

TRIVIALITY OF SOME REPRESENTATIONS OF $\text{MCG}(S_g)$ IN $GL(n, \mathbb{C})$, $\text{Diff}(S^2)$ AND $\text{Homeo}(\mathbb{T}^2)$

JOHN FRANKS AND MICHAEL HANDEL

(Communicated by Daniel Ruberman)

ABSTRACT. We show the triviality of representations of the mapping class group of a genus g surface in $GL(n, \mathbb{C})$, $\text{Diff}(S^2)$ and $\text{Homeo}(\mathbb{T}^2)$ when appropriate restrictions on the genus g and the size of n hold. For example, if S is a surface of finite type with genus $g \geq 3$ and $\phi : \text{MCG}(S) \rightarrow GL(n, \mathbb{C})$ is a homomorphism, then ϕ is trivial provided $n < 2g$. We also show that if S is a closed surface with genus $g \geq 7$, then every homomorphism $\phi : \text{MCG}(S) \rightarrow \text{Diff}(S^2)$ is trivial and that if $g \geq 3$, then every homomorphism $\phi : \text{MCG}(S) \rightarrow \text{Homeo}(\mathbb{T}^2)$ is trivial.

1. INTRODUCTION AND STATEMENT OF RESULTS

If S is a surface of genus g with a (perhaps empty) finite set of punctures and boundary components, we will denote by $\text{MCG}(S)$ the group of isotopy classes of homeomorphisms of S which pointwise fix the boundary and punctures of S . In case the boundary is non-empty the isotopies defining $\text{MCG}(S)$ must pointwise fix the boundary. This is a well-studied object (see the recent book by Farb and Margalit [4] for an extensive survey of known results). In this article we show the triviality of representations of $\text{MCG}(S)$ in $GL(n, \mathbb{C})$, $\text{Diff}(S^2)$ and $\text{Homeo}(\mathbb{T}^2)$ under various additional hypotheses.

For a closed surface S of genus g there is a natural representation of $\text{MCG}(S)$ into the group of symplectic matrices of size $n = 2g$ obtained by taking the induced action on $H_1(S, \mathbb{R})$. It is natural to ask if there are linear representations of lower dimension. This is one of the questions we address.

Theorem 1.1. *Suppose that S is a genus g surface of finite type (perhaps with boundary and punctures). Suppose also that $g \geq 2$, that $n < 2g$, and that*

$$\phi : \text{MCG}(S) \rightarrow GL(n, \mathbb{C})$$

is a homomorphism. Then if $g = 2$, the image of ϕ is finite cyclic, and if $g \geq 3$, ϕ is trivial.

This result improves a result of L. Funar [5] which shows that homomorphisms from the genus $g \geq 1$ mapping class group to $GL(n, \mathbb{C})$ have finite image provided $n \leq \sqrt{g+1}$. Funar's main result applies to a more general context. The application to $\text{MCG}(S)$ makes use of the fact, due to Bridson [2], that $\text{MCG}(S)$ has a fixed

Received by the editors July 26, 2011 and, in revised form, November 14, 2011.
2010 *Mathematics Subject Classification*. Primary 20F65, 37E30; Secondary 20F29.
The first author was supported in part by NSF grant DMS0099640.
The second author was supported in part by NSF grant DMS0103435.

point whenever it acts by semisimple isometries on a complete $CAT(0)$ space of dimension less than the genus of S .

It has come to our attention that our Theorem 1.1 has also been proved by Mustafa Korkmaz [7] based on an earlier version of this article which proved a weaker result.

Since the induced map on homology gives a non-trivial homomorphism from $MCG(S)$ to $GL(2g, \mathbb{C})$ for a closed surface S of genus g , it is clear that one cannot improve on the restriction that $n < 2g$.

In light of the result of Morita which asserts that if S is a closed surface of genus $g \geq 5$ there is no lifting of $MCG(S)$ to $\text{Diff}(S)$, i.e. no homomorphism $\phi : MCG(S) \rightarrow \text{Diff}(S)$, with $\phi(\sigma)$ a representative of the isotopy class σ , it is natural to ask about the existence of homomorphisms $\phi : MCG(S) \rightarrow \text{Diff}(S')$, where S' is a surface with a different genus, g' . An immediate corollary of a recent result of Aramayona and Souto, [1], shows that in some ranges of values for g and g' any such homomorphism is trivial.

More precisely they show:

Theorem 1.2 (Aramayona and Souto [1]). *Suppose that S and S' are closed surfaces, where S has genus $g \geq 6$ and S' has genus less than $2g - 1$. Then any non-trivial homomorphism $\phi : MCG(S) \rightarrow MCG(S')$ is an isomorphism and in particular S and S' have the same genus.*

A nearly immediate corollary is that representations of $MCG(S)$ into the group $\text{Homeo}(S')$ of orientation preserving homeomorphisms of a surface S' are necessarily trivial for many choices of the genus of S and S' . More precisely, we have the following.

Corollary 1.3. *Suppose that S and S' are closed surfaces, where S has genus $g \geq 6$ and S' has genus g' , with $1 < g' < 2g - 1$. Then every homomorphism $\phi : MCG(S) \rightarrow \text{Homeo}(S')$ is trivial.*

Proof. By Theorem 1.2 the image of the map to $MCG(S')$ induced by ϕ is trivial, so the image of ϕ lies in $\text{Homeo}_0(S')$, the subgroup of homeomorphisms isotopic to the identity. If $g \in MCG(S)$ has finite order, then $\phi(g) \in \text{Homeo}_0(S')$ has finite order. Since S' has genus at least two and $\phi(g)$ is both finite order and isotopic to the identity, it follows that $\phi(g) = id$. Since $MCG(S)$ is generated by elements of finite order (see Theorem 7.16 of [4]), we conclude that ϕ is trivial. \square

This motivates the question of whether homomorphisms from $MCG(S)$ to $\text{Homeo}(S')$ or $\text{Diff}(S')$ are trivial when $S' = S^2$ or \mathbb{T}^2 .

