INVERSION FORMULAE
FOR THE cosh-WEIGHTED HILBERT TRANSFORM

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Abstract. In this paper we develop formulae for inverting the so-called cosh-weighted Hilbert transform $H_\mu$, which arises in Single Photon Emission Computed Tomography (SPECT). The formulae are theoretically exact, require a minimal amount of data, and are similar to the classical inversion formulae for the finite Hilbert transform (FHT) $H_0$. We also find the null-space and the range of $H_\mu$ in $L^p$ with $p > 1$. Similarly to the FHT, the null-space turns out to be one-dimensional in $L^p$ for any $p \in (1,2)$ and trivial – for $p \geq 2$. We prove that $H_\mu$ is a Fredholm operator of index $-1$ when it acts between the $L^p$ spaces, $p \in (1,\infty)$, $p \neq 2$. Finally, in the case where $p = 2$ we find the range condition for $H_\mu$, which is similar to that for the FHT $H_0$. Our work is based on the method of the Riemann-Hilbert problem.

1. Introduction

A relationship between the cone-beam transform of a function $f$ and the Hilbert transform of $f$ along lines was found by Gelfand and Graev in [GG91]. In combination with a formula for the finite Hilbert transform (FHT) inversion, the two results led to a development of new, accurate, and flexible algorithms for image reconstruction in transmission tomography (CT); for some of the references see [ZP04, NCP04, PNC05, ZZYW05]. The key property of FHT inversion is that it can be performed using an efficient convolution-type formula and requires only a minimal amount of data. More precisely, for each line $L$ one needs to compute the Hilbert transform of $f$ from the CT data only for points in $I := L \cap \text{supp } f$ (assuming this intersection is an interval). Since the amount of CT data required is minimal (i.e., any algorithm that uses data on a subinterval of $I$ is severely unstable), reconstruction algorithms based on FHT inversion have the potential to reduce the x-ray dose to the patients.

It was shown recently by Rullgard that in the case of Single Photon Emission Computed Tomography (SPECT) with constant attenuation there is a relation between the attenuated projections of $f$ and a modified Hilbert transform of $f$ along lines [Rul04]. This relation is analogous to the one in transmission CT. Let $\mu$ be the constant attenuation coefficient of the medium. It is known that by using simple weighting of the attenuated projections of $f$, one can compute the
the functions $(1.3)$

$$R_\mu f(\alpha, p) := \int_{-\infty}^{\infty} f(\alpha p + \alpha^\perp t)e^{\mu t}dt.$$  

Here $\alpha$ is a unit vector, and $\alpha^\perp$ is obtained by rotating $\alpha$ counterclockwise by $90^\circ$. The identity, which was found in [Rul04], gives us

$$(1.2) \quad \int_0^\pi \frac{\partial R_\mu f(\alpha, p = \alpha \cdot x)}{\partial p} e^{-\mu x \cdot \alpha^\perp} d\alpha = 2 \int_{-\infty}^{\infty} f(x_1 + t, x_2) \cosh \frac{\mu t}{t} dt.$$  

Thus we can compute an integral transform of $f$ along lines $L$ knowing the attenuated projections of $f$. Inverting the integral transform for $f$ along a family of lines which cover the support of $f$ solves the image reconstruction problem. From a theoretical perspective, the lines $L$ are not related to each other. Hence we can consider only one such line $L$, assume that it coincides with the $x$-axis, and assume that $f$ is a function of $x$ only. Thus, using (1.2) and rescaling $\text{supp} f$ so that it coincides with the interval $[-1, 1]$, we arrive at the following problem.

**Problem 1.1.** Let

$$(1.3) \quad H_\mu f := \frac{1}{\pi} \int_{-1}^{1} \frac{\cosh \mu(x-y)}{x-y} f(x)dx = g(y), \mu \geq 0.$$  

Given $g(y)$ on $[-1, 1]$, reconstruct the function $f(x), |x| \leq 1$.

Here and henceforth, singular integrals are understood in the Cauchy principal value sense. These integrals are well defined for Hölder-continuous (on $[-1,1]$) functions $f$, but can be extended to $f \in L^p := L^p([−1,1])$ with $p \in (1, \infty)$. To avoid certain technical issues, we first derive the inversion formulae for the case when $f$ and, thus, $g = H_\mu f$, are Hölder-continuous, and then extend the results to $L^p, p \in (1, \infty)$, functions. Therefore we assume that, unless specified otherwise, the functions $f$ and $g$ in (1.3) are Hölder-continuous on $[-1, 1]$.

It was shown in [Rul04] that the transform $H_\mu$ can be inverted with the help of a distribution, which can be computed numerically. However, this approach requires the data far outside the support of $f$. In [NDPC07] the problem of inverting $H_\mu$ from the minimal amount of data (which, for each line $L$, is the interval $I = L \cap \text{supp} f$) was investigated. In particular, a numerical method for inverting $H_\mu$ based on solving an integral equation was proposed. Additional results on numerical implementation of $H_\mu$ inversion are in [HYZG09].

In this paper we develop several inversion formulae for $H_\mu$. They are theoretically exact, require a minimal amount of data, and are similar to the classical formulae for the FHT inversion. In particular, the inversion formulae are convolution-based; i.e. they admit efficient numerical implementation based on FFT (fast Fourier transform). We also find the null-space of $H_\mu$. Similarly to the FHT, the null-space turns out to be one-dimensional in $L^p$ for any $p \in (1, 2)$, and trivial for $p \geq 2$. Finally, we prove that $H_\mu$ is a Fredholm operator when it acts between the $L^p$ spaces, $p \in (1, \infty), p \neq 2$. In the case where $p = 2$, we find the range condition for $H_\mu$, which is similar to that for the FHT $H_0$. Our work is based on the reduction of equation (1.3) to a vector Riemann-Hilbert problem (RHP).

The paper is organized as follows. In Section 2 we reduce Problem 1.1 to a non-homogeneous vector RHP. We also formulate a homogeneous matrix RHP and show how to obtain a solution to the nonhomogeneous problem from a solution.
Inversion of the \( \cosh \)-weighted Hilbert transform to the homogeneous one. In Section 3 we study the null-space of \( H_\mu : L^p \to L^p \) for all \( p > 1 \). Two inversion formulae for \( H_\mu \) are obtained in Section 4 under the assumption that \( f \) is Hölder-continuous. The action of \( H_\mu \) on \( L^p \) spaces is studied in Section 5. We also obtain two more inversion formulae. One of them applies in the case \( p \in (1, 2) \), and the other in the case \( p \in (2, \infty) \). In Section 6 we show that \( H_\mu : L^2 \to L^2 \) is not Fredholm and describe its range. Some technical lemmas are contained in Appendix A. A different approach leading to the derivation of a two-parameter family of inversion formulae for \( H_\mu \) directly from the solution of a general RHP is presented in Appendix B.

