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ON A THEOREM OF HAZRAT AND HOOBLER

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Abstract. We use cycle complexes with coefficients in an Azumaya algebra, as developed by Kahn and Levine, to compare the G-theory of an Azumaya algebra to the G-theory of the base scheme. We obtain a sharper version of a theorem of Hazrat and Hoobler in certain cases.

1. Introduction

Let $\mathbf{K}_*(X;\mathcal{A})$ be the K-theory of left $\mathcal{A}$-modules which are locally free and finite rank coherent $\mathcal{O}_X$-modules and let $\mathbf{G}_*(X;\mathcal{A})$ be the K-theory of left $\mathcal{A}$-modules which are coherent $\mathcal{O}_X$-modules.

We prove the following theorem.

Theorem 1.1. *Let X be a d-dimensional scheme of finite type over a field k, and let $\mathcal{A}$ be an Azumaya algebra on X of constant degree n. Let $B_{\mathcal{A}} : \mathbf{G}_i(X) \to \mathbf{G}_i(X;\mathcal{A})$ and $B^{\mathbf{K}}_{\mathcal{A}} : \mathbf{K}_i(X) \to \mathbf{K}_i(X;\mathcal{A})$ be the homomorphisms induced by the functor $\mathcal{F} \mapsto \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}$. Then,*

(1) *the kernel and cokernel of $B_{\mathcal{A}} : \mathbf{G}_i(X) \to \mathbf{G}_i(X;\mathcal{A})$ are torsion groups of exponents dividing n^{2d+2};*
(2) *the kernel and cokernel of $B^{\mathbf{K}}_{\mathcal{A}} : \mathbf{K}_i(X) \to \mathbf{K}_i(X;\mathcal{A})$ are torsion groups of exponents dividing n^{2d+2} if X is regular.*

Corollary 1.2. *If $\mathcal{A}$ is an Azumaya algebra of constant degree n over a scheme X of finite type over a field k, then the base extension homomorphism*

$$B_{\mathcal{A}} : \mathbf{G}_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{n}\right] \to \mathbf{G}_*(X;\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{n}\right]$$

is an isomorphism

The theorem above should be compared to the following two theorems, which motivated us in the first place.

Theorem 1.3 (Hazrat-Millar [9])**.** *If $\mathcal{A}$ is an Azumaya algebra of constant degree n which is free over a noetherian affine scheme X, then*

$$B^{\mathbf{K}}_{\mathcal{A}} : \mathbf{K}_i(X) \to \mathbf{K}_i(X;\mathcal{A})$$

has torsion kernel and cokernel of exponents at most n^4.

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Theorem 1.4 (Hazrat-Hoobler [8]). *Let X be a d-dimensional noetherian scheme, and let $\mathcal{A}$ be an Azumaya algebra on X of constant degree n. Then:*

(1) *the kernel of $B_{\mathcal{A}} : \mathbf{G}_i(X) \to \mathbf{G}_i(X;\mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$, and the cokernel is torsion of exponent dividing n^{4d+2};*
(2) *the kernel of $B^{\mathbf{K}}_{\mathcal{A}} : \mathbf{K}_i(X) \to \mathbf{K}_i(X;\mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$ if X is regular, and the cokernel is torsion of exponent dividing n^{4d+2} in this case;*
(3) *the kernel and cokernel of $B^{\mathbf{K}}_{\mathcal{A}} : \mathbf{K}_i(X) \to \mathbf{K}_i(X;\mathcal{A})$ are torsion groups of exponent dividing n^{2d+2} if X has an ample line bundle.*

Since a degree n Azumaya algebra is locally split by degree n extensions, it is expected that the base extension map

$$B^{\mathbf{K}}_{\mathcal{A}} : \mathbf{K}_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{n}\right] \to \mathbf{K}_*(X;\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{n}\right] \tag{1}$$

should be an isomorphism.

Here is a partial history of results and techniques in this direction.