A classical result of Nielsen states that if S has finite negative Euler characteristic and is not closed, then there are injective homomorphisms from $MCG(S)$ to $\text{Homeo}(S^1)$, and hence by coning such examples it follows that there are also injective homomorphisms from $MCG(S)$ to $\text{Homeo}(\mathbb{D}^2)$ and $\text{Homeo}(S^2)$. Similarly, taking the diagonal action we obtain an injective homomorphism from $MCG(S)$ to $\text{Homeo}(\mathbb{T}^2)$. We therefore assume, in the following two results, that our surfaces are closed.

Theorem 1.4. *Suppose that S is a closed surface with genus $g \geq 7$. Then every homomorphism $\phi : MCG(S) \rightarrow \text{Diff}(S^2)$ is trivial.*

Theorem 1.5. *Suppose that S is a closed oriented surface with genus $g \geq 3$. Then if $n = 1, 2$, every homomorphism $\phi : MCG(S) \rightarrow \text{Homeo}(\mathbb{T}^n)$ is trivial.*

A special case of this result was previously shown by K. Parwani [9]. He showed that actions on the circle (i.e. $n = 1$) are trivial provided they are C^1 and the genus $g \geq 6$.

In the case of $\text{Diff}(S^2)$ we can deal with punctured surfaces S provided the number of punctures is bounded by $2g - 10$ where g is the genus.

Theorem 1.6. *Suppose that S is an oriented surface with genus g with k punctures and no boundary. If $g \geq 7$ and $0 \leq k \leq 2g - 10$, then every homomorphism $\phi : MCG(S) \rightarrow \text{Diff}(S^2)$ is trivial.*

Remark 1.7. Note that by the coning construction described above, if S is a surface of genus ≥ 3 with any number of punctures, $MCG(S)$ acts faithfully by homeomorphisms on S^2 . But Theorem 1.6 asserts that if the genus g is ≥ 7 and the number of punctures is $\leq 2g - 10$, then any action by diffeomorphisms is trivial.

2. CRITERIA FOR TRIVIALITY

In this section we recall or establish a few elementary properties of $MCG(S)$ which we will need. In particular, Proposition 2.6 provides useful elementary criteria which imply that a homomorphism from $MCG(S)$ to a group G is trivial.

Proposition 2.1. *Let S denote a surface of genus $g \geq 2$ which has finitely many (perhaps 0) punctures and/or boundary components. Then the abelianization of $MCG(S)$ is trivial if $g \geq 3$, and $\mathbb{Z}/10\mathbb{Z}$ if $g = 2$.*

This result in the case $g \geq 3$ is due to J. Powell [10]. A proof can be found as Theorem 5.1 of [8].

Lemma 2.2. *If G is a perfect group and H is solvable, then any homomorphism $\phi : G \rightarrow H$ is trivial.*

Proof. Since G is perfect, any homomorphism from G to an abelian group is trivial. Since H is solvable, we can inductively define H_k by $H_0 = H$ and $H_k = [H_{k-1}, H_{k-1}]$, and then there exists n such that H_n is the trivial group. Since H_k/H_{k+1} is abelian, any homomorphism from G to it is trivial. Induction on k using the exact sequence $1 \rightarrow H_{k+1} \rightarrow H_k \rightarrow H_k/H_{k+1} \rightarrow 1$ shows that $\phi(G) \subset H_k$ for all k , and hence ϕ is trivial. \square

The following lemmas are well known; see Sections 1.3.2 -1.3.3 and 4.4.2-4.4.3 of [4].

Lemma 2.3. *Let S be an orientable surface of finite type (perhaps with boundary and punctures) which has genus $g \geq 2$. Then $MCG(S)$ acts transitively on*

- (1) *the set of isotopy classes of non-separating simple closed curves in S ,*
- (2) *the set \mathcal{P} of ordered pairs of disjoint simple closed curves in S whose union does not separate S ,*
- (3) *the set of ordered pairs of simple closed curves in S that intersect once.*

Corollary 2.4. *Suppose S is as in the previous lemma and α and β are simple closed curves in S . If α and β intersect transversely in a single point, there is a simple closed curve γ such that both (α, γ) and (β, γ) are in \mathcal{P} . Alternatively, if $(\alpha, \beta) \in \mathcal{P}$, then there exists a simple closed curve γ which intersects both α and β transversely in a single point.*

Proof. It is straightforward to construct one example of a pair α' and β' which intersects in a single point and a γ' such that both (α', γ') and (β', γ') are in \mathcal{P} . By Lemma 2.3 there is a homeomorphism h which takes α' to α and β' to β . Letting $\gamma = h(\gamma')$ proves the first assertion.

Similarly, it is straightforward to construct one example of a pair $(\alpha', \beta') \in \mathcal{P}$ and a γ' which intersects both α' and β' transversely in a single point. By Lemma 2.3 there is a homeomorphism g which takes α' to α and β' to β . Letting $\gamma = g(\gamma')$ proves the second assertion. \square

For an essential simple closed curve α in S , we denote the *left Dehn twist about* α by T_α . Note that by Lemma 2.3 (1), T_α is conjugate to T_β for all non-separating simple closed curves α, β in S .

Lemma 2.5. *Suppose S is a surface of finite type, perhaps with boundary or punctures. There is a finite set \mathcal{C}_0 of non-separating simple closed curves in S such that $\{T_\alpha : \alpha \in \mathcal{C}_0\}$ generates $\text{MCG}(S)$ and such that for each $\alpha, \beta \in \mathcal{C}_0$, either*

- (1) $(\alpha, \beta) \in \mathcal{P}$ or
- (2) α and β intersect transversely in a single point or
- (3) there is $\gamma \in \mathcal{C}_0$ which intersects both α and β transversely in a single point.

