THE EISENSTEIN FAMILY

ROBERT F. COLEMAN

(Communicated by Lev Borisov)

Abstract. Based on the work of Buzzard, Kilford, and Roe, we state a general conjecture about the family of overconvergent Eisenstein series.

In [CM98], Mazur and Coleman defined a rigid analytic object called the eigencurve. This object parametrizes finite slope overconvergent normalized \( p \)-adic eigenforms. The Eisenstein family is a family of overconvergent modular forms whose existence was essential for this construction.

The geometry of the eigencurve is still poorly understood. In [BK05], when \( p = 2 \) (a case not actually dealt with in [BK05], but see [Bu07]), it was proven that the 2-adic level one eigencurve at the “boundary of weight space” is a disjoint union of infinitely many annuli. This has been extended to the 3-adic level one eigencurve in [Ro09]. While it might be too optimistic to expect this feature to be possessed by many more eigencurves, we believe results obtained and used in [BK05] and [Ro09] about the domain of analyticity of the Eisenstein family will generalize. In this note, we reformulate these results and state a general conjecture.

1. The family

Let \( q = 4 \) if \( p = 2 \) and \( p \) otherwise. Let \( \mathcal{C}_p \) be the completion of an algebraic closure of \( \mathbb{Q}_p \), \( \mathcal{R}_p \) the ring of integers of \( \mathcal{C}_p \) and \( \mathcal{F} \) the residue field of \( \mathcal{R}_p \). Let \( \mathcal{W} \subset \mathcal{R}_p \) denote the Witt vectors of \( \mathcal{F} \). Also let \( v \) denote the valuation on \( \mathcal{C}_p^* \) such that \( v(p) = 1 \). Let \( Z \) be the connected component of the ordinary locus of \( X_1(q) \) containing the cusp \( \infty \). Let \( \mathcal{W} \) be the analytic group of continuous \( \mathcal{C}_p \)-valued characters on \( \mathcal{Z}_p^* \) and \( \mathcal{B} \) the subgroup of characters trivial on \( \mu(q) \). Let \( \Lambda = \mathcal{Z}_p[[\mathcal{Z}_p^*]], \Lambda = \mathcal{Z}_p[[1 + q\mathcal{Z}_p^*]] \subset \Lambda \), and for \( a \in \mathcal{Z}_p^* \), let \([a]\) denote its image in \( \Lambda \).

If \( \kappa \in \mathcal{W} \) and \( \alpha = r_1[a_1] + \cdots + r_n[a_n] \in \mathcal{Z}_p[\mathcal{Z}_p^*] \), let

\[ \kappa(\alpha) = \sum_{i=1}^{n} r_i \kappa(a_i). \]

This extends to a continuous homomorphism \( \Lambda \to \mathcal{R}_p \) and induces a homomorphism

\[ \kappa: \Lambda[[q]] \to \mathcal{R}_p[[q]]. \]

Let \( w_p \) be the function on \( \mathcal{W} \), \( \kappa \mapsto \kappa([1 + q] - 1) \).
Let \( E(q) \in \Lambda[[q]] \) be such that
\[
\kappa(E(q)) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n \geq 1} \sigma_n(\kappa) q^n,
\]
for \( \kappa \in \mathcal{B}\setminus\{1\} \), where \( \zeta^* \) is the \( p \)-adic zeta function on \( W \) (see Chapter 4, §3, of [La78]) and
\[
\sigma_n(\kappa) = \sum_{d|n} \kappa(d)/d.
\]
If \( F(q) = \sum_{n \geq 0} a_n q^n \) is a series in \( q \), we set \( V(F)(q) = \sum_{n \geq 0} a_n q^{pn} \). By Corollary 2.1.1 of [Co97ii], there is an element \( \mathcal{E}_p \in A^1(\mathbb{Z}/\mathbb{B})^p \), the ring of rigid analytic functions bounded by one on \( \mathbb{Z}/\mathbb{B} := \mathbb{Z} \times \mathcal{B} \) overconvergent over \( \mathcal{B} \), whose \( q \)-expansion, \( \mathcal{E}_p(q) \), is \( E(q)/V(E)(q) \).

Let \( H \) denote the level one, weight \( p-1 \), Hasse invariant modular form over \( F_p \).

If \( E \) is an elliptic curve over \( \mathbb{R}_p \), let
\[
h(E) = \begin{cases} 0 & \text{if } H(\tilde{E}, \tilde{\omega}) \neq 0, A \in \mathbb{R}_p \text{ and } H(\tilde{E}, \tilde{\omega}) = A \bmod p, \\ 1 & \text{if } H(\tilde{E}, \tilde{\omega}) = 0, \end{cases}
\]
where \( \omega \) generates \( \Omega_{E/\mathbb{R}_p} \), \( \tilde{E} \) is \( E \bmod p \) and \( \tilde{\omega} \) is \( \omega \bmod p \). (This is independent of the choice of \( \omega \).) If \( P \in X_1(N)(\mathbb{C}_p) \), let \( (h(P), A(P)) = (h(E), |\text{Aut}(E)|)/2 \) if \( P \) corresponds to an elliptic curve \( E \) with good reduction, \( \tilde{E} \), and \( (h(P), A(P)) = (0, 1) \) otherwise.

