths of length \( n \) with \( r(\gamma) = r(\nu) = i \). Since \( i \) is not a sink, we can enlarge the paths \( \gamma \) and \( \nu \) using the edges that \( i \) emits, obtaining paths of length \( n + 1 \), and the last relation in the definition of \( L(E) \) gives

\[
\gamma\nu^* = \sum_{\{\alpha \in \hat{E}_1 \mid s(\alpha) = i\}} (\gamma\alpha)(\nu\alpha)^*.
\]

Assume \( E \) has no sources. For each \( i \in E_0 \), choose an arrow \( \alpha_i \) such that \( r(\alpha_i) = i \). Consider the elements

\[
t_+ = \sum_{i \in E_0} \alpha_i, \quad t_- = t_+^*.
\]
defines a surjective linear map
\[ \phi : L \to L, \quad \phi(x) = t_+ x t_- \]
is homogeneous of degree 0 with respect to the \( \mathbb{Z} \)-grading. In particular, it restricts to an endomorphism of \( L_0 \). By [6, Lemma 2.4], we have
\[ L = L_0[t_+, t_-, \phi]. \]

Consider the matrix \( N_E' = [n_{i,j}] \in M_{e_0 \mathbb{Z}} \) given by
\[ n_{i,j} = \#\{\alpha \in E_1 : s(\alpha) = i, \ r(\alpha) = j\}. \]
Let \( e_0' = |\text{Sink}(E)| \). We assume that \( E_0 \) is ordered so that the first \( e_0' \) elements of \( E_0 \) correspond to its sinks. Accordingly, the first \( e_0' \) rows of the matrix \( N_E' \) are 0. Let \( N_E \) be the matrix obtained by deleting these \( e_0' \) rows. The matrix that enters the computation of the Hochschild homology of the Leavitt path algebra is
\[ \begin{pmatrix} 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} - N_E', \ Z^{e_0-e_0'} \to Z^{e_0}. \]

By a slight abuse of notation, we will write \( 1 - N_E' \) for this matrix. Note that \( 1 - N_E' \in M_{e_0 \times (e_0-e_0')}(\mathbb{Z}) \). Of course, \( N_E = N_E' \) in case \( E \) has no sinks.

**Theorem 4.4.** Let \( E \) be a finite quiver without sources, and let \( N = N_E \). For each \( i \in E_0 \setminus \text{Sink}(E) \) and \( m \geq 1 \), let \( V_{i,m} \) be the vector space generated by all closed paths \( c \) of length \( m \) with \( s(c) = r(c) = i \). Let \( \mathbb{Z}(\sigma) \) act on
\[ V_m = \bigoplus_{i \in E_0 \setminus \text{Sink}(E)} V_{i,m} \]
by rotation of closed paths. We have
\[ mHH_n(L(E)) = \begin{cases} \text{coker}(1 - \sigma : V_{|m|} \to V_{|m|}) & n = 0, m \neq 0 \\
\text{coker}(1 - N') & n = m = 0 \\
\text{ker}(1 - \sigma : V_{|m|} \to V_{|m|}) & n = 1, m \neq 0 \\
\text{ker}(1 - N') & n = 1, m = 0 \\
0 & n \notin \{0, 1\} \end{cases} \]