For a proof of this result see the book of Farb and Margalit [4] and, in particular, Figure 4.10 and the discussion after Corollary 4.16. Note that the three possibilities are not mutually exclusive. Indeed, Corollary 2.4 implies that if (1) holds, there is a simple closed curve γ (not necessarily in \mathcal{C}_0) which intersects both α and β transversely in a single point.

Proposition 2.6. *Suppose that S is an orientable surface of finite type (perhaps with boundary and punctures) with genus $g \geq 2$ and that $\phi : \text{MCG}(S) \rightarrow G$ is a homomorphism to some group G . Suppose that one of the following holds:*

- (1) there are disjoint simple closed curves α_0 and β_0 in S whose union does not separate (i.e. $(\alpha_0, \beta_0) \in \mathcal{P}$) and $\phi(T_{\alpha_0}) = \phi(T_{\beta_0})^{\pm 1}$ or
- (2) there are simple closed curves γ_0 and δ_0 in S intersecting transversely in one point such that $\phi(T_{\gamma_0})$ commutes with $\phi(T_{\delta_0})$.

Then the image of ϕ is a finite cyclic group. If $g \geq 3$, then ϕ is trivial.

Proof. We first prove the result when (1) holds. Let \mathcal{C}_0 be the set of simple closed curves described in Lemma 2.5, so Dehn twists around elements of \mathcal{C}_0 form a set of generators of $\text{MCG}(S)$. We want to show that Dehn twists around any two elements of \mathcal{C}_0 have the same image under ϕ or images which are inverses in G . We suppose $\alpha, \beta \in \mathcal{C}_0$ and consider the three possibilities enumerated in Lemma 2.5.

If $(\alpha, \beta) \in \mathcal{P}$, then by Lemma 2.3 (2) there exists $\bar{f} \in \text{MCG}(S)$ such that $T_\alpha = \bar{f}T_{\alpha_0}\bar{f}^{-1}$ and $T_\beta = \bar{f}T_{\beta_0}\bar{f}^{-1}$. Thus

$$\begin{aligned} \phi(T_\alpha) &= \phi(\bar{f}T_{\alpha_0}\bar{f}^{-1}) \\ &= \phi(\bar{f})\phi(T_{\alpha_0})\phi(\bar{f}^{-1}) \\ &= \phi(\bar{f})\phi(T_{\beta_0})^{\pm 1}\phi(\bar{f}^{-1}) \\ &= \phi(\bar{f}T_{\beta_0}\bar{f}^{-1})^{\pm 1} \\ &= \phi(T_\beta)^{\pm 1}. \end{aligned}$$

This proves that $\phi(T_\alpha) = \phi(T_\beta)^{\pm 1}$ when $(\alpha, \beta) \in \mathcal{P}$. Note we used only the fact that $(\alpha, \beta) \in \mathcal{P}$ and not the fact that $\alpha, \beta \in \mathcal{C}_0$.

Next suppose that α and β intersect in a single point. By Corollary 2.4 there exists a simple closed curve δ (not necessarily in \mathcal{C}_0) such that $(\alpha, \delta), (\delta, \beta) \in \mathcal{P}$. Thus, by what we showed above, $\phi(T_\alpha) = \phi(T_\delta)^{\pm 1}$ and $\phi(T_\beta) = \phi(T_\delta)^{\pm 1}$, so we have the desired result in case (2) of Lemma 2.5.

Finally, for case (3) of Lemma 2.5, suppose there exists $\gamma \in \mathcal{C}_0$ which intersects both α and β in a single point. Then by what we have just shown, $\phi(T_\alpha) = \phi(T_\gamma)^{\pm 1}$ and $\phi(T_\beta) = \phi(T_\gamma)^{\pm 1}$.

Hence in all three possibilities enumerated in Lemma 2.5 we have $\phi(T_\alpha) = \phi(T_\beta)^{\pm 1}$. Since \mathcal{C}_0 is a generating set, the image of ϕ is contained in a cyclic subgroup of G . Hence ϕ factors through the abelianization of $MCG(S)$, and the result follows from Proposition 2.1. This proves the result when (1) holds.

To prove the result when (2) holds, we observe that there are generators of $MCG(S)$ consisting of Dehn twists about non-separating simple closed curves, each pair of which are either disjoint or intersect transversely in a single point (see Corollary 4.16 and ff. of [4]). It suffices to show that the ϕ -images of these generators commute, for then $\phi(MCG(S))$ is abelian and ϕ factors through the abelianization of $MCG(S)$, so the result follows from Proposition 2.1.

It is obvious that Dehn twists about disjoint curves commute, so it suffices to show that if γ and δ intersect transversely in a single point, then $\phi(T_\gamma)$ and $\phi(T_\delta)$ commute. By hypothesis this is true for one pair γ_0 and δ_0 which intersects in a single point. By Lemma 2.3 (3) there exists $\bar{f} \in MCG(S)$ such that $T_\gamma = \bar{f}T_{\gamma_0}\bar{f}^{-1}$ and $T_\delta = \bar{f}T_{\delta_0}\bar{f}^{-1}$. From this it follows easily that $\phi(T_\gamma)$ and $\phi(T_\delta)$ commute. \square

Remark 2.7. Suppose γ and δ are simple closed curves which intersect transversely in one point. Then a standard relation in the mapping class group (see Proposition 3.16 of [4]) says

$$T_\gamma T_\delta T_\gamma = T_\delta T_\gamma T_\delta.$$

From this it is clear that the hypothesis of part (2) of Proposition 2.6, that $\phi(T_\gamma)$ and $\phi(T_\delta)$ commute, is equivalent to assuming $\phi(T_\gamma) = \phi(T_\delta)$.

3. LINEAR REPRESENTATIONS OF $MCG(S)$

In this section we prove the first of our main theorems.