2. Reduction to a Vector RHP

Define \( Q(z) \) with the help of the complex integral transform

\[
Q(z) := \frac{1}{2\pi i} \int_{-1}^{1} \frac{\cosh \mu (\zeta - z)}{\zeta - z} f(\zeta) d\zeta.
\]

It is clear that \( Q(z) \) is analytic in \( \mathbb{C} \setminus [-1, 1] \) and attains some limiting values \( Q_{\pm}(z), z \in [-1, 1] \), on the upper and lower sides of the cut \([-1, 1]\). Here and in what follows, the subscripts \( \pm \) denote the boundary values on the upper/lower sides of the branchcut \([-1, 1]\), respectively. It is understood that these values can be infinite at the ends of the cut. By the Sokhotski-Plemelj theorem,

\[
-ig(z) = Q_+(z) + Q_-(z), \quad z \in [-1, 1],
\]

where \( g(z) \) is the same as in (1.3). Formula (2.1) can be written as

\[
Q(z) = \frac{\cosh \mu z}{2\pi i} \int_{-1}^{1} \frac{\cosh \mu \zeta}{\zeta - z} f(\zeta) d\zeta - \frac{\sinh \mu z}{2\pi i} \int_{-1}^{1} \frac{\sinh \mu \zeta}{\zeta - z} f(\zeta) d\zeta.
\]

In a similar fashion, (1.3) can be written in the form

\[
\cosh \mu z [(F_c)_+ + (F_c)_-] - \sinh \mu z [(F_s)_+ + (F_s)_-] = -ig(z),
\]

where

\[
F_c = C(\cosh \mu z \cdot f(z)), \quad F_s = C(\sinh \mu z \cdot f(z))
\]

and \( C \) denotes the Cauchy operator

\[
(C\phi)(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\phi(\zeta)}{\zeta - z} d\zeta.
\]

In the rest of the paper, we will routinely use the standard identities \( C_+ + C_- = -iH \) and \( C_+ - C_- = \text{Id} \) on \([-1, 1]\). Combining (2.4) with the obvious consequence of the last identity,

\[
\sinh \mu z [(F_c)_+ - (F_c)_-] - \cosh \mu z [(F_s)_+ - (F_s)_-] = 0,
\]

we obtain after some algebra the following nonhomogeneous RHP for the vector \( F = (F_c, F_s) \).

Problem 2.1. Find a vector-function \( F(z) \) with the following properties:

(a) \( F(z) \) is analytic in \( \mathbb{C} \setminus [-1, 1] \);
(b) $F(z)$ satisfies the jump condition
\[
F_+ = F_- M + G \quad \text{on } (-1, 1),
\]
where
\[
M = \begin{bmatrix}
-\cosh(2\mu z) & -\sinh(2\mu z) \\
\sinh(2\mu z) & \cosh(2\mu z)
\end{bmatrix}, \quad G = -ig(\cosh \mu z, \sinh \mu z);
\]
\[
(c) F(z) = \mathcal{O}(z^{-1}) \quad \text{as } z \to \infty; \quad \text{and}
\]
\[
(d) F(z) = \mathcal{O}((z \mp 1)^{-1+\varepsilon}) \quad \text{as } z \to \pm 1, \quad \text{where } \varepsilon \text{ is a small positive number.}
\]

**Remark 2.2.** If $f$ is H"older-continuous on $[-1, 1]$, then $F(z)$ has no more than $\mathcal{O}(\ln(z \mp 1))$ behavior at the endpoints $z = \pm 1$. The more general limiting condition (d), used in the RHP 2.1, does not affect our analysis.

To solve the RHP 2.1, we first consider the following homogeneous RHP.

**Problem 2.3.** Find a matrix-function $\Gamma(z)$ with the following properties:

(a) $\Gamma$ is analytic and invertible in $\mathbb{C} \setminus [-1, 1]$;

(b) $\Gamma$ satisfies the jump condition
\[
\Gamma_+ = \Gamma_- M \quad \text{on } (-1, 1);
\]

(c) $\Gamma(z) = 1 + \mathcal{O}(z^{-1})$, $z \to \infty$, where $1$ is the identity matrix; and

(d) $\Gamma(z) = \mathcal{O}((z \mp 1)^{-1+\varepsilon})$ as $z \to \pm 1$, where $\varepsilon$ is a small positive number.

**Proposition 2.4.** If $\Gamma(z)$ is a solution to the RHP 2.3 and $\Gamma^{-1}(z)$ satisfies endpoint conditions (2.12), then
\[
F(z) = C(\Gamma^{-1}) \Gamma
\]
satisfies conditions (a)-(c) of the nonhomogeneous RHP 2.1.

**Proof.** Since $G(z)$ is Hölder-continuous on $[-1, 1]$, the Cauchy operator in (2.13) is well defined. Then
\[
F_+ = C_+(G\Gamma^{-1}_+)\Gamma_+ = [C_-(G\Gamma^{-1}_+) + G\Gamma^{-1}_+]\Gamma_+ = C_-(G\Gamma^{-1}_+)\Gamma_- M + G = F_- M + G.
\]
Condition (d) of the RHP 2.1 will be addressed in Remark 2.6 below. For convenience of matrix calculations below, we will use the Pauli matrices
\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

**Proposition 2.5.** The matrix
\[
\Gamma(z) = \frac{\sigma_1}{2} \det(1, i\beta)(\sigma_3 + \sigma_1)e^{\mu(z - \sqrt{z^2 - 1})\sigma_3}(1 + i\sigma_2)
\]
\[
= \begin{bmatrix} i\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh \mu r & \sinh \mu r \\ \sinh \mu r & \cosh \mu r \end{bmatrix} = \begin{bmatrix} i\beta & 0 \\ 0 & 1 \end{bmatrix} (\cosh \mu r \mathbf{1} + \sinh \mu r \sigma_1)
\]
is a solution of the RHP \(2.3\). Here \(1\) is the identity matrix, \(r := z - \sqrt{z^2 - 1}\) and \(\beta = \beta(z) := \frac{1 - z}{1 + z}\), with the determination of \(\beta\) such that \(\beta_+ > 0\) for \(z \in (-1, 1)\).

**Proof.** It is a straightforward verification by observing \(\beta_+ = -\beta_-\) and then elementary matrix algebra. \(\square\)

**Remark 2.6.** A simple calculation yields the matrix

\[
\Gamma^{-1}(z) = (\cosh \mu r 1 - \sinh \mu r \sigma_1) \left[ \begin{array}{cc} (i\beta)^{-1} & 0 \\ 0 & 1 \end{array} \right],
\]

which satisfies condition \(2.12\) in Proposition \(2.3\). Moreover, if \(\Gamma(z)\) is given by \(2.15\), then \(F(z) = C(\Gamma^{-1})z\) satisfies all the conditions of the RHP \(2.1\).