**Proof.** Let \( L = L(E) \), \( P = P(E) \subset L \) be the path algebras of \( E \) and \( W_m \subset P \) be the subspace generated by all paths of length \( m \). For each fixed \( n \geq 1 \) and \( m \in \mathbb{Z} \), consider the following \( L_{0,n} \)-bimodule:
\[ L_{m,n} = \begin{cases} L_{0,n} W_m L_{0,n} & m > 0 \\
L_{0,n} W_m^* L_{0,n} & m < 0 \end{cases} \]
Write \( L = L(E) \), and let \( mL \) be the homogeneous part of degree \( m \); we have
\[ mL = \bigcup_{n \geq 1} L_{m,n}. \]
If \( m \) is positive, then there is a basis of \( L_{m,n} \) consisting of the products \( \alpha \theta \beta^* \) where each of \( \alpha, \beta \) and \( \theta \) is a path in \( E \), \( r(\alpha) = s(\theta) \), \( r(\beta) = r(\theta) \), \( |\alpha| = |\beta| = n \) and \( |\theta| = m \). Hence the formula
\[ \pi(\alpha \theta \beta^*) = \begin{cases} \theta & \text{if } \alpha = \beta \\
0 & \text{else} \end{cases} \]
defines a surjective linear map \( L_{m,n} \to V_m \). One checks that \( \pi \) induces an isomorphism
\[ HH_0(L_{0,n}, L_{m,n}) \cong V_m \quad (m > 0). \]
Similarly, if \( m < 0 \), then
\[
HH_0(L_{0,n}, L_{m,n}) = V_{|m|}^* \cong V_{-m}.
\]
Next, by (4.1), we have
\[
HH_0(L_{0,n}) = k[E \setminus \text{Sink}(E)] \oplus \bigoplus_{i \in \text{Sink}(E)} k^{r(i,n)}.
\]
Here
\[
r(i,n) = \max\{r \leq n : P(r,i) \neq \emptyset\}.
\]
Now note that because \( L_{0,n} \) is a product of matrix algebras, it is separable, and thus \( HH_1(L_{0,n}, M) = 0 \) for any bimodule \( M \). As observed in (4.3), for the automorphism \( \alpha \), we have \( L = L_0[t_+, t_-, \phi] \). Hence in view of Proposition 8.4 and Lemma 2.3, it only remains to identify the maps \( HH_0(L_{0,n}, L_{m,n}) \to HH_0(L_{0,n+1}, L_{m,n+1}) \) induced by inclusion and by the homomorphism \( \phi \). One checks that for \( m \neq 0 \), these are respectively the cyclic permutation and the identity \( V_{|m|} \to V_{|m|} \). The case \( m = 0 \) is dealt with in the same way as in [5, Proof of Theorem 5.10].

**Corollary 4.5.** Let \( E \) be a finite quiver with at least one non-trivial closed path.

1. \( HH_n(L(E)) = 0 \) for \( n \notin \{0,1\} \).
2. \( mHH_0(L(E)) \cong -mHH_1(L(E)) \) (\( m \in \mathbb{Z} \)).
3. There exist \( m > 0 \) such that \( mHH_0(L(E)) \) and \( mHH_1(L(E)) \) are both non-zero.

**Proof.** We first reduce to the case where the graph does not have sources. By the proof of [3, Theorem 6.3], there is a finite complete subgraph \( F \) of \( E \) such that \( F \) has no sources, \( F \) contains all the non-trivial closed paths of \( E \), and \( L(F) \) is a full corner in \( L(E) \) with respect to the homogeneous idempotent \( \sum_{v \in E_0} p_v \). It follows that \( HH_0(L(E)) \) and \( HH_1(L(F)) \) are graded-isomorphic. Therefore we can assume that \( E \) has no sources.

The first two assertions are already part of Theorem 4.4. For the last assertion, let \( \alpha \) be a primitive closed path in \( E \), and let \( m = |\alpha| \). Let \( \sigma \) be the cyclic permutation; then \( \{\sigma^i \alpha : i = 0, \ldots, m-1\} \) is a linearly independent set. Hence \( N(\alpha) = \sum_{i=0}^{m-1} \sigma^i \alpha \) is a non-zero element of \( V_{m}^\sigma = mHH_1(L(E)) \). Since on the other hand \( N \) vanishes on the image of \( 1-\sigma : V_m \to V_m \), it also follows that the class of \( \alpha \) in \( mHH_0(L(E)) \) is non-zero.

\[
5. \textbf{Applications}
\]

**Theorem 5.1.** Let \( E_1, \ldots, E_n \) and \( F_1, \ldots, F_m \) be finite quivers. Assume that \( n \neq m \) and that each of the \( E_i \) and the \( F_j \) has at least one non-trivial closed path. Then the algebras \( L(E_1) \otimes \cdots \otimes L(E_n) \) and \( L(F_1) \otimes \cdots \otimes L(F_m) \) are not Morita equivalent.

