THE EISENSTEIN FAMILY

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Abstract. Based on the work of Buzzard, Kilford, and Roe, we state a general conjecture about the family of overconvergent Eisenstein series.

In [CM98], Mazur and Coleman defined a rigid analytic object called the eigencurve. This object parametrizes finite slope overconvergent normalized \( p \)-adic eigenforms. The Eisenstein family is a family of overconvergent modular forms whose existence was essential for this construction.

The geometry of the eigencurve is still poorly understood. In [BK05], when \( p = 2 \) (a case not actually dealt with in [BK05], but see [Bu07]), it was proven that the 2-adic level one eigencurve at the “boundary of weight space” is a disjoint union of infinitely many annuli. This has been extended to the 3-adic level one eigencurve in [Ro09]. While it might be too optimistic to expect this feature to be possessed by many more eigencurves, we believe results obtained and used in [BK05] and [Ro09] about the domain of analyticity of the Eisenstein family will generalize. In this note, we reformulate these results and state a general conjecture.

1. The family

Let \( q = 4 \) if \( p = 2 \) and \( p \) otherwise. Let \( C_p \) be the completion of an algebraic closure of \( \mathbb{Q}_p \), \( R_p \) the ring of integers of \( C_p \) and \( F \) the residue field of \( R_p \). Let \( W \subset R_p \) denote the Witt vectors of \( F \). Also let \( v \) denote the valuation on \( C_p^\star \) such that \( v(p) = 1 \). Let \( Z \) be the connected component of the ordinary locus of \( X_1(q) \) containing the cusp \( \infty \). Let \( W \) be the analytic group of continuous \( C_p \)-valued characters on \( Z_p^\ast \) and \( B \) the subgroup of characters trivial on \( \mu(q) \) of \( Z_p^\ast \). Let \( \lambda = Z_p[[Z_p^\ast]], \lambda = Z_p[[1 + qZ_p]] \subset \Lambda \), and for \( a \in Z_p^\ast \), let \( [a] \) denote its image in \( \Lambda \). If \( \kappa \in W \) and \( \alpha = r_1[a_1] + \cdots + r_n[a_n] \in Z_p[Z_p^\ast] \), let

\[
\kappa(\alpha) = \sum_{i=1}^n r_i \kappa(a_i).
\]

This extends to a continuous homomorphism \( \Lambda \to R_p \) and induces a homomorphism

\[
\kappa: \Lambda[[q]] \to R_p[[q]].
\]

Let \( w_p \) be the function on \( W \), \( \kappa \mapsto \kappa([1 + q] - 1) \).

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1See [BC06] for another result about the geometry of the 2-adic eigencurve.

2We will extend \( v \) to \( C_p \cup \{\infty\} \) and let \( v(0) = \infty \) and \( v(\infty) = -\infty \).
Let $E(q) \in \Lambda[[q]]$ be such that
$$\kappa(E(q)) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n \geq 1} \sigma_n(\kappa) q^n,$$
for $\kappa \in \mathcal{B}\setminus\{1\}$, where $\zeta^*$ is the $p$-adic zeta function on $\mathcal{W}$ (see Chapter 4, §3, of [La78]) and
$$\sigma_n(\kappa) = \sum_{d|n, (d,p)=1} \kappa(d)/d.$$

If $F(q) = \sum_{n \geq 0} a_n q^n$ is a series in $q$, we set $V(F)(q) = \sum_{n \geq 0} a_n q^n$. By Corollary 2.1.1 of [Co97i], there is an element $\mathcal{E}_p \in \mathcal{A}^1(\mathcal{Z_B}/\mathcal{B})^p$, the ring of rigid analytic functions bounded by one on $\mathcal{Z_B} := \mathcal{Z} \times \mathcal{B}$ overconvergent over $\mathcal{B}$, whose $q$-expansion, $\mathcal{E}_p(q)$, is $E(q)/V(E)(q)$.

Let $H$ denote the level one, weight $p-1$, Hasse invariant modular form over $\mathcal{F}_p$.\footnote{This, in fact, can be deduced more directly using the arguments which established Corollary 4.1.2 of [Co97i]. The point is, the function labeled there, $e^*$, naturally extends to $\mathcal{B} \times \bigcup_{v \in t_3^*} X(v)$, where $|\kappa - 1|_{X(v)} < 1$.}

If $E$ is an elliptic curve over $\mathcal{R}_p$, let $h(E) = \begin{cases} v(A) & \text{if } H(\tilde{E}, \tilde{w}) \neq 0, A \in \mathcal{R}_p \text{ and } H(\tilde{E}, \tilde{w}) = A \mod p, \\ 1 & \text{if } H(\tilde{E}, \tilde{w}) = 0, \end{cases}$

where $\omega$ generates $\Omega_{E/\mathcal{R}_p}$, $\tilde{E}$ is $E \mod p$ and $\tilde{w}$ is $\omega \mod p$. (This is independent of the choice of $\omega$.) If $P \in X_1(N)(\mathcal{C}_p)$, let $(h(P), A(P)) = (h(E), |\text{Aut}(E)|/2)$ if $P$ corresponds to an elliptic curve $E$ with good reduction, $\tilde{E}$, and $(h(P), A(P)) = (0, 1)$ otherwise.