Wedderburn's theorem [10] easily implies that $\mathbf{K}_0(k) \to \mathbf{K}_0(A)$ is injective with cokernel isomorphic to $\mathbb{Z}/m$, where $A \cong M_m(D)$ for a central k-division algebra D.

Green-Handelman-Roberts [5] proved that the map $B^{\mathbf{K}}_{\mathcal{A}}$ in equation (1) is an isomorphism when $\mathcal{A}$ is a central simple algebra of degree n over a field. They used the Skolem-Noether theorem. That case has also been proven by Hazrat [7] using the fact that A is étale locally a matrix algebra.

The theorem of Hazrat-Millar quoted above uses the opposite algebra. The theorem of Hazrat-Hoobler uses Bass-style stable range arguments and Zariski descent for G-theory.

Our result uses twisted versions of Bloch's cycle complexes. These twisted cycle complexes and the twisted motivic spectral sequence that relates them to G-theory are due to Kahn and Levine [11]. It is possible that our result could be extended to essentially smooth schemes over Dedekind rings by a combination of the work of Kahn and Levine [11] and Geisser [4].

The following is an interesting corollary of our approach: there are natural filtrations of length d on $\mathbf{G}_i(X)$ and $\mathbf{G}_i(X;\mathcal{A})$ coming from [11]. The map $B_{\mathcal{A}} : \mathbf{G}_i(X) \to \mathbf{G}_i(X;\mathcal{A})$ respects the filtrations. We show that the induced maps on each of the $d+1$ slices have kernel and cokernel groups of exponent at most n^2.

It is worth mentioning two related functors on Azumaya algebras with values in abelian groups where the base extension maps are isomorphisms. Dwyer and Friedlander [3, 2.4, 3.1] showed that

$$\mathbf{K}^{\text{ét}}_*(R;\mathbb{Z}/m) \to \mathbf{K}^{\text{ét}}_*(R;A;\mathbb{Z}/m)$$

is an isomorphism in some cases (all of which are Azumaya algebras over a noetherian ring), where $\mathbf{K}^{\text{ét}}$ denotes étale K-theory, as, for instance, in Thomason [12]. In this direction, it is possible to show (for instance, in the setting of Antieau [1]) that $\mathbf{K}^{\text{ét}}(X;\mathcal{A})$ is an invertible object (in the sense of the Picard group) over $\mathbf{K}^{\text{ét}}(X)$ in the category of étale sheaves of $\mathbf{K}^{\text{ét}}$-module spectra on a scheme X.

Finally, Cortiñas and Weibel [2] proved that the base extension maps induce isomorphisms in Hochschild homology over a field k.

2. Twisted higher Chow groups and twisted G-theory

Let X in $\mathbf{Sch}/k$ be an integral k-scheme of finite type, and let $\mathcal{A}$ be a sheaf of Azumaya algebras on X of rank n^2. The degree of $\mathcal{A}$ is defined to be the integer n. Let $\mathcal{E}$ be a left $\mathcal{A}$-module which is locally free and finite rank na as an $\mathcal{O}_X$-module. For generalities on Azumaya algebras, which as $\mathcal{O}_X$-modules are always locally free and of finite rank, see [6].

As in Kahn-Levine [11], define the cycle complex of X with coefficients in $\mathcal{A}$ as follows. Let $S^X_{(s)}(t)$ denote the set of closed subsets $W \subset X \times_k \Delta^t$ such that

$$\dim_k W \cap X \times_k F \leq s + \dim_k F$$

for all faces F of Δ^n. Taking inverse images, $S^X_{(s)}(*)$ becomes a simplicial set. Let $X_s(t)$ denote the subset of irreducible W in $S^X_{(s)}(t)$ such that $\dim_k W = s+t$. Define, for $t \geq 0$,

$$z_s(X, t; \mathcal{A}) = \bigoplus_{W \in X_s(t)} \mathbf{K}_0(k(W); \mathcal{A}).$$