**Proof.** Immediate from Lemma 2.3 and Corollary 4.5 (iii).

**Example 5.2.** It follows from Theorem 5.1 that \( L_2 \) and \( L_2 \otimes_k L_2 \) are not Morita equivalent. There is another way of proving this, due to Jason Bell and George Bergman [8]. By Theorem 3.3 of [9], \( \text{gl.dim} L_2 \leq 1 \). Using a module-theoretic construction, Bell and Bergman show that \( \text{gl.dim}(L_2 \otimes_k L_2) \geq 2 \), which forces \( L_2 \) and \( L_2 \otimes_k L_2 \) to not be Morita equivalent. Bergman then asked Warren Dicks whether general results were known about global dimensions of tensor products and was pointed to Proposition 10(2) of [12], which is an immediate consequence of
Theorem XI.3.1 of [10] and says that if $k$ is a field and $R$ and $S$ are $k$-algebras, then $\text{l.gl.dim } R + \text{w.gl.dim } S \leq \text{l.gl.dim}(R \otimes_k S)$. Consequently, if $\text{l.gl.dim } R < \infty$ and $\text{w.gl.dim } S > 0$, then $\text{l.gl.dim } R < \text{l.gl.dim}(R \otimes_k S)$; in particular, $R$ and $R \otimes_k S$ are then not Morita equivalent. To see that $\text{w.gl.dim } L_2 > 0$, write $x_1, x_2, x_1^*, x_2^*$ for the usual generators of $L_2$ and use normal-form arguments to show that $\{a \in L_2 \mid ax_1 = a + 1\} = \emptyset$ and $\{b \in L_2 \mid x_1b = b\} = \{0\}$. Hence, in $L_2$, $x_1 - 1$ does not have a left inverse and is not a left zerodivisor (or see [4]); thus, $\text{w.gl.dim } L_2 > 0$.

We denote by $L_\infty$ the unital algebra presented by generators $x_1, x_1^*, x_2, x_2^*, \ldots$ and relations $x_i^*x_j = \delta_{i,j}1$.

**Proposition 5.3.** Let $E$ be any finite quiver having at least one non-trivial closed path. Then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent. Similarly, $L_\infty \otimes L_\infty$ and $L_\infty$ are not Morita equivalent.

**Proof.** Let $C_n$ be the algebra presented by generators $x_1, x_1^*, \ldots, x_n, x_n^*$ and relations $x_i^*x_j = \delta_{i,j}1$, for $1 \leq i, j \leq n$. Then

$$L_\infty = \lim_{\longrightarrow} C_n$$

and $C_n \cong L(E_n)$, where $E_n$ is the graph having two vertices $v, w$ and $2n$ arrows $e_1, \ldots, e_n, f_1, \ldots, f_n$, with $s(e_i) = r(e_i) = v = s(f_i)$ and $r(f_i) = w$ for $1 \leq i \leq n$. (The isomorphism $C_n \to L(E_n)$ is obtained by sending $x_i$ to $e_i$ or $f_i$ and $x_i^*$ to $e_i^* + f_i^*$.) It follows from Theorem 4.3 and (5.4) that the formulas in Theorem 4.3 for $\text{HH}_0(L_\infty)$, $m \neq 0$, hold, taking as $V_{i,m}$ the vector space generated by all the words in $x_1, x_2, \ldots$ of length $m$, and that $\text{HH}_0(L_\infty) = k$ and $\text{HH}_n(L_\infty) = 0$ for $n \geq 1$. As before, Lemma 2.3 gives the result.

**Theorem 5.5.** Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m, \ldots$ be a finite and an infinite sequence of quivers. Assume that the number of indices $i$ such that $F_i$ has at least one non-trivial closed path is infinite. Then the algebras $L(E_1) \otimes \cdots \otimes L(E_n)$ and $\bigotimes_{i=1}^{\infty} L(F_i)$ are not Morita equivalent.

**Proof.** Immediate from Lemma 2.3 and Corollary 4.5(iii).

**Example 5.6.** Let $L(\infty) = \bigotimes_{i=1}^{\infty} L_2$, and let $E$ be any quiver having at least one non-trivial closed path. Then $L(\infty) \otimes L(E)$ and $L(E)$ are not Morita equivalent.

