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REGULARITY OF NONLINEAR EQUATIONS FOR FRACTIONAL LAPLACIAN

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Abstract. In this paper, we prove that any $H^s(\Omega)$ solution u of the problem

$$(-\Delta)^s u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{0.1}$$

belongs to $L^\infty(\Omega)$ for the nonlinearity of $f(t)$ being subcritical and critical. This implies that the solution u is classical if $f(t)$ is $C^{1,\gamma}$ for some $0 < \gamma < 1$.

1. Introduction

In this paper, we consider the regularity of solutions to the problem involving fractional Laplacian operators

$$\begin{cases} (-\Delta)^s u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, $0 < s < 1$. The fractional operator $(-\Delta)^s$ is defined as follows. Let φ_k be an eigenfunction of $-\Delta$ given by

$$\begin{cases} -\Delta\varphi_k = \mu_k\varphi_k, & x \in \Omega, \\ \varphi_k = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where μ_k is the corresponding eigenvalue of φ_k, $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \to +\infty$. Then, $\{\varphi_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ satisfying $\int_\Omega \varphi_j\varphi_k\, dx = \delta_{j,k}$. We define the operator $(-\Delta)^s$ for any $u \in C_c^\infty(\Omega)$ by

$$(-\Delta)^s u = \sum_{k=1}^\infty \mu_k^s u_k \varphi_k,$$

where

$$u = \sum_{k=1}^\infty u_k\varphi_k \quad \text{and} \quad u_k = \int_\Omega u\varphi_k dx.$$

This operator can be extended by density for u in the Hilbert space

$$H = \{u \in L^2(\Omega) : \|u\|_H^2 = \sum_{k=1}^\infty \mu_k^s |u_k|^2 < +\infty\}.$$

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It is known that

$$H = \begin{cases} H^s(\Omega), & \text{if} \quad s \in (0, \frac{1}{2}), \\ H^{\frac{1}{2}}_{00}(\Omega), & \text{if} \quad s = \frac{1}{2}, \\ H^s_0(\Omega), & \text{if} \quad s \in (\frac{1}{2}, 1). \end{cases} \tag{1.3}$$

The fractions of the Laplacian are the infinitesimal generators of Lévy stable diffusion processes and appear in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquid, and American options in finances.

An important feature of the operator $(-\Delta)^s$ is its nonlocal character, which can be realized as the boundary operator of a suitable extension in the half-cylinder $\Omega \times (0, \infty)$. Such an interpretation was demonstrated in [5] for the fractional Laplacian in $\mathbb{R}^N$. Their construction can be extended to the case of bounded domains as in [6]. Indeed, let us define

$$\mathcal{C} = \Omega \times (0, +\infty), \quad \partial_L \mathcal{C} = \partial\Omega \times [0, +\infty).$$

We write points in the cylinder $\mathcal{C}$ by $(x, y) \in \mathcal{C} = \Omega \times (0, +\infty)$. Given $s \in (0, 1)$, consider the space $H^1_{0,L}(y^{1-2s})$ of measurable functions $v : \mathcal{C} \to \mathbb{R}$ such that $v \in H^1(\Omega \times (s, t))$ for all $0 < s < t < +\infty, v = 0$ on $\partial_L \mathcal{C}$ and for which the following norm is finite:

$$\|v\|^2_{H^1_{0,L}(y^{1-2s})} = \int_{\mathcal{C}} y^{1-2s} |\nabla v|^2 dxdy.$$

Hence, we can study problem (1.1) by variational methods for a local problem. More precisely, problem (1.1) can be reduced to the problem

$$\begin{cases} div(y^{1-2s}\nabla v) = 0, \text{ in} & \mathcal{C}, \\ v = 0, & \text{on} \quad \partial_L \mathcal{C}, \\ y^{1-2s}\frac{\partial v}{\partial \nu} = f(u), & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{1.4}$$

where ν is the unit outer normal to $\Omega \times \{0\}$. If v satisfies (1.4), then the trace v on $\Omega \times \{0\}$ of the function v will be a solution of problem (1.4).