If $\lambda \in \mathbb{C}$ is an eigenvalue of $L \in GL(n, \mathbb{C})$, then

$$\ker(L - \lambda I) \subset \ker(L - \lambda I)^2 \subset \dots \subset \ker(L - \lambda I)^k$$

is an increasing sequence of subspaces beginning with the *eigenspace* $\ker(L - \lambda I)$ and ending with the *generalized eigenspace* $\ker(L - \lambda I)^k$ for k large. If L' commutes with L , then L' preserves each $\ker(L - \lambda I)^j$.

Theorem 1.1. *Suppose that S is a genus g surface of finite type (perhaps with boundary and punctures). Suppose also that $g \geq 2$, that $n < 2g$, and that*

$$\phi : MCG(S) \rightarrow GL(n, \mathbb{C})$$

is a homomorphism. Then if $g = 2$, the image of ϕ is finite cyclic, and if $g \geq 3$, ϕ is trivial.

Proof. It suffices to show that H , the image of ϕ , is abelian, because in that case ϕ factors through the abelianization of $\text{MCG}(S)$ which is trivial if $g \geq 3$ and finite cyclic if $g \leq 2$ by Proposition 2.1.

The proof that H is abelian is by double induction, first on g and then on n . For this induction (proving only that H is abelian, not that it is cyclic) we may start the induction with $g = 1$ and $n = 1$. Since $GL(1, \mathbb{C})$ is abelian, the fact that H is abelian is obvious if $n = 1$ and hence also if $g = 1$, since $n < 2g$. It therefore suffices to prove that H is abelian, assuming that $g \geq 2$ and that the following inductive hypothesis holds: If N is a genus $g' \geq 1$ surface of finite type (perhaps with boundary and punctures) and if either $g' < g$ and $m < 2g'$ or if $g' = g$ and $m < n < 2g$, then the image of any homomorphism $\text{MCG}(N) \rightarrow GL(m, \mathbb{C})$ is abelian.

Let α and β be non-separating simple closed curves in S that intersect transversely in one point, let $M = M(\alpha, \beta)$ be a genus one subsurface which is a regular neighborhood of $\alpha \cup \beta$ and let $S' = S'(\alpha, \beta)$ be the genus $g_0 = g - 1$ subsurface which is the closure of $S \setminus M$. We consider $\text{MCG}(S')$ to be a subgroup of $\text{MCG}(S)$ via the embedding induced by the inclusion of S' in S . Note that $\text{MCG}(S')$ lies in the intersection of the centralizers of T_α and T_β . Letting $L_\alpha = \phi(T_\alpha)$, $L_\beta = \phi(T_\beta)$ and $H' = \phi(\text{MCG}(S'))$, every element of H' commutes with both L_α and L_β . \square

Sublemma 3.1. *Assume notation and inductive hypothesis as above. If at least one of the following conditions is satisfied, then H is abelian:*

- (1) $\mathbb{C}^n = V_1 \oplus V_2$, where V_i is an H' -invariant subspace with dimension at most $n - 2$.
- (2) L_α has a unique eigenvalue and there exists an H' -invariant subspace V of \mathbb{C}^n with dimension $2 \leq d \leq n - 2$.
- (3) L_α has a unique eigenvalue and there is an H -invariant subspace V of \mathbb{C}^n with dimension $1 \leq d \leq n - 1$.

Proof. It suffices to show that H' is abelian. This is because S' has genus at least 1, and hence there exist simple closed curves γ, δ in S' intersecting transversely in a single point. Then $\phi(T_\gamma)$ and $\phi(T_\delta)$ commute since they lie in $H' = \phi(\text{MCG}(S'))$, which is abelian. Therefore hypothesis (2) of Proposition 2.6 is satisfied, and that proposition implies that $H = \phi(\text{MCG}(S))$ is abelian.

For case (1), identify $\text{Aut}(V_i)$ with $GL(d_i, \mathbb{C})$ where d_i is the dimension of V_i , and let $\phi_i : \text{MCG}(S') \rightarrow GL(d_i, \mathbb{C})$ be the homomorphism induced by ϕ restricted to V_i . Since $n - 2 < 2g - 2 = 2g_0$, the inductive hypothesis implies that the image of ϕ_i , and hence H' , is abelian.

For case (2) let λ be the unique eigenvalue L_α and note that $g_0 \geq 2$ because $2g_0 = 2g - 2 > n - 2 \geq 2$. Identify $\text{Aut}(V)$ with $GL(d, \mathbb{C})$ and let $\phi_1 : \text{MCG}(S') \rightarrow GL(d, \mathbb{C})$ be the homomorphism induced by ϕ . Similarly, identify $\text{Aut}(\mathbb{C}^n/V)$ with $GL(n - d, \mathbb{C})$ and let $\phi_2 : \text{MCG}(S') \rightarrow GL(n - d, \mathbb{C})$ be the homomorphism induced by ϕ . Since $d, n - d \leq n - 2 < 2g - 2 = 2g_0$, the inductive hypothesis implies that $\phi_1(\eta)$ and $\phi_2(\eta)$ have finite order, and hence (considering the Jordan canonical form) are diagonalizable, for all $\eta \in \text{MCG}(S')$. If $\phi(\eta)$ is conjugate to L_α , then λ is its unique eigenvalue, so $\phi_1(\eta) = \lambda I_d$ and $\phi_2(\eta) = \lambda I_{n-d}$. In other words, there is a basis for \mathbb{C}^n , with respect to which any such $\phi(\eta)$ is represented by a matrix that differs from λI only on the on upper right $d \times (n - d)$ block. It is easy to check that any two such matrices commute. Since $\text{MCG}(S')$ has a generating set consisting of

elements that are conjugate to T_α , H' has a generating set consisting of elements that are conjugate to L_α and H' is abelian.