It would be interesting to know the answer to the following question:

**Question 5.7.** Is there a unital homomorphism $\phi: L_2 \otimes L_2 \to L_2$?

Observe that to build a unital homomorphism $\phi: L_2 \otimes L_2 \to L_2$, it is enough to exhibit a non-zero homomorphism $\psi: L_2 \otimes L_2 \to L_2$ because $eL_2e \cong L_2$ for every non-zero idempotent $e$ in $L_2$.

6. **K-theory**

To conclude the paper we note that algebraic $K$-theory cannot distinguish between $L_2$ and $L_2 \otimes L_2$ or between $L_\infty$ and $L_\infty \otimes L_\infty$. For this we need a lemma which might be of independent interest. Recall that a unital ring $R$ is said to be regular supercoherent in case all the polynomial rings $R[t_1, \ldots, t_n]$ are regular coherent in the sense of [13].
Lemma 6.1. Let E be a finite graph. Then $L(E)$ is regular supercoherent.

Proof. Let $P(E)$ be the usual path algebra of $E$. It was observed in the proof of [3, Lemma 7.4] that the algebra $P(E)[t]$ is regular coherent. The same proof gives that all the polynomial algebras $P(E)[t_1, \ldots, t_n]$ are regular coherent. This shows that $P(E)$ is regular supercoherent. By [3, Proposition 4.1], the universal localization $P(E) \to L(E) = \Sigma^{-1}P(E)$ is flat on the left. It follows that $L(E)$ is left regular supercoherent (see [5, page 23]). Since $L(E) \otimes k[t_1, \ldots, t_n]$ admits an involution, it follows that $L(E)$ is regular supercoherent. □

Proposition 6.2. Let $R$ be regular supercoherent. Then the algebraic $K$-theories of $L_2$ and of $L_2 \otimes R$ are both trivial.

Proof. Let $E$ be the quiver with one vertex and two arrows. Then $L_2 \cong L(E)$, and we have

$$L_2 \otimes R = L_R(E).$$

Applying [5, Theorem 7.6] we obtain that $K_*(L_R(E)) = K_*(L(E)) = 0$. The result follows. □

We finally obtain a $K$-absorbing result for Leavitt path algebras of finite graphs, indeed for any regular supercoherent algebra.

Proposition 6.3. Let $R$ be a regular supercoherent algebra. Then the natural inclusion $R \to R \otimes L_\infty$ induces an isomorphism $K_i(R) \to K_i(R \otimes L_\infty)$ for all $i \in \mathbb{Z}$.

Proof. Adopting the notation used in the proof of Proposition [5,3] we see that it is enough to show that the natural map $R \to R \otimes L(E_n)$ induces isomorphisms $K_i(R) \to K_i(R \otimes L(E_n))$ for all $i \in \mathbb{Z}$ and all $n \geq 1$. Since $R$ is regular supercoherent, the $K$-theory of $R \otimes L(E_n) \cong L_R(E_n)$ can be computed by using [5, Theorem 7.6]. By the explicit form of the quiver $E_n$, we thus obtain that

$$K_i(R \otimes L(E_n)) \cong (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R).$$

The natural map $R \to L_R(E_n)$ factors as

$$R \to Rv \oplus Rw \to L_R(E_n).$$

The first map induces the diagonal homomorphism $K_i(R) \to K_i(R) \oplus K_i(R)$, sending $x$ to $(x, x)$. The second map induces the natural surjection

$$K_i(R) \oplus K_i(R) \to (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R).$$

Therefore the natural homomorphism $R \to L_R(E_n)$ induces an isomorphism

$$K_i(R) \xrightarrow{\sim} K_i(L_R(E_n)).$$

This concludes the proof. □

Corollary 6.4. The natural maps $k \to L_\infty \to L_\infty \otimes L_\infty$ induce $K$-theory isomorphisms $K_*(k) = K_*(L_\infty) = K_*(L_\infty \otimes L_\infty)$. 


Proof. A first application of Proposition 6.3 gives $K_*(k) = K_*(L_\infty)$. A second application shows that for $E_n$ as in the proof above, the inclusion $L(E_n) \to L(E_n) \otimes L_\infty$ induces a $K$-theory isomorphism; passing to the limit, we obtain the corollary. □

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