Recently, problem (1.1) has been studied in [2], [3], [4], [5] and [6], etc., where the existence of solutions and various properties of solutions were considered. In particular, the regularity of weak solutions of (1.1) was considered in [3] and [6]. It proved in [3] and [6] the following results.

Theorem 1.1. *Let $f = h \in H'$ and $u \in H$ be a solution of* (1.1).
(*i*) *If $h \in L^\infty(\Omega)$, then $u \in C^\alpha(\bar{\Omega})$, where $0 < \alpha < 1$.*
(*ii*) *If $h \in C^\alpha(\Omega)$, then $u \in C^{2,\alpha}(\Omega) \cap C^\alpha(\bar{\Omega})$ for some $0 < \alpha < 1$.*

According to Theorem 1.1, if a weak solution u of (1.1) belongs to $L^\infty(\Omega)$, u will be Hölder continuous provided that f is continuous. We may also verify from the proof of Proposition 2.9 in [9] that $u \in C^\alpha_{loc}(\Omega)$ if $u \in L^\infty_{loc}(\Omega)$ and so on. In [4], it was shown for the half Laplacian, i.e. $s = \frac{1}{2}$, and the subcritical case that solutions are bounded in L^q norm for all $q < \infty$ by the Brézis-Kato argument [1]. It seems that the L^q bound in [4] cannot be simply improved to be a L^∞ bound by their argument. For the general case $s \neq \frac{1}{2}$, in [6] the authors suppose, among other things, that $f(t)$ is nondecreasing and that

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty.$$

They showed the extremal solution of (1.1) is bounded for the lower dimensional case, and for the higher dimensional case, the solution is possibly singular; see Theorem 1.4 in [6]. Actually, if $f(t) = e^t$, $n = 10$, the extremal solution is singular if $\Omega = B_1(0)$. We will prove in this paper that any weak solution of (1.1) is classical in subcritical and critical cases for the dimension $n \geq 2$. By a weak solution of (1.1) we mean a solution belonging to $H^s(\Omega)$, or alternatively, an $H^1_{0,L}(y^{1-2s})$ solution of (1.5). We suppose in this paper that $f \in C^{1,\gamma}(\mathbb{R})$, $\gamma > \max\{0, 1-2s\}$, satisfies

$$|f(t)| \leq C(1 + |t|^p), \tag{1.5}$$

where $1 < p \leq \frac{N+2s}{N-2s}$. By the Pohozaev identity, we know that $2^*_s = \frac{2N}{n-2s}$ is the critical exponent for $(-\Delta)^s$. The restriction of the exponent p is natural in the sense of variational methods. Our main result is the following.

Theorem 1.2. *Suppose condition* (1.5) *holds. Let $u \in H^s(\Omega)$ be a solution of* (1.1). *Then, $u \in L^\infty(\Omega)$ if $1 < p < \frac{N+2s}{N-2s}$; and $u \in L^\infty_{loc}(\Omega)$ if $p = \frac{N+2s}{N-2s}$. Consequently, $u \in C^{2,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$.*

By Theorem 1.2, we see that if the growth of $f(t)$ is not so fast at infinity, solutions of (1.1) are classical without the restriction on the dimension of the whole space $\mathbb{R}^N$. In the critical case, it is a Brézis-Kato type result for a fractional Laplacian. Our argument is the use of Moser-Nash iteration for problem (1.4). Although the weighted function y^{1-2s} is possibly singular or degenerates at $y = 0$, we still may establish an inverse Hölder inequality for $v(\cdot, 0) = u(\cdot)$, and we may iterate the inequality for u. The proof of Theorem 1.2 is given in section 2.

2. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. First, we recall a result in [6].

Lemma 2.1. *There exists a trace operator from $H^1_{0,L}(y^{1-2s})$ into $H^s_0(\Omega)$. Furthermore, the space H given by* (1.2) *is characterized by*

$$H = \{u = tr_\Omega v : v \in \ H^1_{0,L}(y^{1-2s})\}.$$

Lemma 2.1 was proved in [6]. In its proof, we see in fact that the mapping $tr : H^1_{0,L}(y^{1-2s}) \hookrightarrow H^s(\Omega)$ is continuous. Next, we have the Sobolev embedding theorem.