### 3. Null-space of the transform \(H_\mu\)

In the case \(\mu = 0\), it is a well-known fact that if \(Hv = 0\) on \((-1, 1)\) and \(v \in L^p\) for some \(p \in (1, 2)\), then \(v\) is a constant multiple of the function \(v_0 = (1 - z^2)^{-\frac{1}{2}}\) (see [Gak66]). In this section we prove that, similarly to \(H = H_0\), the transform \(H_\mu : L^p \to L^p\) with \(\mu > 0\) has a nontrivial one-dimensional null space \(\mathcal{N}_\mu\) for any \(p \in (1, 2)\) and construct a function \(v_\mu \in \mathcal{N}_\mu\). This function is not in \(L^2\), and the null-space of \(H_\mu\) in the case \(p \in [2, \infty)\) is trivial. We start with a result describing analytic properties of \(\phi \in \mathcal{N}_\mu\).

**Proposition 3.1.** If \(\phi \in \mathcal{N}_\mu\) and \(\phi \in L^p\) for some \(p \in (1, 2)\), then \(\phi(z) = \frac{\Phi(z)}{\sqrt{1 - z^2}}\), where \(\Phi(z)\) is an entire function. If \(\phi \in \mathcal{N}_\mu\) and \(\phi \in L^p\) for some \(p \in [2, \infty)\), then \(\phi(z) = \Phi(z)\sqrt{1 - z^2}\), where \(\Phi(z)\) is an entire function.

**Proof.** Pick any \(\phi \in \mathcal{N}_\mu\) and suppose \(\phi \in L^p\) for some \(p \in (1, 2)\). Then

\[
(3.1) \quad \int_{-1}^{1} \cosh \mu (x - y) \frac{\phi(x)}{x - y} \, dx = 0, \quad |y| < 1.
\]

Denote \(K(t) := (\cosh(\mu t) - 1) / t\). Then \(K(z), z \in \mathbb{C}\), is entire. Rewrite (3.1) as follows:

\[
(3.2) \quad (H\phi)(y) = -b(y), \quad |y| < 1, \quad b(y) := \int_{-1}^{1} K(x - y) \phi(x) \, dx.
\]

Clearly \(b(z), z \in \mathbb{C}\), is entire as well. In particular, \(b \in L^p\) for any \(p \in (1, 2)\). Thus, by Corollary 2.5 of [OE91],

\[
(3.3) \quad \phi(x) = \frac{C}{\sqrt{1 - x^2}} + \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \int_{-1}^{1} b(y) \sqrt{1 - y^2} \, dy, \quad |x| < 1,
\]

for some constant \(C\). Equation (3.3) can be rewritten as

\[
(3.4) \quad \phi(x) = \frac{1}{\sqrt{1 - x^2}} \left( C + \frac{1}{\pi} \int_{-1}^{1} \frac{b(y) - b(x)}{y - x} \sqrt{1 - y^2} \, dy + \frac{b(x)}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - y^2}}{y - x} \, dy \right),
\]

where \(|x| < 1\). A simple residue calculation (see also formula 2.2.5.10 in [PBMS6]) shows that the last integral in parentheses in (3.4) equals \(-\pi x\). Thus, the desired assertion is proved.
Next, suppose $p \in [2, \infty)$. Similarly to (3.3), by Proposition 2.6 of [OE91] for $p \in (2, \infty)$ and by Theorem 4.2 of [OE91] for $p = 2$, we get from (3.2):

\begin{equation}
\phi(x) = \frac{1}{\pi} \sqrt{1 - x^2} \int_{-1}^{1} \frac{b(y)}{\sqrt{1 - y^2}(y - x)} dy, \ |x| < 1.
\end{equation}

Since $\int_{-1}^{1} (\sqrt{1 - y^2}(y - x))^{-1} dy = 0$ for $|x| < 1$, we can write (3.5) in the form

\begin{equation}
\phi(x) = \frac{1}{\pi} \sqrt{1 - x^2} \int_{-1}^{1} \frac{b(y) - b(x)}{\sqrt{1 - y^2}(y - x)} dy, \ |x| < 1,
\end{equation}

and the desired assertion follows immediately. \hfill \Box

We continue with the observation that if $f \in \mathcal{N}_\mu$, then the corresponding vector $F = (F_c, F_s)$ (cf. (2.5)) has the jump $F_+ = F_-M = \Gamma_+ M \Gamma_-^{-1}$ on $[\mathbb{R}, \mathbb{R}]$; that is, $U$ does not have a jump on $[-1, 1]$. Thus, $U$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and bounded at infinity. According to (2.16) and the endpoint behavior (2.12) of $\tilde{\Gamma}(z)$, it has at most simple poles at the endpoints $\pm 1$. Thus, $U$ has the form (3.7).

Proposition 3.2. Let $\tilde{\Gamma}(z)$ be a solution to the RHP (2.3) If $U(z) = \tilde{\Gamma}(z) \Gamma^{-1}(z)$, where $\Gamma(z)$ is given by (2.15), then

\begin{equation}
U(z) = 1 + \frac{A}{z-1} + \frac{B}{z+1},
\end{equation}

where $A, B$ are constant matrices.

Proof. Matrix $U$ is analytic in $C \setminus [-1, 1]$ and $U_+ = \Gamma_+ \Gamma_-^{-1} = \Gamma_- M (M^{-1} \Gamma_-^{-1}) = U_-$ on $[-1, 1]$; that is, $U$ does not have a jump on $(-1, 1)$. Thus, $U$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and bounded at infinity. According to (2.16) and the endpoint behavior (2.12) of $\tilde{\Gamma}(z)$, it has at most simple poles at the endpoints $\pm 1$. Thus, $U$ has the form (3.7).

Let $\phi \in \mathcal{N}_\mu$. By Proposition 3.1 the row $F = C((\cosh \mu z, \sinh \mu z) \phi(z))$ has the following properties: (a) $F$ is analytic in $\mathbb{C} \setminus [-1, 1]$ with, at worst, $O((z \mp 1)^{-\frac{1}{2}})$ behaviour at the endpoints $\pm 1$; and (b) $F(z) = O(z^{-1})$ as $z \to \infty$.