The argument for case (3) is the same as that for case (2), except that it is applied directly to H and not H' . We can apply this to dimension 1 and $n - 1$ because S has genus g and not $g - 1$. \square

With Sublemma 3.1 in hand we prove Theorem 1.1 by a case analysis.

Case 1 (L_α has three distinct eigenvalues or has two distinct eigenvalues whose generalized eigenspaces do not have dimension 1 or $n - 1$). Since each generalized eigenspace of L_α is H' -invariant and since \mathbb{C}^n is the product of the generalized eigenspaces of L_α , this case follows from Sublemma 3.1(1).

Case 2 (L_α has a unique eigenvalue λ). The proof in this case requires a subcase analysis depending on the dimension r of the eigenspace W_α . We will make use of the fact that L_β is conjugate to L_α (and so has unique eigenvalue λ and eigenspace W_β) and that both W_α and W_β are H' -invariant.

Case 2A ($r \neq 1, n - 1$). If $r = n$, then $L_\alpha = \lambda I$. Since H is generated by conjugates of L_α , each of which is multiplication by a scalar, it is abelian. If $2 \leq r \leq n - 2$, then Sublemma 3.1(2) applied to $V = W_\alpha$ implies that H is abelian.

Case 2B ($W_\alpha = W_\beta$). For any pair α', β' of non-separating simple closed curves intersecting in a point, $W_{\alpha'} = W_{\beta'}$ by Lemma 2.3(3). Corollary 2.4 implies that if a non-separating simple closed curve γ is disjoint from α , then there is a non-separating simple closed curve δ that intersects both α and γ in a single point. Applying the above result with the pair (α, β) replaced first by (α, δ) and then by (δ, γ) , we conclude that $W_\alpha = W_\delta = W_\gamma$. It follows that $V = W_\alpha$ is an eigenspace for a generating set of H and in particular is H -invariant. Sublemma 3.1(3) completes the proof.

Case 2C ($r = 1$ and $W_\alpha \neq W_\beta$). If $n \geq 4$, then Sublemma 3.1(2) applied to $V = W_\alpha \oplus W_\beta$ completes the proof. If $n = 2$, then $\mathbb{C}^2 = W_\alpha \times W_\beta$. Since W_α and W_β are H' -invariant, there is a basis of \mathbb{C}^2 with respect to which each element of H' is diagonal, so H' is abelian.

Suppose then that $n = 3$ and hence that there is a basis for \mathbb{C}^3 with respect to which L_α has matrix $\lambda I + N$, where N is the 3×3 matrix equal to $E_{12} + E_{23}$, where E_{ij} is the matrix all of whose entries are 0 except the ij^{th} , which is 1. The subspaces $\ker(L - \lambda I) \subset \ker(L - \lambda I)^2 \subset \ker(L - \lambda I)^3 = \mathbb{C}^3$ are all distinct and H' -invariant. The matrices for elements of H' are therefore upper triangular in the same basis for which the matrix of L is $\lambda I + N$. There are generators of H' that are conjugate to L_α and so have unique eigenvalue λ . The diagonal entries for the matrices for these generators are all λ . A straightforward computation using the fact that these generators commute with L_α shows that their matrices have the form $\lambda I + aN + bN^2$ but the group generated by matrices of this form is abelian since all elements are polynomials in N . This proves that H' is abelian.

Case 2D ($r = n - 1$ and $W_\alpha \neq W_\beta$). If $n \geq 4$, then Sublemma 3.1(2) applied to $V = W_\alpha \cap W_\beta$ completes the proof. If $n = 2$, then $r = 1$ and Case 2C applies.

Suppose then that $n = 3$ and $r = 2$.

The image $U_\alpha = (L_\alpha - \lambda I)(\mathbb{C}^3)$ of $L_\alpha - \lambda I$ is contained in $W_\alpha = \ker(L_\alpha - \lambda I)$ because $(L_\alpha - \lambda I)^2 = 0$; since W_α has dimension 2, U_α has dimension 1. Moreover,

if W is any two-dimensional L_α -invariant subspace of \mathbb{C}^3 , then $(L_\alpha - \lambda I)(W) \subset W$ and also $(L_\alpha - \lambda I)(W) \subset U_\alpha$. Indeed, $(L_\alpha - \lambda I)(W) = U_\alpha$ unless $W = W_\alpha$. Hence U_α is a one-dimensional subspace of every L_α -invariant two-dimensional subspace of \mathbb{C}^3 . We conclude that if $W_1 \neq W_2$ are two-dimensional L_α -invariant subspaces, then $U_\alpha = W_1 \cap W_2$.

Suppose that α' is a simple closed non-separating curve in S . If α' is disjoint from α , then $L_{\alpha'} = \phi(T_{\alpha'})$ commutes with L_α and so preserves U_α . We claim that the same is true if α' intersects α in a single point. By Lemma 2.3(3) we may assume that $\alpha' = \beta$. Choose simple closed non-separating curves $\gamma, \delta \subset S'$ that intersect in a single point and let W_γ and W_δ be the unique eigenspaces for $L_\delta = \phi(T_\delta)$ and $L_\gamma = \phi(T_\gamma)$. Both W_γ and W_δ are L_α -invariant. Lemma 2.3(3) implies that $W_\gamma \neq W_\delta$ and hence that $U_\alpha = W_\gamma \cap W_\delta$. For the same reason, $U_\beta = W_\gamma \cap W_\delta$. This proves that $U_\beta = U_\alpha$ and hence that U_β is L_α -invariant, as claimed.

We have now shown that there is a generating set for H that preserves U_α , so Sublemma 3.1(3) completes the proof.