Lemma 2.2. *Given $s > 0$ and $\frac{1}{\sigma} \geq 1$ so that $\frac{1}{\sigma} \geq \frac{1}{2} - \frac{s}{N}$, the inclusion map $i : H^s(\Omega) \to L^\sigma(\Omega)$ is well defined and bounded. If the above inequality is strict, then the inclusion is compact.*

By Lemma 2.1 and Lemma 2.2, we know that there exists a continuous linear mapping from $H^1_{0,L}(y^{1-2s})$ to $L^q(\Omega)$ if $2 \leq q \leq \frac{2N}{N-2s}$.

Proof of Theorem 1.2. We first deal with the subcritical case $1 \leq p < \frac{N+2s}{N-2s}$.

Let $\bar{t} = |t| + k$ and $\bar{t}^+ = t^+ + k$. For $k > 0$ large, we have

$$|f(t)| \leq C|\bar{t}|^p,$$

where $1 < p < \frac{N+2s}{N-2s}$. Denote $v^+ = \max\{0, v\}$, $v^- = -\min\{0, v\}$. We deal only with v^+; it can be done in the same way for v^-. Let

$$\bar{v}^+_L = \begin{cases} \bar{v}^+ & \text{if} \quad \bar{v}^+ < L, \\ L & \text{if} \quad \bar{v}^+ \geq L. \end{cases} \tag{2.1}$$

For any $\varphi \in H^1_{0,L}(y^{1-2s})$, by (1.4),

$$\int_{\mathcal{C}} y^{1-2s}\nabla v\nabla\varphi\,dxdy = \int_{\Omega\times\{0\}} y^{1-2s}\frac{\partial v}{\partial y}\varphi\,dx = \int_{\Omega\times\{0\}} f(u)\varphi\,dx. \tag{2.2}$$

For $\beta > 1$ to be determined, we choose in (2.2) that

$$\varphi = \bar{v}^+(\bar{v}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1},$$

and since

$$\nabla\varphi = (\bar{v}_L^+)^{2(\beta-1)}\nabla\bar{v}^+ + 2(\beta-1)\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)-1}\nabla\bar{v}_L^+,$$

we obtain

$$\begin{aligned}&\int_{\mathcal{C}} y^{1-2s}\nabla v\big((\bar{v}_L^+)^{2(\beta-1)}\nabla\bar{v}^+ + 2(\beta-1)\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)-1}\nabla\bar{v}_L^+\big)\,dxdy\\ &= \int_{\mathcal{C}} y^{1-2s}\big((\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}^+|^2 + 2(\beta-1)\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}_L^+|^2\big)\,dxdy.\end{aligned} \tag{2.3}$$

Let $w_L = \bar{v}^+(\bar{v}_L^+)^{\beta-1}$ and then

$$\nabla w_L = (\bar{v}_L^+)^{\beta-1}\nabla v^+ + (\beta-1)\bar{v}^+(\bar{v}_L^+)^{\beta-2}\nabla v_L^+;$$

we deduce from (2.2) and (2.3) for $\beta > 1$ that

$$\begin{aligned}&\int_{\mathcal{C}} y^{1-2s}|\nabla w_L|^2\,dxdy\\ &\le C\beta\int_{\mathcal{C}} y^{1-2s}\nabla v\big((\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}^+|^2 + 2(\beta-1)(\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}_L^+|^2\big)\,dxdy\\ &= C\beta\int_{\Omega\times\{0\}}\big(\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1}\big)f(v)\,dxdy\\ &\le C\beta\int_{\Omega\times\{0\}}\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)}|f(v)|\,dxdy.\end{aligned} \tag{2.4}$$

By the assumption on f, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ so that

$$|f(t)| \le C|\bar{t}|^p \le \varepsilon|\bar{t}|^{2_s^*-1} + C_\varepsilon|\bar{t}|.$$

