ON A THEOREM OF HAZRAT AND HOOBLER

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Abstract. We use cycle complexes with coefficients in an Azumaya algebra, as developed by Kahn and Levine, to compare the $G$-theory of an Azumaya algebra to the $G$-theory of the base scheme. We obtain a sharper version of a theorem of Hazrat and Hoobler in certain cases.

1. Introduction

Let $K^*(X; A)$ be the $K$-theory of left $A$-modules which are locally free and finite rank coherent $O_X$-modules and let $G^*(X; A)$ be the $K$-theory of left $A$-modules which are coherent $O_X$-modules.

We prove the following theorem.

Theorem 1.1. Let $X$ be a $d$-dimensional scheme of finite type over a field $k$, and let $A$ be an Azumaya algebra on $X$ of constant degree $n$. Let $B_A : G_i(X) \to G_i(X; A)$ and $B_{K,A} : K_i(X) \to K_i(X; A)$ be the homomorphisms induced by the functor $F \mapsto A \otimes O_X F$. Then,

1. the kernel and cokernel of $B_A : G_i(X) \to G_i(X; A)$ are torsion groups of exponents dividing $n^{2d+2}$;
2. the kernel and cokernel of $B_{K,A} : K_i(X) \to K_i(X; A)$ are torsion groups of exponents dividing $n^{2d+2}$ if $X$ is regular.

Corollary 1.2. If $A$ is an Azumaya algebra of constant degree $n$ over a scheme $X$ of finite type over a field $k$, then the base extension homomorphism

$$B_A : G_*(X) \otimes \mathbb{Z} \left[ \frac{1}{n} \right] \to G_*(X; A) \otimes \mathbb{Z} \left[ \frac{1}{n} \right]$$

is an isomorphism.

The theorem above should be compared to the following two theorems, which motivated us in the first place.

Theorem 1.3 (Hazrat-Millar [9]). If $A$ is an Azumaya algebra of constant degree $n$ which is free over a noetherian affine scheme $X$, then $B_{K,A} : K_i(X) \to K_i(X; A)$ has torsion kernel and cokernel of exponents at most $n^4$. 
Theorem 1.4 (Hazrat-Hoobler [8]). Let $X$ be a $d$-dimensional noetherian scheme, and let $\mathcal{A}$ be an Azumaya algebra on $X$ of constant degree $n$. Then:

1. the kernel of $B_\mathcal{A} : G_i(X) \to G_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$, and the cokernel is torsion of exponent dividing $n^{4d+2}$;
2. the kernel of $B^K_\mathcal{A} : K_i(X) \to K_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$ if $X$ is regular, and the cokernel is torsion of exponent dividing $n^{4d+2}$ in this case;
3. the kernel and cokernel of $B^K_\mathcal{A} : K_i(X) \to K_i(X; \mathcal{A})$ are torsion groups of exponent dividing $n^{2d+2}$ if $X$ has an ample line bundle.

Since a degree $n$ Azumaya algebra is locally split by degree $n$ extensions, it is expected that the base extension map

\[
B^K_\mathcal{A} : K_*(X) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{n}] \to K_*(X; \mathcal{A}) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{n}]
\]

should be an isomorphism.

Here is a partial history of results and techniques in this direction.

Wedderburn’s theorem [10] easily implies that $K_0(k) \to K_0(A)$ is injective with cokernel isomorphic to $\mathbb{Z}/m$, where $A \cong M_m(D)$ for a central $k$-division algebra $D$.

Green-Handelman-Roberts [5] proved that the map $B^K_\mathcal{A}$ in equation (1) is an isomorphism when $\mathcal{A}$ is a central simple algebra of degree $n$ over a field. They used the Skolem-Noether theorem. That case has also been proven by Hazrat [7] using the fact that $A$ is étale locally a matrix algebra.

The theorem of Hazrat-Millar quoted above uses the opposite algebra. The theorem of Hazrat-Hoobler uses Bass-style stable range arguments and Zariski descent for $G$-theory.

Our result uses twisted versions of Bloch’s cycle complexes. These twisted cycle complexes and the twisted motivic spectral sequence that relates them to $G$-theory are due to Kahn and Levine [11]. It is possible that our result could be extended to essentially smooth schemes over Dedekind rings by a combination of the work of Kahn and Levine [11] and Geisser [4].

The following is an interesting corollary of our approach: there are natural filtrations of length $d$ on $G_i(X)$ and $G_i(X; \mathcal{A})$ coming from [11]. The map $B_\mathcal{A} : G_i(X) \to G_i(X; \mathcal{A})$ respects the filtrations. We show that the induced maps on each of the $d+1$ slices have kernel and cokernel groups of exponent at most $n^2$.