If $r \in \mathcal{R}_{>0}$, let $\mathcal{B}_r$ be the annulus $\{\kappa \in \mathcal{B}: v(w_p(\kappa)) < r\}$.

**Conjecture 1.1.** (a) The restriction of $\mathcal{E}_p$ to $\mathcal{Z}_p := \mathcal{Z} \times \mathcal{B}_{c_p}$, where $c_p = 3$ if $p = 2$ and $1$ otherwise, analytically continues to a function whose absolute value is bounded by $1$ on the rigid connected component $\mathcal{V}_p$ of

$$\{(P, \kappa) \in X_1(\mathcal{q}) \times \mathcal{B}: h(P) < \frac{p}{p+1}, A(P) \cdot h(P) < v(w_p(\kappa)) < c_p \}$$

containing $\mathcal{Z}_p$.\footnote{We use the definition on page 97 of [Ka73].}$^\dagger$

(b) Moreover, if $\kappa \in \mathcal{B}_{c_p}$, the restriction of this function to $\mathcal{V}_p|_{\kappa} \subset X_1(\mathcal{q})$ does not analytically continue to any larger connected region in $X_1(\mathcal{q})$.

**Theorem 1.2.** Conjecture 1.1 is true if $p$ equals $2$ or $3$.

**Proof.** We will use the notation $E_{t,i}$ for the Eisenstein series of §B1 of [Co97i].

We first prove (a). Suppose $p = 2$. If $P \in X_1(4) = X_0(4)$, $A(P) = 12$ if $P$ corresponds to an elliptic curve with good supersingular reduction and $1$ otherwise.

\footnote{\mathcal{B}_{c_p} \text{ (for all } p \text{) is the largest annulus of the form } \mathcal{B}_r \text{ containing no points corresponding to classical level } 1 \text{ forms.}
Let $y_2$ be the function on $X_1(4)$, 
$$y_2 = \frac{E_{(2,0)}/V(E_{(2,0)}) - 1}{24}.$$ 

As explained in [BK05], $y_2$ yields an isomorphism $X_1(4) \to \mathbb{P}^1$. By Theorem 7 and a remark on page 613 (the end of the proof) of [BK05], 
(1) 
$$E_2 \in \mathbb{Z}_2[[w_2/8, 8y_2]] \cap \mathbb{Z}_2[[w_2, y_2]].$$

In other words, 
$$E_2 = \sum_{j \geq 0} \left( \sum_{i<j} b_{i,j} \left( \frac{8}{w_2} \right)^{j-i} + \sum_{i \geq j} b_{i,j} w_2^{i-j} \right) (w_2y_2)^j,$$
where $b_{i,j} \in \mathbb{Z}_2$. So $E_2$ continues to and is bounded by one on the connected component $D_2$ of 
$$\{(P, \kappa) \in X_1(4) \times \mathcal{B}: -v(y_2(P)) < v(w_2(\kappa)) < 3\}$$
containing $Z_2$.

By Lemma 2 (iii) of [BK05], if $F_2 = V(\Delta)/\Delta$, which is a modular function of level 2 that yields an isomorphism $X_1(2) = X_0(2) \to \mathbb{P}^1$, then 
(2) 
$$F_2 = \frac{y_2(1+8y_2)}{(1-8y_2)^2}.$$ 

Thus if $v(y_2) > -3$, $|F_2/y_2 - 1| < 1$.

One can show that 
(3) 
$$\frac{(2^8 F_2 + 1)^3}{F_2} = j.$$ 

Now, 
(4) 
$$v(j(E)) = 12h(E)$$
if $12 > v(j(E)) \geq 0$, by Theorem 4.2. So if $v(y_2) > -3$, then $v(F_2) > -3$ and $v(F_2) = -12h$.

This implies $D_2$ is contained in $\mathcal{V}_2$.