See [11, Definition 5.6.1]. Kahn and Levine show that this actually becomes a complex, $z_s(X, *; \mathcal{A})$, and they define the higher Chow groups with coefficients in $\mathcal{A}$ as

$$\mathbf{CH}_s(X, t; \mathcal{A}) = \mathrm{H}_t(z_s(X, *; \mathcal{A})).$$

There are maps relating the complex $z_r(X, *; \mathcal{A})$ to $z_r(X, *)$, the untwisted complex that computes Bloch's higher Chow groups. These are induced by the base-change map $B_{\mathcal{E}}$ and the forgetful map F on K-theory:

$$B^{\mathbf{K}}_{\mathcal{E}} : \mathbf{K}_0(k(W)) \to \mathbf{K}_0(k(W); \mathcal{A}),$$
$$F : \mathbf{K}_0(k(W), \mathcal{A}) \to \mathbf{K}_0(k(W)).$$

The map $B_{\mathcal{E}}$ takes a $k(W)$-vector space and tensors with $\mathcal{E}_{k(W)}$ to produce a left $\mathcal{A}_{k(W)}$-module. The norm map F simply forgets the $\mathcal{A}\otimes_{k(W)}$-module structure on a vector space. The kernels of both of these maps are zero.

Lemma 2.1. *The compositions $F \circ B^z_{\mathcal{E}}$ and $B^z_{\mathcal{E}} \circ F$ are multiplication by na on $z_s(X, t)$ and $z_s(X, t; \mathcal{A})$.*

Proof. Indeed, since the rank of $\mathcal{E}$ is na as an $\mathcal{O}_X$-module, this follows immediately. □

Corollary 2.2. *The cokernel of $F : z_s(X, t; \mathcal{A}) \to z_s(X, t)$ is a torsion group of exponent bounded above by n^2, and $B^z_{\mathcal{E}} : z_s(X, t) \to z_s(X, t; \mathcal{A})$ is a torsion group of exponent bounded above by na.*

Proof. In the first case, one always has $ind(\mathcal{A}_{k(W)})|n$, where $ind(\mathcal{A}_{k(W)}$ is the degree of the unique division algebra over $k(W)$ such that $\mathcal{A}_{k(W)} \cong M_m(D)$ for some m. Similarly,

$$\left(\frac{na}{ind(\mathcal{A}_{k(W)})^2}\right)|na,$$

so the second statement follows. □

Proposition 2.3. *The kernels and cokernels of*

$$B_{\mathcal{E}}^{\mathbf{CH}} : \mathbf{CH}_s(X,t) \to \mathbf{CH}_s(X,t;\mathcal{A})$$

and of

$$F : \mathbf{CH}_s(X,t;\mathcal{A}) \to \mathbf{CH}_s(X,t)$$

are torsion groups of exponent at most na.

Proof. This follows immediately from Lemma 2.1. □

Here is our main theorem. Theorem 1.1 follows from it by taking $\mathcal{E} = \mathcal{A}$.

Theorem 2.4. *Let X be a d-dimensional scheme of finite type over a field, and let $\mathcal{A}$ be an Azumaya algebra on X. Then, the kernels and cokernels of*

$$B_{\mathcal{E}} : \mathbf{G}_r(X) \to \mathbf{G}_r(X;\mathcal{A})$$

and of

$$F : \mathbf{G}_r(X;\mathcal{A}) \to \mathbf{G}_r(X)$$

are groups of exponent bounded above by $(na)^{d+1}$ for all $r \geq 0$.