Let $\tilde{F}$ be a matrix with the rows $F$ and $0$ respectively. Then $\tilde{\Gamma} = \Gamma + \tilde{F}$ satisfies RHP (2.3) with $\varepsilon = \frac{1}{2}$. Thus, according to Proposition 3.2, $\tilde{\Gamma} = \Gamma \Gamma U \Gamma$, where $U$ has the form (3.7), so that $\tilde{F} = \left(\frac{A}{z-1} + \frac{B}{z+1}\right) \Gamma$. Since $\Gamma$ is invertible, it is clear that the second rows of $A, B$ are zeros. Taking into account (2.15), we obtain

\begin{equation}
F = \begin{pmatrix}
(ik\beta, l) \\
(\mu z, n)
\end{pmatrix} \begin{pmatrix}
cosh \mu (z - \sqrt{z^2 - 1}) & \sinh \mu (z - \sqrt{z^2 - 1}) \\
\sinh \mu (z - \sqrt{z^2 - 1}) & \cosh \mu (z - \sqrt{z^2 - 1})
\end{pmatrix},
\end{equation}

where $(k, l)$ and $(m, n)$ denote the first rows of the constant matrices $A, B$ respectively. It is now clear that the requirement $F = O((z \mp 1)^{-\frac{1}{2}})$ as $z \to \pm 1$ respectively implies $l = m = n = 0$. Thus, up to a constant factor,

\begin{equation}
F(z) = (1 - z^2)^{-\frac{1}{2}} (\cosh \mu (z - \sqrt{z^2 - 1}), \sinh \mu (z - \sqrt{z^2 - 1})).
\end{equation}

Now using (2.8), we obtain

\begin{equation}
\phi(z) = \cosh \mu z [(F_c)_+ - (F_c)_-] - \sinh \mu z [(F_s)_+ - (F_s)_-]
\end{equation}

\begin{equation}
= \left(\begin{pmatrix}
cosh \mu z \\
\sinh \mu z
\end{pmatrix} \right)_+ - \left(\begin{pmatrix}
cosh \mu z \\
\sinh \mu z
\end{pmatrix} \right)_- = \frac{2 \cos(\mu \sqrt{1 - z^2})}{\sqrt{1 - z^2}}.
\end{equation}
If $p \in [2, \infty)$, Proposition 3.1 implies that $\phi(z) \in \mathcal{N}_\mu$ is bounded near $z = \pm 1$. Hence the matrices $A$ and $B$ in (3.7) are zero, and $\phi(z) \equiv 0$. Thus, we have obtained the following result.

**Theorem 3.3.** For any $p \in (1, 2)$, the null-space of $H_\mu : L^p \to L^p$ is one-dimensional and is spanned by

$$v_\mu = \frac{\cos(\mu \sqrt{1 - z^2})}{\sqrt{1 - z^2}}.$$ (3.11)

If $p \in [2, \infty)$, the null-space of $H_\mu$ is trivial.

### 4. Inversion Formula for the Transform $H_\mu$

In the case $\mu = 0$ the integral transform $H_\mu$ coincides with the FHT $H = H_0$ on $[-1, 1]$. The inverse transforms $H_L^{-1}, H_R^{-1}$ for $H$ with the square root singularities at the left/right endpoint of the contour, respectively, are given by

$$(4.1)\quad H_L^{-1}g = -\frac{1}{\pi} \sqrt{\frac{1 - z}{1 + z}} \int_{-1}^{1} \frac{1 + \zeta g(\zeta) d\zeta}{1 - \zeta - z}, \quad H_R^{-1}g = -\frac{1}{\pi} \sqrt{\frac{1 + z}{1 - z}} \int_{-1}^{1} \frac{1 - \zeta g(\zeta) d\zeta}{1 + \zeta - z}.$$ (4.2)

In this section we derive analogous formulae for inverting $H_\mu$, $\mu > 0$, using the solution $\Gamma(z)$ to the RHP 2.3 given by (2.15). Here we still assume that $g$ is Hölder continuous.

Let $H_\mu f = g$ on $[-1, 1]$. Then, according to (2.15), (2.19) and (2.21),

$$f(z) = -i[C_+(g(\cosh \mu \zeta, \sinh \mu \zeta) \Gamma_+^{-1})(z) \Gamma_+(z) \Gamma_-(z)](\cosh \mu z, -\sinh \mu z)^T.$$ (4.3)

We start with the calculation of

$$\Gamma(z)(\cosh \mu z, -\sinh \mu z)^T = (i\beta \cosh(\mu \sqrt{z^2 - 1}), -\sinh(\mu \sqrt{z^2 - 1}))^T =: P^T(z).$$ (4.4)

Equation (4.3) implies that $P_+(z) = -P_-(z)$ on the cut $[-1, 1]$. Then (4.2) becomes

$$(4.5)\quad f = -i[C_+(g(\cosh \mu \zeta, \sinh \mu \zeta) \Gamma_+^{-1})(z) + C_-(g(\cosh \mu \zeta, \sinh \mu \zeta) \Gamma_+^{-1})(z)]P^T_+(z) = H(g(\cosh \mu \zeta, \sinh \mu \zeta) \Gamma_+^{-1})(z)P^T_+(z).$$

Next we use (2.16) to calculate

$$(4.6)\quad (\cosh \mu z, \sinh \mu z) \Gamma^{-1} = (-i\beta^{-1} \cosh(\mu \sqrt{z^2 - 1}), \sinh(\mu \sqrt{z^2 - 1})): = Q(z).$$

Combining (4.3), (4.5) and (4.6), we obtain

$$f(z) = -H(Q_+(\zeta) g(\zeta))(z)P^T_+(z)$$

$$= \cos(\mu \sqrt{1 - z^2})H_L^{-1}(\cos(\mu \sqrt{1 - \zeta^2})g) - \sin(\mu \sqrt{1 - z^2})H(\sin(\mu \sqrt{1 - \zeta^2})g),$$

where the inverse Hilbert transforms $H_L^{-1}, H_R^{-1}$ were defined in (1.1). Let $\tilde{G}$ be obtained from (2.15) via replacing $\beta = \sqrt{\frac{1 + z}{1 - z}}$ by $1/\beta$. Then $\tilde{G}$ is a solution of the RHP 2.3. Using this solution instead of $\Gamma$ in (4.2), we can obtain another inversion formula by replacing $H_L^{-1}$ by $H_R^{-1}$ in (4.6).
Thus, we obtained the following expressions for the transforms \((H_\mu)_L^{-1}\) and \((H_\mu)_R^{-1}\) with the square root singularities at \(\mp 1\) respectively that are inverse to the cosh transform \(H_\mu\):

\[
(H_\mu)_L^{-1}g = \cos(\mu \sqrt{1 - z^2})H_{L,R}^{-1}(\cos(\mu \sqrt{1 - \zeta^2})g) - \sin(\mu \sqrt{1 - z^2})H(\sin(\mu \sqrt{1 - \zeta^2})g).
\]

(4.7)

In the case \(\mu = 0\) they coincide with \(H_{L,R}^{-1}g\). The precise meaning of the inversion formula will be discussed in Section 5.

