ON A THEOREM OF HAZRAT AND HOOBLER

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Abstract. We use cycle complexes with coefficients in an Azumaya algebra, as developed by Kahn and Levine, to compare the $G$-theory of an Azumaya algebra to the $G$-theory of the base scheme. We obtain a sharper version of a theorem of Hazrat and Hoobler in certain cases.

1. Introduction

Let $K^*_s(X;A)$ be the $K$-theory of left $A$-modules which are locally free and finite rank coherent $O_X$-modules and let $G^*_s(X;A)$ be the $K$-theory of left $A$-modules which are coherent $O_X$-modules.

We prove the following theorem.

Theorem 1.1. Let $X$ be a $d$-dimensional scheme of finite type over a field $k$, and let $A$ be an Azumaya algebra on $X$ of constant degree $n$. Let $B_A : G^*_i(X) \to G^*_i(X;A)$ and $B^K_A : K^*_i(X) \to K^*_i(X;A)$ be the homomorphisms induced by the functor $F \mapsto A \otimes O_X F$. Then,

1. the kernel and cokernel of $B_A : G^*_i(X) \to G^*_i(X;A)$ are torsion groups of exponents dividing $n^{2d+2}$;
2. the kernel and cokernel of $B^K_A : K^*_i(X) \to K^*_i(X;A)$ are torsion groups of exponents dividing $n^{2d+2}$ if $X$ is regular.

Corollary 1.2. If $A$ is an Azumaya algebra of constant degree $n$ over a scheme $X$ of finite type over a field $k$, then the base extension homomorphism

$$B_A : G^*_s(X) \otimes \mathbb{Z} \left[ \frac{1}{n} \right] \to G^*_s(X;A) \otimes \mathbb{Z} \left[ \frac{1}{n} \right]$$

is an isomorphism.

The theorem above should be compared to the following two theorems, which motivated us in the first place.

Theorem 1.3 (Hazrat-Millar [9]). If $A$ is an Azumaya algebra of constant degree $n$ which is free over a noetherian affine scheme $X$, then

$$B^K_A : K^*_i(X) \to K^*_i(X;A)$$

has torsion kernel and cokernel of exponents at most $n^4$. 

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Theorem 1.4 (Hazrat-Hoobler [8]). Let $X$ be a $d$-dimensional noetherian scheme, and let $\mathcal{A}$ be an Azumaya algebra on $X$ of constant degree $n$. Then:

1. the kernel of $B_{\mathcal{A}} : G_i(X) \to G_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$, and the cokernel is torsion of exponent dividing $n^{4d+2}$;
2. the kernel of $B^K_{\mathcal{A}} : K_i(X) \to K_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$ if $X$ is regular, and the cokernel is torsion of exponent dividing $n^{4d+2}$ in this case;
3. the kernel and cokernel of $B^K_{\mathcal{A}} : K_i(X) \to K_i(X; \mathcal{A})$ are torsion groups of exponent dividing $n^{2d+2}$ if $X$ has an ample line bundle.

Since a degree $n$ Azumaya algebra is locally split by degree $n$ extensions, it is expected that the base extension map

$B^K_{\mathcal{A}} : K_*(X) \otimes_{\mathbb{Z}} \mathbb{Z} \left[ \frac{1}{n} \right] \to K_*(X; \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z} \left[ \frac{1}{n} \right]$  

should be an isomorphism.

Here is a partial history of results and techniques in this direction.

Wedderburn's theorem [10] easily implies that $K_0(k) \to K_0(A)$ is injective with cokernel isomorphic to $\mathbb{Z}/m$, where $A \cong M_m(D)$ for a central $k$-division algebra $D$.

Green-Handelman-Roberts [5] proved that the map $B^K_{\mathcal{A}}$ in equation (1) is an isomorphism when $\mathcal{A}$ is a central simple algebra of degree $n$ over a field. They used the Skolem-Noether theorem. That case has also been proven by Hazrat [7] using the fact that $\mathcal{A}$ is étale locally a matrix algebra.

The theorem of Hazrat-Millar quoted above uses the opposite algebra. The theorem of Hazrat-Hoobler uses Bass-style stable range arguments and Zariski descent for $G$-theory.

Our result uses twisted versions of Bloch's cycle complexes. These twisted cycle complexes and the twisted motivic spectral sequence that relates them to $G$-theory are due to Kahn and Levine [11]. It is possible that our result could be extended to essentially smooth schemes over Dedekind rings by a combination of the work of Kahn and Levine [11] and Geisser [4].