It is worth mentioning two related functors on Azumaya algebras with values in abelian groups where the base extension maps are isomorphisms. Dwyer and Friedlander [3, 2.4, 3.1] showed that

\[
K^\text{ét}_*(R; \mathbb{Z}/m) \to K^\text{ét}_*(R; A; \mathbb{Z}/m)
\]

is an isomorphism in some cases (all of which are Azumaya algebras over a noetherian ring), where $K^\text{ét}$ denotes étale $K$-theory, as, for instance, in Thomason [12]. In this direction, it is possible to show (for instance, in the setting of Antieau [1]) that $K^\text{ét}(X; \mathcal{A})$ is an invertible object (in the sense of the Picard group) over $K^\text{ét}(X)$ in the category of étale sheaves of $K^\text{ét}$-module spectra on a scheme $X$.

Finally, Cortiñas and Weibel [2] proved that the base extension maps induce isomorphisms in Hochschild homology over a field $k$.  

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2. Twisted higher Chow groups and twisted \( G \)-theory

Let \( X \) in \( \text{Sch}/k \) be an integral \( k \)-scheme of finite type, and let \( \mathcal{A} \) be a sheaf of Azumaya algebras on \( X \) of rank \( n^2 \). The degree of \( \mathcal{A} \) is defined to be the integer \( n \). Let \( \mathcal{E} \) be a left \( \mathcal{A} \)-module which is locally free and finite rank \( na \) as an \( \mathcal{O}_X \)-module. For generalities on Azumaya algebras, which as \( \mathcal{O}_X \)-modules are always locally free and of finite rank, see [6].

As in Kahn-Levine [11], define the cycle complex of \( X \) with coefficients in \( \mathcal{A} \) as follows. Let \( S^X_{(s)}(t) \) denote the set of closed subsets \( W \subset X \times_k \Delta^t \) such that

\[
\dim_k W \cap X \times_k F \leq s + \dim_k F
\]

for all faces \( F \) of \( \Delta^n \). Taking inverse images, \( S^X_{(s)}(*) \) becomes a simplicial set. Let \( X^s(t) \) denote the subset of irreducible \( W \) in \( S^X_{(s)}(t) \) such that \( \dim_k W = s + t \). Define, for \( t \geq 0 \),

\[
z_s(X, t; \mathcal{A}) = \bigoplus_{W \in X^s(t)} K_0(k(W); \mathcal{A}).
\]

See [11 Definition 5.6.1]. Kahn and Levine show that this actually becomes a complex, \( z_s(X, *; \mathcal{A}) \), and they define the higher Chow groups with coefficients in \( \mathcal{A} \) as

\[
\text{CH}_s(X, t; \mathcal{A}) = H_2(z_s(X, *; \mathcal{A})�).\]

There are maps relating the complex \( z_r(X, *; \mathcal{A}) \) to \( z_r(X, *) \), the untwisted complex that computes Bloch’s higher Chow groups. These are induced by the base-change map \( \mathcal{B} \mathcal{E} \) and the forgetful map \( F \) on \( K \)-theory:

\[
B^\mathcal{E}_\mathcal{E} : K_0(k(W)) \to K_0(k(W); \mathcal{A}),
F : K_0(k(W), \mathcal{A}) \to K_0(k(W)).
\]

The map \( B^\mathcal{E}_\mathcal{E} \) takes a \( k(W) \)-vector space and tensors with \( \mathcal{E}_{k(W)} \) to produce a left \( \mathcal{A}_{k(W)} \)-module. The norm map \( F \) simply forgets the \( \mathcal{A} \otimes_{k(W)} \)-module structure on a vector space. The kernels of both of these maps are zero.

**Lemma 2.1.** The compositions \( F \circ B^\mathcal{E}_\mathcal{E} \) and \( B^\mathcal{E}_\mathcal{E} \circ F \) are multiplication by \( na \) on \( z_s(X, t) \) and \( z_s(X, t; \mathcal{A}) \).

**Proof.** Indeed, since the rank of \( \mathcal{E} \) is \( na \) as an \( \mathcal{O}_X \)-module, this follows immediately. \( \square \)

**Corollary 2.2.** The cokernel of \( F: z_s(X, t; \mathcal{A}) \to z_s(X, t) \) is a torsion group of exponent bounded above by \( n^2 \), and \( B^\mathcal{E}_\mathcal{E} : z_s(X, t) \to z_s(X, t; \mathcal{A}) \) is a torsion group of exponent bounded above by \( na \).