Remark 4.1. Direct calculations show that

\[
(H_\mu)^{-1}_R g - (H_\mu)^{-1}_L g = \frac{2}{\pi}(v_\mu, g)v_\mu(z),
\]

where \(v_\mu\) is given by (3.11), and

\[
(v_\mu, g) = \int_{-1}^{1} v_\mu(\zeta)g(\zeta)d\zeta.
\]

(4.9)

Thus the following three conditions are equivalent: (a) both \((H_\mu)_L^{-1}\) and \((H_\mu)_R^{-1}\) are bounded on \([-1, 1]\); (b) \((H_\mu)_L^{-1}g = (H_\mu)_R^{-1}g\) on \([-1, 1]\); and (c) \((v_\mu, g) = 0\).

Remark 4.2. Inversion formulae (4.7) appear to reconstruct a Hölder continuous function \(f\) up to a multiple of the null-space function \(v_\mu\), since \((H_\mu)_L^{-1}Rg\) appear to have square root singularities at \(z = \mp 1\). As shown in Remark 4.1 these singularities disappear if \(g\) satisfies \((v_\mu, g) = 0\). It will be shown below that this condition is, in fact, the range condition for \(H_\mu\) in \(L^p\) with \(p > 2\).

5. Analysis of \(H_\mu\) in \(L^p\) Spaces

In this section we study the action of \(H_\mu\) in \(L^p\)-spaces and obtain two more inversion formulae. We start the section by reminding the reader of a few facts about Fredholm operators. Let \(X\) be a Banach space. The space of linear continuous operators \(X \rightarrow X\) is denoted \(L(X)\). An operator \(A \in L(X)\) is said to be Fredholm if (a) \(N(A)\) and \(N(A^*)\), the null-spaces of \(A\) and its adjoint \(A^*\), are finite-dimensional; and (b) the range of \(A\) is closed. If \(A \in L(X)\) is Fredholm, denote

\[
\alpha(A) := \dim N(A), \quad \beta(A) := \dim N(A^*).
\]

The number \(\chi(A) := \alpha(A) - \beta(A)\) is called the index of \(A\). It is proven in Jor82, §13, that the FHT \(H : L^p \rightarrow L^p\) is Fredholm for \(p \in (1, \infty), p \neq 2\). Moreover,

\[
\alpha(H) = 1, \quad \beta(H) = 0, \quad p \in (1, 2), \quad \alpha(H) = 0, \quad \beta(H) = 1, \quad p \in (2, \infty).
\]

(5.2)

From Theorem 3.3

\[
\alpha(H_\mu) = 1, \quad p \in (1, 2); \quad \alpha(H_\mu) = 0, \quad p \in (2, \infty).
\]

(5.3)

Using the kernel \(K\), which was introduced in the proof of Proposition 3.1, we see that \(H_\mu - H\) is a compact operator. Hence, by Theorem 5.12 of Jor82: (a) \(H_\mu : L^p \rightarrow L^p\) is also Fredholm for \(p \in (1, \infty), p \neq 2\), and (b) \(H\) and \(H_\mu\) have the same index. From (5.2) and (5.3),

\[
\beta(H_\mu) = 0, \quad p \in (1, 2); \quad \beta(H_\mu) = 1, \quad p \in (2, \infty).
\]

(5.4)
Theorem 5.1. The map $H_\mu : L^p \rightarrow L^p$ is Fredholm for all $p \in (1, \infty), p \neq 2$. For $p \in (1, 2)$: (a) the null-space of $H_\mu$ is one-dimensional and spanned by $v_\mu$; (b) the range of $H_\mu$ is all of $L_p$; (c) the pseudo-inverse of $H_\mu$ modulo the null-space is given by

\begin{equation}
H_\mu^{-1} g = \frac{1}{\pi} \int_{-1}^{1} \frac{\cos(\sqrt{1-z^2})}{\sqrt{1-z^2}} \frac{1+\zeta}{1-\zeta^2} g(\zeta) d\zeta
\end{equation}

and (d) $H_\mu^{-1} : L^p \rightarrow L^p$ given by (5.8) is continuous.
For \( p \in (2, \infty) \): (a) the null-space of \( H_\mu \) is trivial; (b) the range of \( H_\mu \) has co-dimension one and consists of all \( g \in L_p \) such that \( \int_{-1}^1 g(\zeta)v_\mu(\zeta) d\zeta = 0 \); (c) the pseudo-inverse of \( H_\mu \) on its range is given by

\[
H_\mu^{-1} g = -\frac{1}{\pi} \cos(\mu \sqrt{1-z^2}) \sqrt{1-z^2} \int_{-1}^1 \frac{\cos(\mu \sqrt{1-\zeta^2}) g(\zeta)}{\zeta - z} d\zeta
\]

and (d) \( H_\mu^{-1} : L^p \to L^p \) given by (5.9) is continuous.

6. THE RANGE OF \( H_\mu \) IN \( L^2 \)

In this section we provide a characterization of the range of \( H_\mu \) and, thus, the domain of its inverse. Our range description is the generalization of Theorem 3.2 in [OE91] to the \( H_\mu \) transform. Similarly to Lemma 3.1 of [OE91], we see that \( H_\mu \) is not Fredholm on \( L^2 \).

**Lemma 6.1.** The operator \( H_\mu : L^2 \to L^2 \) is injective, and its range is a proper dense subspace of \( L^2 \); i.e. \( H_\mu : L^2 \to L^2 \) is not Fredholm.

**Proof.** The injectivity of \( H_\mu \) follows from the fact that \( v_\mu \notin L^2 \). Since \( H_\mu^* = -H_\mu \) and the dual space of \( L^2 \) is identified with itself, we see that \( H_\mu(L^2) \) is dense. Next, let \( g \in \mathcal{C}_0^\infty([-1,1]) \) be an even nonnegative function which is not identically zero. From formula (5.8), \( (H_\mu^{-1} g)(y) \sim \pm c_1 / \sqrt{1 - y^2} \) as \( y \to \pm 1 \), where \( c_1 \neq 0 \) is some constant. Clearly, there does not exist \( c_2 \) such that \( H_\mu^{-1} g - c_2 v_\mu \in L^2 \). Hence \( g \notin H_\mu(L^2) \).