This implies that

$$\begin{aligned}&\int_{\Omega\times\{0\}}\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)}|f(v)|\,dx\\ &\le C\int_{\Omega\times\{0\}}|\bar{v}^+|^{p+1}(\bar{v}_L^+)^{\beta-1}\,dx\\ &\le C\int_{\Omega\times\{0\}}\big[\varepsilon(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)} + C_\varepsilon(\bar{v}_L^+)^{2(\beta-1)}(\bar{v}^+)^2\big]\,dx.\end{aligned} \tag{2.5}$$

By (2.4), (2.5) and the Sobolev embedding theorem,

$$\begin{aligned}&\Big(\int_{\Omega\times\{0\}}|w_L|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \le C\int_{\mathcal{C}} y^{1-2s}|\nabla w_L|^2\,dxdy\\ &\le C\beta\int_{\Omega\times\{0\}}\big[\varepsilon(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)} + C_\varepsilon(\bar{v}_L^+)^{2(\beta-1)}(\bar{v}^+)^2\big]\,dx.\end{aligned}$$

That is,

$$\begin{aligned}&\Big(\int_\Omega (\bar u^+(\bar u_L^+)^{\beta-1})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\\ &\le C\beta\int_\Omega \big[\varepsilon(\bar u^+)^{2_s^*}(\bar u_L^+)^{2(\beta-1)}+C_\varepsilon(\bar u_L^+)^{2(\beta-1)}(\bar u^+)^2\big]\,dx.\end{aligned}\tag{2.6}$$

We claim that $\bar u^+\in L^{\frac{(2_s^*)^2}{2}}(\Omega)$. Indeed, choosing $\beta=\frac{2_s^*}{2}$, we have

$$\begin{aligned}&\Big(\int_\Omega (\bar u^+(\bar u_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\\ &\le \int_\Omega \big[\varepsilon(\bar u^+)^{2_s^*}(\bar u_L^+)^{2_s^*-2}+C_\varepsilon(\bar u_L^+)^{2_s^*-2}(\bar u^+)^2\big]\,dx\\ &\le \varepsilon\Big(\int_\Omega (\bar u^+(\bar u_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\Big(\int_\Omega (\bar u^+)^{2_s^*}\,dx\Big)^{\frac{2_s^*-2}{2_s^*}}+C_\varepsilon\int_\Omega (\bar u_L^+)^{2_s^*-2}(\bar u^+)^2\,dx.\end{aligned}\tag{2.7}$$

Choosing $\varepsilon>0$ properly small, we obtain

$$\Big(\int_\Omega (\bar u^+(\bar u_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\le C\int_\Omega (\bar u_L^+)^{2_s^*-2}(\bar u^+)^2\,dx.$$

Let $L\to+\infty$. It yields

$$\Big(\int_\Omega (\bar u^+)^{\frac{(2_s^*)^2}{2}}\,dx\Big)^{\frac{2}{2_s^*}}\le C\int_\Omega (\bar u^+)^{2_s^*}\,dx<+\infty.\tag{2.8}$$

Now let $t=\frac{(2_s^*)^2}{2(2_s^*-2)}$; it follows that $\frac{2t}{t-1}<2_s^*$. We estimate the right-hand side of (2.6). By the Hölder inequality,

$$\begin{aligned}&\int_\Omega (\bar u^+)^{2_s^*}(\bar u_L^+)^{2(\beta-1)}\,dx\\ &\le\Big(\int_\Omega (\bar u^+)^{(2_s^*-2)t}\,dx\Big)^{\frac{1}{t}}\Big(\int_\Omega (\bar u^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}}\\ &\le C\Big(\int_\Omega (\bar u^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}}\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&\int_\Omega (\bar u^+)^{2\beta}\,dx\\ &\le\Big(\int_\Omega (\bar u^+)^{(2_s^*-2)t}\,dx\Big)^{\frac{1}{t}}\Big(\int_\Omega (\bar u^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}}\\ &\le C\Big(\int_\Omega (\bar u^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}}.\end{aligned}\tag{2.10}$$