**Proof.** In the first case, one always has \( \text{ind}(\mathcal{A}_{k(W)}) | n \), where \( \text{ind}(\mathcal{A}_{k(W)}) \) is the degree of the unique division algebra over \( k(W) \) such that \( \mathcal{A}_{k(W)} \cong M_m(D) \) for some \( m \). Similarly,

\[
\left( \frac{na}{\text{ind}(\mathcal{A}_{k(W)})^2} \right) \approx na,
\]

so the second statement follows. \( \square \)
Proposition 2.3. The kernels and cokernels of
\[ B_{\mathcal{E}}^{CH} : \text{CH}_s(X, t) \to \text{CH}_s(X, t; A) \]
and of
\[ F : \text{CH}_s(X, t; A) \to \text{CH}_s(X, t) \]
are torsion groups of exponent at most na.

Proof. This follows immediately from Lemma 2.1. □

Here is our main theorem. Theorem 1.1 follows from it by taking \( \mathcal{E} = A \).

Theorem 2.4. Let \( X \) be a \( d \)-dimensional scheme of finite type over a field, and let \( A \) be an Azumaya algebra on \( X \). Then, the kernels and cokernels of
\[ B_{\mathcal{E}} : G_r(X) \to G_r(X; A) \]
and of
\[ F : G_r(X; A) \to G_r(X) \]
are groups of exponent bounded above by \((na)^{d+1}\) for all \( r \geq 0 \).

Proof. Kahn and Levine \[11\] show that there is a convergent spectral sequence
\[ E_2^{p,q}(A) = \text{CH}_q(X, -p - q; A) \Rightarrow \text{G}_{-p-q}(X; A). \]
There is also the motivic spectral sequence
\[ E_2^{p,q} = \text{CH}_q(X, -p - q) \Rightarrow \text{G}_{-p-q}(X). \]
The functors \( B_{\mathcal{E}} : G(X) \to G(X; A) \) and \( F : G(X; A) \to G(X) \) are compatible with these spectral sequences and the functors \( B_{\mathcal{E}}^{CH} \) and \( F \) on higher Chow groups. Note that \( E_2^{p,q} = E_2^{p,q}(A) = 0 \) whenever \( q < 0, -p < 0, \) or \( q > d \).

We will prove the theorem for the kernel of the functor \( B_{\mathcal{E}} \). The other cases are entirely similar. On the \( E_\infty \)-page, the composition functor \( F \circ B_{\mathcal{E}}^{CH} \) is still multiplication by \( na \), so the kernels and cokernels of \( B_{\mathcal{E}}^{CH} \) on \( E_\infty \) are still of exponent at most \( na \). The spectral sequences abut to filtrations \( F^s G_r(X; A) \) and \( F^s G_r(X) \), where
\[ F^{(s/s+1)} G_r(X; A) = F^s G_r(X; A)/F^{s+1} G_r(X; A) \cong E_{-r+s,-s}(A), \]
\[ F^{(s/s+1)} G_r(X) = F^s G_r(X)/F^{s+1} G_r(X) \cong E_{-r+s,-s}. \]
The filtration looks like
\[ 0 = F^0 G_r(X) \subseteq F^{-1} G_r(X) \subseteq \cdots \subseteq F^{-d} G_r(X) = G_r(X). \]
The filtration \( F^s G_r(X) \) is of length \( d \) by the vanishing statements. Let \( z \in G_r(X) \) be in the kernel of \( F \), and let \( \overline{z} \) be the image of \( z \) in \( E_{-r-d} \). Then, by hypothesis, \( \overline{z} \) is in the kernel of \( F \), so that \( na \cdot \overline{z} = 0 \). Thus, \( na \cdot z \) is contained in \( F^{-d+1} G_r(X) \). Continuing in this way, we see that \((na)^{d+1} \cdot z \) is contained in \( F^0 G_r(X) = 0 \), so \((na)^{d+1} \cdot z = 0 \).

Corollary 2.5. The same result holds for K-theory when \( X \) is regular.

Corollary 2.6. The maps
\[ B_{\mathcal{E}}^{(s/s+1)} : F^{(s/s+1)} G_r(X) \to F^{(s/s+1)} G_r(X; A), \]
\[ F : F^{(s/s+1)} G_r(X; A) \to F^{(s/s+1)} G_r(X) \]
have torsion kernels and cokernels of exponent at most \( na \).
Proof. This follows from the proof of the theorem. □

Corollary 2.7. For any commutative ring R in which na is invertible, the maps

\[ B^2 \zeta : z_\ast(X, *; R) \to z_\ast(X, *; A; R), \]
\[ B^2 : G_r(X; R) \to G_r(X; A; R), \]
\[ F : z_\ast(X, *; A; R) \to z_\ast(X, *; R), \]
\[ F : G_r(X; A; R) \to G_r(X; R) \]

are isomorphisms.

It is interesting that this method proves the isomorphisms by means of an isomorphism of cycle complexes, not just a quasi-isomorphism.

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REFERENCES


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