To expedite the computation and shorten the notation we introduce

\[
\begin{align*}
  w &:= \sqrt{1-z^2} , \\
  c &:= \cos(\mu \sqrt{1-z^2}) , \\
  s &:= \sin(\mu \sqrt{1-z^2}) , \\
  S &:= \cosh(\mu z) , \\
  C &:= \sinh(\mu z) .
\end{align*}
\]

(6.1)

Then the transform \( H_\mu \) and formula (5.8) read

\[
H_\mu[f] = CH[Cf] - SH[Sf] , \quad H_\mu^{-1}[g] = -\frac{c}{w} W[c wg] - s H[s g]
\]

respectively.

**Theorem 6.2.** For any \( f \in L^p \), \( p \in (1, 2) \), we have

\[
H_\mu^{-1} [H_\mu[f]] = f + \kappa v_\mu , \quad \text{where} \quad \kappa := -\frac{1}{\pi} \int_{-1}^1 \cosh(\mu \zeta) f(\zeta) d\zeta,
\]

and \( H_\mu^{-1} \) is the expression (5.8).

**Proof.** The proof is a direct but somewhat complicated computation. We shall use Lemmas A.1 and A.2 in the appendix. Another result we need is the Poincaré–Bertrand theorem [Tri57], Theorem 4.2.IV: if \( \phi \in L^p \), \( \psi \in L^q \) are in conjugate spaces, then

\[
H [\phi H[\psi]] = H[\phi] H[\psi] - \psi \phi - H [\psi H[\phi]] .
\]
Suppose that \( g \) is in the image of \( H_\mu \); that is, \( g = H_\mu[f] = CH[Cf] - SH[Sf] \). Then
\[
-H_\mu^{-1}[g] = \frac{c}{w} H[cw] + sH[sg]
\]
(6.5)
\[
= \frac{c}{w} H[cw (CH[Cf] - SH[Sf])] + sH[s (CH[Cf] - SH[Sf])].
\]
We apply the Poincaré–Bertrand theorem to each term, and we get the following two expressions:
\[
- H[cwS] H[Sf] + cwS^2 f + H[SfH[cwS]]
= -cw f + H[Cf \left( H[cwC](z) - H[cwC]\right)] - H[Sf \left( H[cwS](z) - H[cwS]\right)]
\]
and
\[
H[s (CH[Cf] - SH[Sf])] = H[sC] H[Cf] - sC^2 f - H[CfH[sC]]
- H[sS] H[Sf] + sS^2 f + H[SfH[sS]]
= -sf + H[Cf \left( H[sC](z) - H[sC]\right)] - H[Sf \left( H[sS](z) - H[sS]\right)].
\]
Plugging these expressions into (6.5) and using Lemmas A.1, A.2 we get
\[
-H_\mu^{-1}[g] = -f + \frac{c}{w} \left( H[f(\xi) \left( C(\xi)H[cwC](z) - S(\xi)H[cwS](z)\right)] - f(\xi) \left( C(\xi)H[sC](z) - S(\xi)H[sS](z)\right]\right)
\]
(6.8)
\[
- f(\xi) \left( C(\xi)H[cwC](\xi) - S(\xi)H[cwS](\xi)\right)] + sH[f(\xi) \left( C(\xi)H[sC](\xi) - S(\xi)H[sS](\xi)\right)]
- sH[f(\xi) \left( C(\xi)H[sC](\xi) - S(\xi)H[sS](\xi)\right)]
\]
We now apply Lemma A.1 to the second term in the last expression of (6.8) and Lemma A.2 to the last two terms. Note that in doing so, each lemma is applied once with \( z = z \) and once with \( z = \xi \). We obtain
\[
-H_\mu^{-1}[g] = -f + \frac{c(z)}{w(z)} H[f(\xi) \left( - S(\xi - z) s(z) w(z) + (\xi - z) C(\xi) \right)]
+ s(z) H[f(\xi) \left( S(\xi - z) c(z) \right)]
\]
(6.9)
\[
= -f - c(z) s(z) H[S(\xi - z)f(\xi)] + \frac{c(z)}{w(z)} H[(\xi - z) C(\xi) f(\xi)]
+ s(z) c(z) H[S(\xi - z)f(\xi)] = -f + \frac{v_\mu}{\pi} \int_{-1}^{1} \cosh(\mu \xi) f(\xi) d\xi.
\]
\[\square\]

**Theorem 6.3** (See Theorem 3.2 in [OE91]). The function \( g \in L^2 \) is in the range of \( H_\mu : L^2 \to L^2 \) if and only if there is a constant \( \kappa \) such that
\[
H_\mu^{-1}[g](z) = \kappa v_\mu(z) \in L^2,
\]
where \( H_\mu^{-1} \) is the expression (5.8). Then, if \( g = H_\mu[f] \), the constant \( \kappa \) is given by (6.3).
Proof. Suppose that the constant \( \varkappa \) exists. Since \( g \in L^2 \), then \( g \in L^{2-\varepsilon} \), where \( \varepsilon \) is a small positive number. Using the fact that \( H_\mu : L^p \to L^p \) is onto for \( p \in (1, 2) \), we have
\[
(6.11) \quad H_\mu [H_\mu^{-1}[g] - \varkappa v_\mu] = g.
\]
This means that \( g \in H_\mu(L^2) \), being the \( H_\mu \)-image of \( H_\mu^{-1}[g] - \varkappa v_\mu \in L^2 \). Vice versa, suppose that \( g = H_\mu[f] \), with \( g, f \in L^2 \). But then \( f, g \in L^{2-\varepsilon} \) as well, so it must be that \( H_\mu^{-1}[g] - f \) is in the null-space, namely
\[
(6.12) \quad H_\mu^{-1}[g] - f = \varkappa v_\mu \Rightarrow H_\mu^{-1}[g] - \varkappa v_\mu = f \in L^2.
\]
The value of the constant is provided by Theorem 6.2. \( \square \)

The constant in Theorems 6.2 and 6.3 is expressed in terms of the preimage of \( g \), which might be inconvenient sometimes. It is interesting to note that in SPECT with constant attenuation, the value of the constant \( \varkappa \) is easily accessible from the data [NDPC07]. By arguing analogously to [OE91], we can give an expression which does not use the preimage but resorts to a limit.

We need some additional notation:

\[
I_n := \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad v_n(z) := v_\mu \chi_n(z), \quad g_n(z) := \chi_n(z) H_\mu^{-1}[g](z).
\]

Here \((\cdot, \cdot)\) is the \( L^2 \) inner product, \( \| \cdot \| \) is the \( L^2 \) norm, and \( \chi_n \) is the characteristic function of \( I_n \). It is clear that the sequence of \( v_n \)’s belongs to \( L^2 \) but their norms diverge. Let us denote
\[
(6.14) \quad \mu_n := \frac{(g_n, v_n)}{\|v_n\|^2}.
\]

**Proposition 6.4.** The vector \( g \in L^2 \) belongs to the image of \( H_\mu : L^2 \to L^2 \) if and only if
\[
(6.15) \quad \sup_{n \in \mathbb{N}} \|g_n - \mu_n v_n\|^2 < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|v_n\|^2 |\mu_n - \varkappa|^2 < \infty
\]
for some constant \( \varkappa \). In this case \( \varkappa = \lim_{n \to \infty} \mu_n \).