The following is an interesting corollary of our approach: there are natural filtrations of length $d$ on $G_i(X)$ and $G_i(X; \mathcal{A})$ coming from [11]. The map $B_{\mathcal{A}} : G_i(X) \to G_i(X; \mathcal{A})$ respects the filtrations. We show that the induced maps on each of the $d+1$ slices have kernel and cokernel groups of exponent at most $n^2$.

It is worth mentioning two related functors on Azumaya algebras with values in abelian groups where the base extension maps are isomorphisms. Dwyer and Friedlander [3] 2.4, 3.1] showed that

$K^\text{ét}_*(R; \mathbb{Z}/m) \to K^\text{ét}_*(R; A; \mathbb{Z}/m)$

is an isomorphism in some cases (all of which are Azumaya algebras over a noetherian ring), where $K^\text{ét}$ denotes étale $K$-theory, as, for instance, in Thomason [12]. In this direction, it is possible to show (for instance, in the setting of Antieau [1]) that $K^\text{ét}(X; \mathcal{A})$ is an invertible object (in the sense of the Picard group) over $K^\text{ét}(X)$ in the category of étale sheaves of $K^\text{ét}$-module spectra on a scheme $X$.

Finally, Cortiñas and Weibel [2] proved that the base extension maps induce isomorphisms in Hochschild homology over a field $k$. 
2. Twisted higher Chow groups and twisted $G$-theory

Let $X$ in $\text{Sch}/k$ be an integral $k$-scheme of finite type, and let $\mathcal{A}$ be a sheaf of Azumaya algebras on $X$ of rank $n^2$. The degree of $\mathcal{A}$ is defined to be the integer $n$. Let $\mathcal{E}$ be a left $\mathcal{A}$-module which is locally free and finite rank $\mathcal{E}$. For generalities on Azumaya algebras, which as $\mathcal{O}_X$-modules are always locally free and of finite rank, see [6].

As in Kahn-Levine [11], define the cycle complex of $X$ with coefficients in $\mathcal{A}$ as follows. Let $S_{(s)}^X(t)$ denote the set of closed subsets $W \subset X \times_k \Delta^n$ such that $\dim_k W \cap X \times_k F \leq s + \dim_k F$ for all faces $F$ of $\Delta^n$. Taking inverse images, $S_{(s)}^X(t) \ast$ becomes a simplicial set. Let $X_s(t)$ denote the subset of irreducible $W$ in $S_{(s)}^X(t)$ such that $\dim_k W = s + t$. Define, for $t \geq 0$, $z_s(X, t; \mathcal{A}) = \bigoplus_{W \in X_s(t)} K_0(k(W); \mathcal{A})$. See [11, Definition 5.6.1]. Kahn and Levine show that this actually becomes a complex, $z_s(X, t; \mathcal{A})$, and they define the higher Chow groups with coefficients in $\mathcal{A}$ as $\text{CH}_s(X, t; \mathcal{A}) = H_t(z_s(X, t; \mathcal{A}))$.

There are maps relating the complex $z_r(X, t; \mathcal{A})$ to $z_r(X, t)$, the untwisted complex that computes Bloch’s higher Chow groups. These are induced by the base-change map $B_\mathcal{E}$ and the forgetful map $F$ on $K$-theory:

$$B^K_\mathcal{E} : K_0(k(W)) \to K_0(k(W); \mathcal{A}),$$
$$F : K_0(k(W), \mathcal{A}) \to K_0(k(W)).$$

The map $B_\mathcal{E}$ takes a $k(W)$-vector space and tensors with $\mathcal{E}_k(W)$ to produce a left $\mathcal{A}_k(W)$-module. The norm map $F$ simply forgets the $\mathcal{A} \otimes_{k(W)}$-module structure on a vector space. The kernels of both of these maps are zero.

**Lemma 2.1.** The compositions $F \circ B_\mathcal{E}$ and $B_\mathcal{E} \circ F$ are multiplication by $na$ on $z_s(X, t)$ and $z_s(X, t; \mathcal{A})$.

**Proof.** Indeed, since the rank of $\mathcal{E}$ is $na$ as an $\mathcal{O}_X$-module, this follows immediately. \hfill $\square$

**Corollary 2.2.** The cokernel of $F : z_s(X, t; \mathcal{A}) \to z_s(X, t)$ is a torsion group of exponent bounded above by $n^2$, and $B_\mathcal{E} : z_s(X, t) \to z_s(X, t; \mathcal{A})$ is a torsion group of exponent bounded above by $na$.