If $m$ and $n$ are positive integers, let $\pi(mn,n) : X_1(mn) \to X_1(n)$ be the “forgetful map”. The image of $\mathcal{V}_2$ in $X_1(2)$ is contained in the connected component $\mathcal{C}$ of $\pi(2,1)^{-1}\{x \in X(1): v(j(x)) < 3\}$ containing the cusp $\infty$ by (4). Also, from (3), we see that $v(F_2) > -3$ on $\mathcal{C}$. Using (2), we see that $v(y_2) > -3$ on the connected component of $\pi(2,2)^{-1}\mathcal{C}$ containing the cusp $\infty \in X_1(4)$. Thus $\mathcal{V}_2 = D_2$.

Now, suppose $p = 3$. Let 
$$y_3 = \frac{E_{(1,0)}/V(E_{(1,0)}) - 1}{6}.$$ 

Then $y_3$ is a level 9 modular function giving an isomorphism $X_0(9) \to \mathbb{P}^1$. By Theorem 4.2 and the proof of Corollary 4.3 of [Ro09], 
$$E_3 \in \mathbb{Z}_3[[w_3/3, 3y_3]] \cap \mathbb{Z}_3[[w_3, y_3]].$$
By Lemma 2.4(3) of [Ro09], if $F_3 = \sqrt{V(\Delta)/\Delta}$ (a level 3 modular function yielding $X_1(3) = X_0(3) \cong \mathbb{P}^1$), then

$$F_3 = \frac{y_3(1 + 3y_3 + 9y_3^2)}{(1 - 3y_3)^2}.$$  

Thus, if $v(y_3) > -1$, then $|F_3/y_3 - 1| < 1$.

As McMurdy pointed out,

$$j = \frac{(1 + 27F_3)(1 + 243F_3)^3}{F_3}.$$  

If $v(F_3) > -3$, then $v(F_3) = -v(j)$. Also, using Theorem 2.2 if $v(y_3) > -1$, we see that $v(y_3) = v(F_3) = -v(j) = -6h$. We can now argue as above to show $E_3$ extends to and is bounded by one on $V_3$. This proves (a).

To establish the rest of the theorem we will use

**Lemma 1.3.** If $f$ is an analytic function on the open unit disk $B(0,1)$ and $f$ analytically continues to a strictly larger connected rigid subspace of the closed unit disk $B^1$, then there exists an open affine $U$ in $\mathbb{A}^1_k$ containing 0 and an analytic function $F$ on the affinoid $Y = \text{red}^{-1}U$ such that $F|_{B(0,1)} = f$. Moreover, if $||f||_{B(0,1)} \leq 1$, then $||F||_Y \leq 1$ and $\mathcal{A}^o(Y) = O_{\mathbb{A}^1_k}(U)$.

**Proof.** This follows from the fact that any connected affinoid in $B^1$ which properly intersects $B(0,1)$ must contain the complement of finitely many residue disks.  

Suppose $p = 2$ and suppose $\kappa \in B$ and $v(w_2(\kappa)) < 3$. Then the fiber above $\kappa$ of $V_2$ is isomorphic to $B(0,1)$ and we may regard $X =: X_\kappa = w_2(\kappa)y_2$ as a uniformizing parameter on it. Thus, as pointed out on page 614 of [BK05], there is a $g_\kappa(z) \in \mathcal{O}_\kappa[[z]]$ such that, after pullback, the restriction of $E_2$ to this fiber equals $g_\kappa(w_2(\kappa)y_2)$.

It follows that if we take $f = Xg_\kappa(X)$ and $f$ extends to an analytic function on a larger connected region in the fiber above $\kappa$ of $X_1(4) \times B_3$ than $V_2|_\kappa$, then there must be an open affine $U$ in $\mathbb{A}^1_k$ containing 0 and an analytic function $F$ on $Y = \text{red}^{-1}U$ such that $F|_{B(0,1)} = f$. Then $F$ must be a rational function in $X$. It follows from the analysis on page 614 of [BK05] that

$$\bar{F}(X)^2 + \bar{F}(X) + X = 0.  \tag{5}$$

But, this means $\bar{F}$ is integral over $F[X]$, which implies $\bar{F} \in F[X]$, but there are no solutions of (5) in $F[X]$. This implies $E_2$ does not continue to a larger connected region in the fiber above $\kappa$.