Case 3 (L_α has two eigenvalues; the dimensions of the generalized eigenspaces are 1 and $n - 1$). Let U_α and W_α be the dimension 1 and dimension $n - 1$ generalized eigenspaces of L_α respectively; define U_β and W_β similarly. All of these spaces are H' -invariant. The argument given for Case 2B applies in this context if either $U_\alpha = U_\beta$ or $W_\alpha = W_\beta$. We may therefore assume that $U_\alpha \neq U_\beta$ and $W_\alpha \neq W_\beta$. Corollary 2.4 and Lemma 2.3(2) imply that if γ is a non-separating simple closed curve that is disjoint from α and whose union with α does not separate S , then $U_\alpha \neq U_\gamma$ and $W_\alpha \neq W_\gamma$ (where U_γ and W_γ are defined in the obvious way).

Case 3A ($n = 2$ or 3). If $n = 2$, then U_α and W_α are one-dimensional and H' -invariant, so there is a basis with respect to which the matrix for each element of H' is diagonal. Thus H' is abelian, and we are done.

If $n = 3$, choose simple closed non-separating curves $\gamma, \delta \subset S'$ that intersect in a point. Since U_α is a one-dimensional L_γ -invariant subspace that is not equal to U_γ , it must be contained in W_γ . The same argument implies that $U_\beta \subset W_\gamma$ and hence that W_γ is spanned by U_α and U_β . For the same reason, W_δ is spanned by U_α and U_β . But then $W_\gamma = W_\delta$, in contradiction to Lemma 2.3(3).

Case 3B ($g > 3$, $n \geq 4$). If $g > 3$, then the induction hypothesis implies that H' acts trivially on $W_\alpha \cap W_\beta$. Hence if γ is a non-separating simple closed curve in S' , the eigenvalue of L_γ with multiplicity $(n - 1)$ must be 1 since the dimension of $W_\alpha \cap W_\beta$ is > 1 . Therefore the eigenvalue of L_α corresponding to the eigenspace W_α is 1 and the determinant of L_α is the eigenvalue corresponding to U_α ; in particular, the determinant of L_α is not 1. But postcomposing ϕ with the determinant defines a homomorphism from $\text{MCG}(S)$ to \mathbb{C}^* , which must be trivial. This contradiction shows that this case cannot happen.

Case 3C ($g \leq 3$, $n = 4$ or 5). We may assume that $g = 3$ because $2g > n$. Varying our notation slightly, we choose three pairs of simple closed curves (α_i, β_i) , $1 \leq i \leq 3$, such that α_i and β_i intersect transversely in a single point and $\alpha_i \cup \beta_i$ is disjoint from $\alpha_j \cup \beta_j$ for $i \neq j$ and such that no two from among these six curves separate S .

We apply the following observation twice: if γ and δ are disjoint simple closed curves whose union does not separate, then $U_\gamma \subset W_\delta$. This follows from the fact that L_γ and L_δ commute and $U_\gamma \neq U_\delta$ is L_δ -invariant and so is contained in an eigenspace of L_δ .

By hypothesis, U_{α_1} and U_{β_1} span a two-dimensional subspace of \mathbb{C}^n . By the above observation, $U_{\alpha_1}, U_{\beta_1} \subset W_{\alpha_2}$, and hence the span V_0 of $U_{\alpha_1}, U_{\beta_1}$ and U_{α_2} is three-dimensional.

Applying the observation above a second time, we note that the span of $U_{\alpha_1}, U_{\beta_1}$ and U_{α_2} is contained in both W_{α_3} and W_{β_3} and hence in their intersection. We conclude that the dimension of $W_{\alpha_3} \cap W_{\beta_3}$ is three, since $n \leq 5$ and $W_{\alpha_3} \neq W_{\beta_3}$ implies it cannot be greater than three. Therefore $V_0 = W_{\alpha_3} \cap W_{\beta_3}$.

It is now straightforward to extend the six curves above to a set of non-separating simple closed curves such that Dehn twists about these curves form a generating set for $MCG(S)$ and each of these curves is either disjoint from $\alpha_1 \cup \beta_1 \cup \alpha_2$ or disjoint from $\alpha_3 \cup \beta_3$ (see Corollary 4.16 and ff. of [4]). It follows from the two descriptions of V_0 (as $W_{\alpha_3} \cap W_{\beta_3}$ and the span of $U_{\alpha_1}, U_{\beta_1}$ and U_{α_2}) that if γ is any one of these generators, then V_0 is L_γ -invariant. Hence $V_0 \subset \mathbb{C}^5$ is H invariant, and Sublemma 3.1(3) completes the proof. \square

4. HOMOMORPHISMS TO $\text{Homeo}(\mathbb{T}^2)$

Lemma 4.1. *If $n = 1$ or 2 and G is a finite group which acts on \mathbb{T}^n by homeomorphisms isotopic to the identity, then the action factors through an abelian group which acts freely on \mathbb{T}^n .*

Proof. We give the proof only for $n = 2$, as the proof for $n = 1$ is essentially identical. Suppose $\phi : G \rightarrow \text{Homeo}(\mathbb{T}^2)$ is a homomorphism. If $g \in G$ and $\phi(g) \in \text{Homeo}(\mathbb{T}^2)$ has a fixed point, then $\phi(g) = id$, since every finite order homeomorphism of the torus which is isotopic to the identity and has a fixed point must be the identity (see [6] or [3]). It follows that if $A = G/\ker(\phi)$, then A acts freely on \mathbb{T}^2 .

Let $\bar{\phi} : A \rightarrow \text{Homeo}(\mathbb{T}^2)$ be the injective homomorphism induced by ϕ and let M be the quotient of \mathbb{T}^2 by this action. Since A acts freely, M is a closed surface, and the projection $p : \mathbb{T}^2 \rightarrow M$ is a k -fold covering for some k . The Euler characteristic of M is $1/k$ times the Euler characteristic of \mathbb{T}^2 and so is zero, proving that M is homeomorphic to a torus and hence has abelian fundamental group. Since A is isomorphic to the group of covering translations for $p : \mathbb{T}^2 \rightarrow M$, A is isomorphic to a quotient of $\pi_1(M)$ and hence is abelian. \square

We can now provide the proof of our result about homomorphisms to $\text{Homeo}(\mathbb{T}^2)$.