Proof. Kahn and Levine [11] show that there is a convergent spectral sequence

$$\mathrm{E}_2^{p,q}(\mathcal{A}) = \mathbf{CH}_q(X, -p-q; \mathcal{A}) \Rightarrow \mathbf{G}_{-p-q}(X;\mathcal{A}).$$

There is also the motivic spectral sequence

$$\mathrm{E}_2^{p,q} = \mathbf{CH}_q(X, -p-q) \Rightarrow \mathbf{G}_{-p-q}(X).$$

The functors $B_{\mathcal{E}} : \mathbf{G}(X) \to \mathbf{G}(X;\mathcal{A})$ and $F : \mathbf{G}(X;\mathcal{A}) \to \mathbf{G}(X)$ are compatible with these spectral sequences and the functors $B_{\mathcal{E}}^{\mathbf{CH}}$ and F on higher Chow groups. Note that $\mathrm{E}_2^{p,q} = \mathrm{E}_2^{p,q}(\mathcal{A}) = 0$ whenever $q < 0$, $-p < 0$, or $q > d$.

We will prove the theorem for the kernel of the functor $B_{\mathcal{E}}$. The other cases are entirely similar. On the E_∞-page, the composition functor $F \circ B_{\mathcal{E}}^{\mathbf{CH}}$ is still multiplication by na, so the kernels and cokernels of $B_{\mathcal{E}}^{\mathbf{CH}}$ on E_∞ are still of exponent at most na. The spectral sequences abut to filtrations $F^s\mathbf{G}_r(X;\mathcal{A})$ and $F^s\mathbf{G}_r(X)$, where

$$F^{(s/s+1)}\mathbf{G}_r(X;\mathcal{A}) = F^s\mathbf{G}_r(X;\mathcal{A})/F^{s+1}\mathbf{G}_r(X;\mathcal{A}) \cong \mathrm{E}_\infty^{-r+s,-s}(\mathcal{A}),$$
$$F^{(s/s+1)}\mathbf{G}_r(X) = F^s\mathbf{G}_r(X)/F^{s+1}\mathbf{G}_r(X) \cong \mathrm{E}_\infty^{-r+s,-s}.$$

The filtration looks like

$$0 = F^0\mathbf{G}_r(X) \subseteq F^{-1}\mathbf{G}_r(X) \subseteq \cdots \subseteq F^{-d}\mathbf{G}_r(X) = \mathbf{G}_r(X).$$

The filtration $F^s\mathbf{G}_r(X)$ is of length d by the vanishing statements. Let $z \in \mathbf{G}_r(X)$ be in the kernel of F, and let $\overline{z}$ be the image of z in $E_\infty^{-r-d,d}$. Then, by hypothesis, $\overline{z}$ is in the kernel of F, so that $na \cdot \overline{z} = 0$. Thus, $na \cdot z$ is contained in $F^{-d+1}\mathbf{G}_r(X)$. Continuing in this way, we see that $(na)^{d+1} \cdot z$ is contained in $F^0\mathbf{G}_r(X) = 0$, so $(na)^{d+1} \cdot z = 0$. □

Corollary 2.5. *The same result holds for K-theory when X is regular.*

Corollary 2.6. *The maps*

$$B_{\mathcal{E}}^{(s/s+1)} : F^{(s/s+1)}\mathbf{G}_r(X) \to F^{(s/s+1)}\mathbf{G}_r(X;\mathcal{A}),$$
$$F : F^{(s/s+1)}\mathbf{G}_r(X;\mathcal{A}) \to F^{(s/s+1)}\mathbf{G}_r(X)$$

have torsion kernels and cokernels of exponent at most na.

Proof. This follows from the proof of the theorem. □

Corollary 2.7. *For any commutative ring R in which na is invertible, the maps*

$$B^z_{\mathcal{E}} : z_s(X, *; R) \to z_s(X, *; \mathcal{A}; R),$$
$$B_{\mathcal{E}} : \mathbf{G}_r(X; R) \to \mathbf{G}_r(X; \mathcal{A}; R),$$
$$F : z_s(X, *; \mathcal{A}; R) \to z_s(X, *; R),$$
$$F : \mathbf{G}_r(X; \mathcal{A}; R) \to \mathbf{G}_r(X; R)$$

are isomorphisms.

It is interesting that this method proves the isomorphisms by means of an isomorphism of cycle complexes, not just a quasi-isomorphism.

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