We deduce from (2.6), (2.9) and (2.10) that

$$\Big(\int_\Omega (\bar u^+(\bar u_L^+)^{\beta-1})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\le C\beta^2\Big(\int_\Omega (\bar u^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}},$$

namely,

$$\Big(\int_\Omega (\bar u^+)^{\beta 2_s^*}\,dx\Big)^{\frac{1}{\beta}}\le C^{\frac{1}{\beta}}\beta^{\frac{2_s^*}{\beta}}\Big(\int_\Omega (\bar u^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{\frac{(t-1)2_s^*}{2t\beta}}.\tag{2.11}$$

Let $\tau\frac{2-s^*(t-1)}{2t}, \tau > 1$, and

$$I_i = \Big(\int_\Omega (\bar{u}^+)^{\frac{2\tau^i t}{t-1}}\,dx\Big)^{\frac{1}{\tau^i}}.$$

Iterating by (2.11), we obtain

$$\begin{aligned} I_{i+1} &\le C^{\frac{1}{\tau^{i+1}}}(\tau^{2_s^* i})^{\frac{1}{\tau^{i+1}}} I_i \\ &\le C^{\sum_{j=0}^{i+1}\frac{1}{\tau^j}} e^{\sum_{j=0}^{i+1} 2_s^* j\frac{1}{\tau^j}} I_0 \end{aligned} \tag{2.12}$$

and

$$I_0 = \int_\Omega (\bar{u}^+)^{\frac{2t}{t-1}}\,dx < \infty.$$

Next, we consider the critical case: $p = \frac{N+2s}{N-2s}$.

For any $x_0 \in \Omega$, we choose $R > 0$ small so that $B_{2R}(x_0) \subset \Omega$. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta(x) \equiv 1$ if $x \in B_R(x_0)$ and $\eta(x) \equiv 0$ if $x \notin B_{R+r}(x_0)$. Furthermore, $|\nabla\eta| \le \frac{2}{r}$, where $0 < r < R$ and $\eta \ge 0$ in Ω. Choose in (2.2) that

$$\varphi = \eta^2(\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1}),$$

and since

$$\begin{aligned} \nabla\varphi = \eta^2\big[&(\bar{v}_L^+)^{2(\beta-1)}\nabla\bar{v}^+ + 2(\beta-1)\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)-1}\nabla\bar{v}_L^+\big] \\ &+ 2\eta\nabla_x\eta(\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1}), \end{aligned}$$

we obtain

$$\begin{aligned} &\int_{\mathcal{C}} y^{1-2s}\eta^2\big((\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}^+|^2 + 2(\beta-1)(\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}_L^+|^2\big)\,dxdy \\ &+ \int_{\mathcal{C}} y^{1-2s}2\eta\nabla v\nabla_x\eta(\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1})\,dxdy \\ &= \int_{\Omega\times\{0\}} \eta^2(\bar{v}^+(\bar{v}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1})f(v)\,dxdy. \end{aligned} \tag{2.13}$$

Hence, by the assumption on f, $|f(t)| \le C|\bar{t}|^{2_s^*-1}$,

$$\begin{aligned} &\int_{\mathcal{C}} y^{1-2s}\eta^2\big((\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}^+|^2 + 2(\beta-1)(\bar{v}_L^+)^{2(\beta-1)}|\nabla\bar{v}_L^+|^2\big)\,dxdy \\ &\le \int_{\Omega\times\{0\}} \eta^2(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)}\,dx + \int_{\mathcal{C}} y^{1-2s}|\nabla_x\eta|^2(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy. \end{aligned} \tag{2.14}$$

Let $w_L = \eta\bar{v}^+(\bar{v}_L^+)^{\beta-1}$. We infer that

$$\begin{aligned} &\int_{\mathcal{C}} y^{1-2s}|\nabla w_L|^2\,dxdy \\ &\le \beta\int_{\Omega\times\{0\}} \eta^2(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)}\,dx + \beta\int_{\mathcal{C}} y^{1-2s}|\nabla_x\eta|^2(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy. \end{aligned} \tag{2.15}$$