Theorem 1.5. *Suppose that S is a closed oriented surface with genus $g \geq 3$. Then if $n = 1, 2$, every homomorphism $\phi : MCG(S) \rightarrow \text{Homeo}(\mathbb{T}^n)$ is trivial.*

Proof. We give the proof only for $n = 2$, as the proof for $n = 1$ is essentially identical. Observe that $MCG(\mathbb{T}^2) \cong SL(2, \mathbb{Z})$, so by Theorem 1.1 the map from $MCG(S)$ to $MCG(\mathbb{T}^2)$ induced by ϕ is trivial. Hence the image of ϕ lies in $\text{Homeo}_0(\mathbb{T}^2)$, the homeomorphisms isotopic to the identity.

Let H be the subgroup of $SO(3)$ generated by u , a rotation of $2\pi/g$ about the z -axis, and v , a rotation of π about the x -axis. Then H is a group of order $2g$ which is non-abelian (this uses $g \geq 3$). In fact, H is a semidirect product of $\mathbb{Z}/g\mathbb{Z}$ with

$\mathbb{Z}/2\mathbb{Z}$, with the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acting on $Z/g\mathbb{Z}$ by sending an element to its inverse. One easily calculates

$$v^{-1}u^{-1}vu = u^2.$$

We want to show that if S is a closed surface of genus $g \geq 3$, then there is a subgroup of $\text{MCG}(S)$ isomorphic to H . Let M_0 be the unit sphere in \mathbb{R}^3 and note that it is H invariant. The H orbit of the point $(1, 0, 0) \in S^2$ consists of g points uniformly spaced about the equator. We remove a small open disk about each of these points, thus creating a genus zero surface M with g boundary components which is H invariant. Next we attach g genus one surfaces (each with one boundary component) to each of the boundary components of M . This creates a surface S of genus g . If we make these attachments in a symmetric fashion, we may assume that S is invariant under the group H acting on \mathbb{R}^3 . The restriction to S of each non-trivial element of H represents a non-trivial element of $\text{MCG}(S)$, and the assignment of an element of H to this representative in $\text{MCG}(S)$ is an embedding of H in $\text{MCG}(S)$.

The restrictions of u and v to S are representatives of elements $U, V \in \text{MCG}(S)$, and U and V generate a subgroup of $\text{MCG}(S)$ isomorphic to H . Suppose $\phi : \text{MCG}(S) \rightarrow \text{Homeo}(\mathbb{T}^2)$ is a homomorphism. Since $U^2 = V^{-1}U^{-1}VU$ is a commutator, Lemma 4.1 implies that $\phi(U^2) = id \in \text{Homeo}(\mathbb{T}^2)$. On the other hand, u^2 induces a non-trivial permutation of the g tori we attached to M . Let α be a non-separating simple closed curve in one of these tori and let $\beta = u^2(\alpha)$. Then α and β are disjoint simple closed curves in S whose union does not separate. Since

$$T_\beta = U^2T_\alpha U^{-2},$$

we have

$$\phi(T_\beta) = \phi(U^2)\phi(T_\alpha)\phi(U^{-2}) = \phi(T_\alpha).$$

Part (1) of Proposition 2.6 now implies that ϕ is trivial. □

5. HOMOMORPHISMS TO $\text{Diff}(S^2)$

Theorem 1.4. *Suppose that S is a closed surface of genus $g \geq 7$. Then every homomorphism $\phi : \text{MCG}(S) \rightarrow \text{Diff}(S^2)$ is trivial.*

Proof. If the genus g is odd we let S_0 be a torus, and if g is even we let it be a surface of genus two. In either case we embed S_0 in \mathbb{R}^3 so that it is invariant under rotation by π about the z -axis and let R denote this rotation. We consider two surfaces Y^\pm of genus $(g - 1)/2$ or genus $(g - 2)/2$ (depending on the parity of g), each with a single boundary component. We remove two open disks from S_0 , which are symmetrically placed so that they are interchanged by R , and attach the surfaces Y^\pm along boundaries to the boundary components of S_0 thus created. We do this in such a way that Y^+ lies in the half space with x -coordinate positive and $T(Y^\pm) = Y^\mp$. Then $S = S_0 \cup Y^+ \cup Y^-$ has genus g and $R : S \rightarrow S$, the restriction of R to S , is an involution which induces an element of order two $\tau \in \text{MCG}(S)$. The element τ induces an inner automorphism which we denote by $\tau_\# : \text{MCG}(S) \rightarrow \text{MCG}(S)$.

There are embeddings $\text{MCG}(Y^\pm) \rightarrow \text{MCG}(S)$ induced by the embeddings of Y^\pm in S . Let G^\pm denote the subgroup of $\text{MCG}(S)$ which is the image of the embedding of $\text{MCG}(Y^\pm)$. Note that elements of G^+ commute with elements of G^- since they have representatives with disjoint support. Also note that $\tau_\#(G^\pm) = G^\mp$.

Let G_0 be the subgroup of $\text{MCG}(S)$ consisting of all elements of the form $\eta\tau_{\#}(\eta)$ for $\eta \in G^+$. It is clear that the correspondence $\eta \mapsto \eta\tau_{\#}(\eta)$ defines an isomorphism from G^+ to G_0 , and hence G_0 is isomorphic to $\text{MCG}(Y^+)$. Observe that

$$\tau_{\#}(\eta\tau_{\#}(\eta)) = \tau_{\#}(\eta)\eta = \eta\tau_{\#}(\eta),$$

so τ commutes with every element of G_0 .