**Proof.** We have

\[
(6.16) \quad |\mu_n - \varkappa|^2 + \frac{\|g_n - \mu_n v_n\|^2}{\|v_n\|^2} = |\mu_n|^2 + |\varkappa|^2 - 2\Re(\mu_n, \varkappa) + \frac{\|g_n\|^2}{\|v_n\|^2} - \mu_n \frac{(v_n, g_n)}{\|v_n\|^2}
\]

\[
- \frac{(g_n, v_n)}{\|v_n\|^2} + |\mu_n|^2 = |\mu_n|^2 + |\varkappa|^2 - 2\Re(\mu_n, \varkappa) + \frac{\|g_n\|^2}{\|v_n\|^2} - \mu_n \frac{(v_n, g_n)}{\|v_n\|^2} - \mu_n \frac{(v_n, g_n)}{\|v_n\|^2} + |\mu_n|^2
\]

\[
= |\varkappa|^2 - 2\Re(\mu_n, \varkappa) + \frac{\|g_n\|^2}{\|v_n\|^2} = \frac{1}{\|v_n\|^2} \left( |\varkappa|^2\|v_n\|^2 - 2\Re(\varkappa, g_n)\right) + \|g_n\|^2
\]

\[
= \frac{1}{\|v_n\|^2} \|g_n - \varkappa v_n\|^2,
\]
that is
\[
(6.17) \quad \|v_n\|^2 |\mu_n - \varkappa|^2 + \|g_n - \mu_n v_n\|^2 = \|g_n - \varkappa v_n\|^2.
\]
On the right-hand side there is the norm of the projection of the vector \( h - \varepsilon v_\mu := H_\mu^{-1}[g] - \varepsilon v_\mu \) on the interval \( I_n \); these projections tend strongly to the identity in \( L^2 \). If \( h - \varepsilon v_\mu \in L^2 \), then the right-hand side of (6.17) has a limit, and thus it must be that

\[
\sup_{n \in \mathbb{N}} \|v_n\| |\mu_n - \varepsilon|^2 < \infty, \quad \sup_{n \in \mathbb{N}} \|g_n - \mu_n v_n\|^2 < \infty.
\]

In particular, since \( \|v_n\| \to +\infty \), we must have \( \mu_n \to \varepsilon \).

Vice versa, if the second inequality in (6.18) holds and there is a constant \( \varepsilon \) such that the first inequality in (6.18) holds as well, then the \( \sup_n \) of the right-hand side of (6.17) is necessarily finite. Since

\[
\|h - \varepsilon v_\mu\|^2 = \sup_n \|g_n - \varepsilon v_n\|^2,
\]

it follows that \( h - \varepsilon v_\mu \in L^2 \). The assertion of the proposition then follows from Theorem 6.3 \( \square \)

Remark 6.5. Replacing the constant \( \varepsilon \) in (6.3) with its expression provided in Proposition 6.4 gives an alternative formula for inverting \( H_\mu : L^2 \to L^2 \).

APPENDIX A. TECHNICAL LEMMAS

Lemma A.1. We have

(A.1) \( C(\xi)H[\text{cwC}](z) - S(\xi)H[\text{cwS}](z) = -S(\xi - z)s(z)w(z) - zC(\xi) + \frac{\mu}{2}S(\xi). \)

Proof. Clearly,

(A.2) \( C(\xi)H[\text{cwC}](z) - S(\xi)H[\text{cwS}](z) = \frac{1}{\pi} \int \frac{C(\xi)w(\xi)C(\xi - \zeta)}{\zeta - z} d\xi. \)

Note that \( \sqrt{1 - \zeta^2} = -i\sqrt{\zeta^2 - 1} \) and define \( r(z) := z - \sqrt{z^2 - 1} \). We first compute the Cauchy transform (\( \gamma \) is a contour clockwise around \([-1, 1]):

(A.3)

\[
\int_{-1}^{1} \frac{\cos(\mu \sqrt{1 - \zeta^2}) \sqrt{1 - \zeta^2} \cosh(\mu(\xi - \zeta))}{\zeta - z} \frac{1}{2\pi i} dz
= \frac{1}{4} \int_{-1}^{1} \frac{\sqrt{1 - \zeta^2} \left( e^{\mu\sqrt{\zeta^2 - 1}} + e^{-\mu\sqrt{\zeta^2 - 1}} \right) (e^{\mu(\xi - \zeta)} + e^{\mu(\xi - \zeta)})}{\zeta - z} \frac{1}{2\pi i} dz
= \frac{i}{4} \int_{-1}^{1} \frac{\sqrt{\zeta^2 - 1} \left( e^{\mu(\xi - r_+)} + e^{\mu(\xi - r_-)} + e^{-\mu(\xi - r_-)} + e^{-\mu(\xi - r_+)} \right) \zeta - z}{\zeta - z} \frac{1}{2\pi i} dz
= -\frac{i}{2} \int_{-1}^{1} \frac{\cosh(\mu(\xi - r_+)) \sqrt{\zeta^2 - 1} + \cosh(\mu(\xi - r_-)) \sqrt{\zeta^2 - 1}}{\zeta - z} \frac{1}{2\pi i} dz
= -\frac{i}{2} \int_{-1}^{1} \frac{\cosh(\mu(\xi - r)) \sqrt{\zeta^2 - 1}}{\zeta - z} \frac{1}{2\pi i} d\zeta
= -\frac{i}{2} \frac{\cosh(\mu(\xi - r(z))) \sqrt{\zeta^2 - 1} - z \cosh(\mu\xi) + \frac{\mu}{2} \sinh(\mu\xi)}{\zeta - z} \frac{1}{2\pi i} d\zeta.
\]
Now (we use $w := \sqrt{1 - z^2}$)

$$
\left[ \cosh(\mu(\xi - r))\sqrt{z^2 - 1} \right]_{\pm} = \cosh(\mu(\xi - z)) \cosh(\mu\sqrt{z^2 - 1}) \sqrt{z^2 - 1}_{\pm}
$$

(A.4)

$$
\pm \cosh(\mu(\xi - z)) \sinh(\mu\sqrt{z^2 - 1}) \sqrt{z^2 - 1}_{\pm}
$$

We have

Lemma A.2. We have

(A.6) $C(\xi)H[sC](z) - S(\xi)H[sS](z) = S(\xi - z)c(z) - S(\xi)$.