Now suppose $p = 3$ and $\kappa \in B$. Then if $v(w_3(\kappa(4))) < 1$, by §5 of [Ro09] there is a $g_\kappa(z) \in \mathcal{O}_\kappa[[z]]$ such that the restriction of $E_3$ to the fiber above $\kappa$ equals $g_\kappa(w_3(\kappa)y_3)$. By the proof of Lemma 5.1 of [Ro09], if $G(X) = X^2\bar{g}_\kappa(X)$, then

$$G(X)^3 + G(X)^2 + XG(X) - X^3 = 0.$$  

This has no solution in $F[X]$, so as above $E_3$ does not continue to a larger connected region in the fiber above $\kappa$. This completes the proof of (b).
2. Valuation of Hasse

We use the notation and formulas of [DT75]. Suppose $R$ is a ring, $p = 0$ in $R$ and $a_1, \ldots, a_9 \in R$. Let

$$E: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

$$\pi = dx/(2y + a_1 x + a_3) = dy/(3x^2 + 2a_2 x + a_4 - a_1 y).$$

$b_2 = a_1^2 + 4a_2, -4b_8 = b_1^2 - b_2b_6, c_4 = b_2^2 - 24b_4$ and $\Delta = b_1^4 - 27b_2^2 + b_6(36b_4 - b_2^2)$. Suppose $\Delta \in R^*$. Then $E$ is an elliptic curve over $R$. $\pi$ is a non-vanishing differential and

$$j(E) = c_4^3/\Delta.$$

The following lemma should be well known (it follows easily from the definition).

**Lemma 2.1.** We have $H(E, \pi) = a_1$ if $p = 2$, and $H(E, \pi) = b_2$ if $p = 3$.

**Theorem 2.2.** Suppose $P$ is a point on the $j$-line $X(1)$ over $R_p$ corresponding to an elliptic curve $E_P$ with good supersingular reduction and $P_0 \in X(1)(W)$ such that $P_0 = P$ and $j(P_0) = 0$ if $p \leq 3$ or $E_{p-1}(P_0) = 0$ if $p > 3$ Then

$$h(E_P) = \frac{v(j(P) - j(P_0))}{|\text{Aut}(E_P)|/2}$$

if either expression is strictly less than one.

**Proof.** When $p$ is 2 or 3, this follows from (6), Theorem 10.1 of chapter III of [Si86] (which implies $|\text{Aut}(E_P)|/2 = 12$ if $p = 2$ and if $p = 3$) and Lemma 2.1. When $p > 3$,

$$H = E_{p-1} \mod p,$$

by §2.1 of [Ka73]. Suppose $N \geq 3, (N, p) = 1$, and let $E(N)$ denote the universal elliptic curve over $Y(N)$ (the open modular curve of level $N$) and $G = \Gamma(1)/\Gamma(N)$. Let $f: Y(N) \rightarrow Y(1)$ be the natural map. Then $G$ acts on $(E(N), Y(N))$ and hence on $\omega = f_*\Omega_Y^1(E(N)/Y(N))$. Now $E_{p-1}$ may be considered a $G$-invariant section of $\omega^{p-1}$. Let $U \subset Y(1)$ be the residue disk of $\bar{P}$ and $V$ one of the residue classes (which are disks over $W$) above $f^{-1}(P)$. Then $f: V \rightarrow U$ is finite, surjective and of degree $d := |\text{Aut}(E_P)|/2$. Moreover, if $d > 1$, $U$ will contain a point $R$ with $j(R)$ equal to 0 or 1728 and $f|_V$ is totally ramified at the unique point $Q$ above $R$. As is well-known, $E_{p-1}(R) = 0$ (as $P$ is supersingular)[3] so $R = P_0$. If $\eta$ is a basis for $\omega(V)$ on the formal scheme attached to $V$, then $E_{p-1}|_V = sp^{p-1}$, where

$$s: V \rightarrow B(0, 1)$$

is an isomorphism which vanishes at $Q$. Moreover,

$$f^*(j|_V) = j(P_0) + s^d g,$$

where $g \in A^d(V)^*$. Now, suppose $A \in V(R_p)$ and $f(A) = P$. Then

$$j(P) - j(P_0) = s(A)^d g(A).$$

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6If $p > 2$, $E$ also has the equation $y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$. (See chapter III. §1, of [Si86].)

7It is known that if $p > 3$, then $E_{p-1}$ has unique zero in each supersingular disk.

8If $\tau$ is in the upper half-plane, $E_k(\tau) = \frac{1}{2} \sum_{(a,b) = 1} (a\tau + b)^{-k}$, so if $\tau$ is a quadratic root of unity, then $\tau^{-k} E_k(\tau) = E_k(\tau)$. 

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Since \( v(g(A)) = 0 \),
\[
v(j(P) - j(P_0)) = dv(s(A)) = (\mathcal{A}(E_P)/2) \cdot h(E_P)
\]
if \( v(j(P) - j(P_0))/(\mathcal{A}(E_P)/2) \) or \( h(E_P) \) is strictly less than one.

Remarks 1. (i) One can show that \( h(E) = v(E) \), where \( v(E) \) is as defined on page 36 of [Bu03] when \( v(E) < p/(p+1) \).

(ii) Proposition 1 of [BC06] follows from Theorem 2.2.

References


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