Now let α be a non-separating simple closed curve in Y^+ and let $\beta = R(\alpha)$. Recall that T_{α} denotes the left Dehn twist about α .

We want to show that $\phi(T_{\alpha}) = \phi(T_{\beta})$ or $\phi(T_{\beta})^{-1}$. Since α and β are disjoint simple closed curves in S whose union does not separate, it will follow from part (1) of Proposition 2.6 that $\phi : \text{MCG}(S) \rightarrow \text{Diff}(S^2)$ is trivial.

To do this we consider $h = \phi(\tau) \in \text{Diff}(S^2)$. The diffeomorphism h of S^2 has order at most two. If h is the identity, then

$$\phi(T_{\alpha}) = h \circ \phi(T_{\alpha}) \circ h^{-1} = \phi(\tau T_{\alpha} \tau^{-1}) = \phi(T_{\beta}).$$

Hence we may assume h has order two. In that case h has exactly two fixed points. To see this note that the derivative of h at a fixed point can only be $-I$ (since h preserves orientation), and hence the index of each fixed point is $+1$. Since the sum of the fixed point indices is 2, the Euler characteristic of S^2 , there must be two fixed points. The set of fixed points must be preserved by each element of $\phi(G_0)$ since these elements commute with h . If some elements of $\phi(G_0)$ permute these fixed points, then that action determines a non-trivial homomorphism from G_0 to $\mathbb{Z}/2\mathbb{Z}$. Such a homomorphism cannot exist since $G_0 \cong \text{MCG}(Y^+)$ is perfect by Proposition 2.1. We conclude there is a global fixed point $p \in S^2$ for G_0 .

By Theorem 1.1 any homomorphism from $G_0 \cong \text{MCG}(Y^+)$ to $GL(2, \mathbb{R})$ is trivial. We can conclude that the derivative $D_p(\phi(\eta)) = I$ for every $\eta \in G_0$.

Since G_0 is perfect, $\phi(G_0)$ admits no non-trivial homomorphisms to \mathbb{R} . Applying Thurston's stability theorem [11] to $\phi(G_0)$, we conclude that $\phi(G_0)$ is trivial. Finally we observe that if we let

$$\eta = T_{\alpha}T_{\beta} = T_{\alpha}\tau_{\#}(T_{\alpha}) \in G_0,$$

then $\phi(\eta) = id$ implies

$$\phi(T_{\alpha}) = \phi(T_{\beta})^{-1}.$$

Once again Proposition 2.6(1) implies that $\phi : \text{MCG}(S) \rightarrow \text{Diff}(S^2)$ is trivial. \square

Theorem 1.6. *Suppose that S is an oriented surface with genus g with k punctures and no boundary. If $g \geq 7$ and $0 \leq k \leq 2g - 10$, then every homomorphism $\phi : \text{MCG}(S) \rightarrow \text{Diff}(S^2)$ is trivial.*

Proof. If S'_0 is a surface of genus $g_0 \geq 1$, then it can be embedded in \mathbb{R}^3 in such a way that it is invariant under R , rotation about the z -axis by π , and so that it intersects the z -axis in $2g_0 + 2$ points. So R has $2g_0 + 2$ fixed points.

If $k \leq 2g_0 + 2$, we let S_0 be S'_0 punctured at k of the points of $\text{Fix}(R)$ so R represents an element of $\text{MCG}(S_0)$. We now repeat the proof of Theorem 1.4 using this S_0 instead of the surface of genus one or two labeled S_0 in that proof.

The surface S is constructed as before by attaching two surfaces of genus ≥ 3 to S_0 . Hence the resulting surface S has genus $g \geq g_0 + 6 \geq 7$. We then conclude as before that ϕ is trivial. The restriction on k is

$$k \leq 2g_0 + 2 \leq 2(g - 6) + 2 = 2g - 10. \quad \square$$

ACKNOWLEDGMENT

The authors would like to thank Kiran Parkhe for pointing out some inaccuracies in an earlier version of this paper.

REFERENCES

- [1] Javier Aramayona and Juan Souto. Homomorphisms between mapping class groups. <http://arxiv.org/abs/1011.1855v1>.
- [2] Martin R. Bridson. On the dimension of CAT(0) spaces where mapping class groups act. *To appear in Journal für die reine und angewandte Mathematik*.
- [3] L.E.J. Brouwer. *Aufzählung der periodischen Transformationen des Torus*. 21:1352–1356, 1919.
- [4] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups*. Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 2011. MR2850125 (2012h:57032)
- [5] Louis Funar. Two questions on mapping class groups. *Proc. Amer. Math. Soc.*, 139(1):375–382, 2011. MR2729098 (2012a:57002)
- [6] C. S. Hartzman. Nonsingular periodic flows on T^3 and periodic homeomorphisms of T^2 . *Canad. Math. Bull.*, 24(1):23–28, 1981. MR611204 (83k:58072)
- [7] Mustafa Korkmaz. Low-dimensional linear representations of mapping class groups. *Preprint*. <http://arxiv.org/abs/11.04.4816>.
- [8] Mustafa Korkmaz. Low-dimensional homology groups of mapping class groups: A survey. *Turkish J. Math.*, 26(1):101–114, 2002. MR1892804 (2003f:57002)
- [9] Kamlesh Parwani. C^1 actions on the mapping class groups on the circle. *Algebr. Geom. Topol.*, 8(2):935–944, 2008. MR2443102 (2010c:37061)
- [10] Jerome Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978. MR0494115 (58:13045)
- [11] William P. Thurston. A generalization of the Reeb stability theorem. *Topology*, 13:347–352, 1974. MR0356087 (50:8558)

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, 2033 SHERIDAN ROAD, EVANSTON, ILLINOIS 60208-2730

DEPARTMENT OF MATHEMATICS, LEHMAN COLLEGE, 250 BEDFORD PARK BOULEVARD WEST, BRONX, NEW YORK 10468