Proof. This is very similar to the proof of Lemma [A.1] and some steps are skipped. We have

(A.7) $C(\xi)H[sC](z) - S(\xi)H[sS](z) = \frac{1}{\pi} \int_{-1}^{1} \frac{\cosh(\mu(\xi - \zeta)) \sin(\mu\sqrt{1 - \zeta^2})}{\zeta - z} \, d\zeta$.

The Cauchy transform gives

(A.8)

$$
\frac{1}{i} \int_{-1}^{1} \frac{\cosh(\mu(\xi - \zeta)) \sinh(\mu\sqrt{\zeta^2 - 1})}{\zeta - z} \, d\zeta = \frac{1}{4i} \int_{-1}^{1} \frac{(e^{\mu(\xi - r_+) + e^{-\mu(\xi - r_-) - e^{\mu(\xi - r_-)} - e^{-\mu(\xi - r_+)}}}{\zeta - z} \, d\zeta
$$

$$
= \frac{1}{2i} \int_{-1}^{1} \frac{\sinh(\mu(\xi - r_+)) - \sinh(\mu(\xi - r_-))}{\zeta - z} \, d\zeta = \frac{1}{2i} \int_{\gamma} \frac{\sinh(\mu(\xi - r))}{\zeta - z} \, d\zeta
$$

Now

(A.9)

$sinh(\mu(\xi - r_+)(z)) = sinh(\mu(\xi - z)) \cosh(\mu\sqrt{1 - z^2}) \pm i \cosh(\mu(\xi - z)) \sin(\mu\sqrt{1 - z^2})$.

Therefore ($H = i(C_+ + C_-)$)

(A.10)

$$
\frac{1}{\pi} \int_{-1}^{1} \frac{C(\xi - \zeta)s(\zeta)}{\zeta - z} \, d\zeta = S(\xi - z)c(z) - S(\xi).
$$

APPENDIX B. GENERAL SOLUTION OF THE RHP 2.3 AND AN ALTERNATIVE EXPRESSION FOR INVERSION FORMULAE

In this section we will give an alternative way of deriving the inversion formulae for $H_\mu(f)$, where $f$ is a Hölder-continuous (on $[-1, 1]$) function. Our derivation is based on the general solution $\Gamma(z)$ to the RHP 2.3. Only brief outlines of proofs are included here. In this section we do not require that $\Gamma(z)$ be invertible.

Proposition B.1. Any solution to Problem 2.33 has growth bounded by $O(1/\sqrt{z^2 - 1})$.
This result follows from the fact that the jump matrix $M$ is analytically reducible (similar) to $\sigma_3$.

**Lemma B.2.** The determinant of any solution to Problem 2.3 is of the form

\[(B.1) \quad \det \Gamma(z) = t \sqrt{\frac{z - 1}{z + 1}} + (1 - t) \sqrt{\frac{z + 1}{z - 1}} = \frac{z - 2t + 1}{\sqrt{z^2 - 1}}, \quad t \in \mathbb{C}.\]

A solution $\Gamma(z)$ is invertible in $\mathbb{C} \setminus \{-1, 1\}$ if and only if $t = 0$ or $t = 1$.

We state the RHP for $\det \Gamma(z)$ and give the general solution to this RHP. Proposition B.1 is used to set endpoint conditions as $\det \Gamma(z) = \mathcal{O}((z \mp 1)^{-1})$, $z \to \pm 1$.

**Proposition B.3.** If $\Gamma$ is a solution of Problem 2.3 with determinant $\det \Gamma$ as in Lemma B.2 then it is of the form

\[(B.2) \quad \Gamma(z; t, \alpha) = \left[ \begin{array}{cc} \frac{z + 1 - 2t}{\sqrt{z^2 - 1}} & 0 \\ \frac{\sqrt{z^2 - 1}}{\sqrt{z^2 - 1}} & 1 \end{array} \right] \left[ \begin{array}{cc} \cosh(r\mu) & \sinh(r\mu) \\ \sinh(r\mu) & \cosh(r\mu) \end{array} \right].\]

The proof is similar to Proposition B.2. In fact, it is still valid if we relax the endpoint conditions (2.12) by $\Gamma(z; t, \alpha) = o((z \mp 1)^{-1})$, $z \to \pm 1$.

**Theorem B.4.** The formula for the inverse of $H_\mu$ is

\[(B.3) \quad f = J^{(\alpha,t)}_\mu g = \frac{(z + 1 - 2t)c(z)}{\sqrt{1 - z^2}} H \left( \frac{\sqrt{1 - \zeta^2 c(\zeta)}}{\zeta + 1 - 2t} g \right) - s(z)H \left( s(\zeta)g \right) + 2\alpha A[g] v_\mu(z),\]

where $t \in \mathbb{C} \setminus (0, 1)$, $\alpha \in \mathbb{C}$, and

\[(B.4) \quad c(z) := \cos \left( \mu \sqrt{1 - z^2} \right), \quad s(z) := \sin \left( \mu \sqrt{1 - z^2} \right), \quad A[g] := \int_{-1}^{1} \frac{g(\zeta)s(\zeta)}{\zeta + 1 - 2t} \frac{d\zeta}{2i\pi}.\]

By Proposition 2.3 we have $F = \left[ G(\zeta)\Gamma^{-1}(\zeta; t, \alpha) \right] \Gamma(z; t, \alpha)$. Now the result follows from calculations that are similar to those from Section 4.

**Remark B.5.** Theorem B.4 provides a two-parameter family of inversion formulæ depending on $\alpha, t$. Direct calculation

\[J^{(\alpha,t)}_\mu (g) - J^{(\beta,s)}_\mu (g) = v_\mu(z) \left( \int_{-1}^{1} \frac{2(s - t)\sqrt{1 - \zeta^2 c(\zeta)} + ((\alpha - \beta)(\zeta + 1) - 2\alpha s + 2\beta t)s(\zeta)}{(\zeta + 1 - 2t)(\zeta + 1 - 2s)} g(\zeta) \, d\zeta \right)\]

shows that, as expected, the difference of any two inversions is a multiple of $v_\mu \in \mathcal{N}_\mu$. Moreover, $(H_\mu)^{-1}_{L,R}$ from (4.17) are given by $(H_\mu)^{-1}_{L} = J^{(0,0)}_\mu$ and $(H_\mu)^{-1}_{R} = J^{(0,1)}_\mu$,

and, sending $t \to \infty$, we have

\[(B.5) \quad J^{(0,\infty)}_\mu [g] = H^{-1}_{\mu}[g] = \frac{c(z)}{\sqrt{1 - z^2}} H \left[ \sqrt{1 - \zeta^2 c(\zeta)g(\zeta)} \right] - s(z)H \left[ s(\zeta)g(\zeta) \right].\]
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