**Proof.** In the first case, one always has $\text{ind}(\mathcal{A}_k(W)) | n$, where $\text{ind}(\mathcal{A}_k(W)$ is the degree of the unique division algebra over $k(W)$ such that $\mathcal{A}_k(W) \cong M_m(D)$ for some $m$. Similarly, $$\left(\frac{na}{\text{ind}(\mathcal{A}_k(W))^2}\right) \mid na,$$
so the second statement follows. \hfill $\square$
Proposition 2.3. The kernels and cokernels of
\[ B^\text{CH}_r : \text{CH}_s(X, t) \to \text{CH}_s(X, t; A) \]
and of
\[ F : \text{CH}_s(X, t; A) \to \text{CH}_s(X, t) \]
are torsion groups of exponent at most na.

Proof. This follows immediately from Lemma 2.1. \qed

Here is our main theorem. Theorem 1.1 follows from it by taking \( E = A \).

Theorem 2.4. Let \( X \) be a d-dimensional scheme of finite type over a field, and let \( A \) be an Azumaya algebra on \( X \). Then, the kernels and cokernels of
\[ B^\text{CH}_r : G_r(X) \to G_r(X; A) \]
and of
\[ F : G_r(X; A) \to G_r(X) \]
are groups of exponent bounded above by \( (na)^{d+1} \) for all \( r \geq 0 \).

Proof. Kahn and Levine \([11]\) show that there is a convergent spectral sequence
\[ E^p,q_2(A) = \text{CH}_q(X, -p - q; A) \Rightarrow G_{-p-q}(X; A). \]
There is also the motivic spectral sequence
\[ E^p,q_2 = \text{CH}_q(X, -p - q) \Rightarrow G_{-p-q}(X). \]
The functors \( B^\text{CH}_E : G(X) \to G(X; A) \) and \( F : G(X; A) \to G(X) \) are compatible with these spectral sequences and the functors \( B_{\text{CH}}^E \) and \( F \) on higher Chow groups. Note that \( E^p,q_2 = E^p,q_2(A) = 0 \) whenever \( q < 0, -p < 0 \), or \( q > d \).

We will prove the theorem for the kernel of the functor \( B^\text{CH}_E \). The other cases are entirely similar. On the \( E_\infty \)-page, the composition functor \( F \circ B^\text{CH}_E \) is still multiplication by \( na \), so the kernels and cokernels of \( B^\text{CH}_E \) on \( E_\infty \) are still of exponent at most \( na \). The spectral sequences abut to filtrations \( F^sG_r(X; A) \) and \( F^sG_r(X) \), where
\[
\begin{align*}
F^{(s/s+1)}G_r(X; A) & = F^sG_r(X; A) / F^{s+1}G_r(X; A) \cong E_{-r+s,-s}^\infty(A), \\
F^{(s/s+1)}G_r(X) & = F^sG_r(X) / F^{s+1}G_r(X) \cong E_{-r+s,-s}^\infty.
\end{align*}
\]
The filtration looks like
\[ 0 = F^0G_r(X) \subseteq F^{-1}G_r(X) \subseteq \cdots \subseteq F^{-d}G_r(X) = G_r(X). \]
The filtration \( F^sG_r(X) \) is of length \( d \) by the vanishing statements. Let \( z \in G_r(X) \) be in the kernel of \( F \), and let \( \overline{z} \) be the image of \( z \) in \( E_{-r,d}^\infty \). Then, by hypothesis, \( \overline{z} \) is in the kernel of \( F \), so that \( na \cdot \overline{z} = 0 \). Thus, \( na \cdot z \) is contained in \( F^{d+1}G_r(X) \). Continuing in this way, we see that \( (na)^{d+1} \cdot z \) is contained in \( F^0G_r(X) = 0 \), so \( (na)^{d+1} \cdot z = 0 \). \qed

Corollary 2.5. The same result holds for K-theory when \( X \) is regular.

Corollary 2.6. The maps
\[
\begin{align*}
B^E_r : F^{(s/s+1)}G_r(X) & \to F^{(s/s+1)}G_r(X; A), \\
F : F^{(s/s+1)}G_r(X; A) & \to F^{(s/s+1)}G_r(X)
\end{align*}
\]
have torsion kernels and cokernels of exponent at most \( na \).
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Proof. This follows from the proof of the theorem. □

Corollary 2.7. For any commutative ring $R$ in which $na$ is invertible, the maps

\[ B_E^z : z_s(X, \ast; R) \to z_s(X, \ast; A; R), \]
\[ B_E : G_r(X; R) \to G_r(X; A; R), \]
\[ F : z_s(X, \ast; A; R) \to z_s(X, \ast; R), \]
\[ F : G_r(X; A; R) \to G_r(X; R) \]

are isomorphisms.

It is interesting that this method proves the isomorphisms by means of an isomorphism of cycle complexes, not just a quasi-isomorphism.

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REFERENCES


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