By the Poincaré type inequality (see also [6]),

$$\int_{\mathcal{C}} y^{1-2s}|w_L|^2\,dxdy \le \int_{\mathcal{C}} y^{1-2s}|\nabla w_L|^2\,dxdy. \tag{2.16}$$

We derive from (2.15) and (2.16) that

$$\begin{aligned}
&\int_{B_R(x_0)\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy \\
&\le \beta\int_{\Omega\times\{0\}} \eta^2(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)}\,dx \\
&+ \beta r^{-2}\int_{(B_{R+r}(x_0)\setminus B_R(x_0))\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy.
\end{aligned} \tag{2.17}$$

Using the filling hole technique, we find that there exist θ, θ_1, $0<\theta,\theta_1<1$ so that

$$\begin{aligned}
&\int_{B_R(x_0)\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy \\
&\le \beta\theta_1\int_{\Omega\times\{0\}} \eta^2(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)}\,dx \\
&+ \theta_1\int_{B_{R+r}(x_0)\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy.
\end{aligned} \tag{2.18}$$

Let

$$J_i = \int_{B_{R+2^{-i}r}(x_0)\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy.$$

Iterating by (2.18), we obtain

$$\int_{B_R(x_0)\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy \le C\beta\int_{\Omega\times\{0\}} \eta^2(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)}\,dx \tag{2.19}$$

provided that J_0 is finite. Since $R>0$ can be chosen such that $0<R<R_0$ for some $R_0>0$, we may have from (2.17) and (2.19) that

$$\begin{aligned}
&\int_{B_R(x_0)\times(0,\infty)} y^{1-2s}(\bar{v}^+)^2(\bar{v}_L^+)^{2(\beta-1)}\,dxdy \\
&\le C\beta^2(1+r^{-2})\int_{\Omega\times\{0\}} \eta^2(\bar{v}^+)^{2_s^*}(\bar{v}_L^+)^{2(\beta-1)}\,dx.
\end{aligned} \tag{2.20}$$

Now, we claim that $\bar{u}^+\in L^{\frac{(2_s^*)^2}{2}}(\Omega)$. Indeed, choosing $\beta=\frac{2_s^*}{2}$, we have

$$\begin{aligned}
&\Big(\int_\Omega \eta^2(\bar{u}^+(\bar{u}_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \\
&\le \Big(\int_\Omega \eta^2(\bar{u}^+(\bar{u}_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\Big(\int_\Omega \eta^2(\bar{u}^+)^{2_s^*}\,dx\Big)^{\frac{2_s^*-2}{2_s^*}}.
\end{aligned} \tag{2.21}$$

Choosing $R=R_0>0$ small enough, we have

$$\int_{B_{2R}(x_0)} (\bar{u}^+)^{2_s^*}\,dx < \frac{1}{2}.$$

By (2.21) and letting $L\to+\infty$, we get

$$\Big(\int_{B_R(x_0)} (\bar{u}^+)^{\frac{(2_s^*)^2}{2}})\,dx\Big)^{\frac{2}{2_s^*}} \le C\int_\Omega (\bar{u}^+)^{2_s^*}\,dx < +\infty. \tag{2.22}$$

Now let $t = \frac{(2_s^*)^2}{2(2_s^*-2)}$, $B_i = B_{R+2^{-i}r}(x_0)$, and

$$I_i = \Big(\int_{B_i} (\bar{u}^+)^{\frac{pt\tau^i}{t-1}}\, dx \Big)^{\frac{1}{\tau^i}}.$$

Iterating I_i as before, we see that $\bar{u}^+ \in \mathbb{L}^\infty(B_R(x_0))$ for $0 < R < R_0$. Hence, $u^+ \in \mathbb{L}^\infty(B_R(x_0))$ for $0 < R < R_0$.

The proof of Theorem 1.2 is complete. □

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