If \( r \in \mathbb{R}_{>0} \), let \( \mathcal{B}_r \) be the annulus \( \{ \kappa \in \mathcal{B} : v(w_p(\kappa)) < r \} \).

**Conjecture 1.1.** (a) The restriction of \( \mathcal{E}_p \) to \( \mathbb{Z}_p := \mathbb{Z} \times \mathcal{B}_c_p \), where \( c_p = 3 \) if \( p = 2 \) and 1 otherwise, analytically continues to a function whose absolute value is bounded by 1 on the rigid connected component \( \mathcal{V}_p \) of
\[
\{(P, \kappa) \in X_1(q) \times \mathcal{B} : h(P) < \frac{p}{p+1}, A(P) \cdot h(P) < v(w_p(\kappa)) < c_p\}
\]
containing \( \mathbb{Z}_p \).

(b) Moreover, if \( \kappa \in \mathcal{B}_c_p \), the restriction of this function to \( \mathcal{V}_p|_{\kappa} \subset X_1(q) \) does not analytically continue to any larger connected region in \( X_1(q) \).

**Theorem 1.2.** Conjecture 1.1 is true if \( p \) equals 2 or 3.

**Proof.** We will use the notation \( E_{i,j} \) for the Eisenstein series of §B1 of [Co97ii].

We first prove (a). Suppose \( p = 2 \). If \( P \in X_1(4) = X_0(4) \), \( A(P) = 12 \) if \( P \) corresponds to an elliptic curve with good supersingular reduction and 1 otherwise.

---

3 This, in fact, can be deduced more directly using the arguments which established Corollary 4.1.2 of [Co97]. The point is, the function labeled there, \( e^* \), naturally extends to
\[
\mathcal{B} \times \bigcup_{v \in \mathcal{I}_2} X(v).
\]

4 We use the definition on page 97 of [Ka73].

5 \( \mathcal{B}_c_p \) (for all \( p \)) is the largest annulus of the form \( \mathcal{B}_r \) containing no points corresponding to classical level 1 forms.
Let $y_2$ be the function on $X_1(4)$,
$$y_2 = \frac{E(2,0)/V(E(2,0)) - 1}{24}.$$

As explained in [BK05], $y_2$ yields an isomorphism $X_1(4) \to \mathbb{P}^1$. By Theorem 7 and a remark on page 613 (the end of the proof) of [BK05],

(1) $$\mathcal{E}_2 \in \mathbb{Z}_2[[w_2/8, 8y_2]] \cap \mathbb{Z}_2[[w_2, y_2]].$$

In other words,

$$\mathcal{E}_2 = \sum_{j \geq 0} \left( \sum_{i<j} b_{ij} \left( \frac{8}{w_2} \right)^{j-i} + \sum_{i \geq j} b_{ij} w_2^{i-j} \right) (w_2 y_2)^j,$$

where $b_{ij} \in \mathbb{Z}_2$. So $\mathcal{E}_2$ continues to and is bounded by one on the connected component $D_2$ of

$$(\{(P, \kappa) \in X_1(4) \times \mathcal{B}: -v(y_2(P)) < v(w_2(\kappa)) < 3\})$$

containing $Z_2$.

By Lemma 2 (iii) of [BK05], if $F_2 = V(\Delta)/\Delta$, which is a modular function of level 2 that yields an isomorphism $X_0(2) = X_1(2) \to \mathbb{P}^1$, then

(2) $$F_2 = \frac{y_2(1 + 8y_2)}{(1 - 8y_2)^2}.$$

Thus if $v(y_2) > -3$, $|F_2/y_2 - 1| < 1$.

One can show that

(3) $$\frac{(2^8 F_2 + 1)^3}{F_2} = j.$$

Now,

(4) $$v(j(E)) = 12h(E)$$

if $12 > v(j(E)) \geq 0$, by Theorem 2.2. So if $v(y_2) > -3$, then $v(F_2) > -3$ and

$$v(F_2) = -12h.$$

This implies $D_2$ is contained in $\mathcal{V}_2$.

If $m$ and $n$ are positive integers, let $\pi(mn, n) : X_1(mn) \to X_1(n)$ be the “forgetful map”. The image of $\mathcal{V}_2$ in $X_1(2)$ is contained in the connected component $C$ of $\pi(2,1)^{-1}\{x \in X(1): v(j(x)) < 3\}$ containing the cusp $\infty$ by (4). Also, from (3), we see that $v(F_2) > -3$ on $C$. Using (2), we see that $v(y_2) > -3$ on the connected component of $\pi(2,2)^{-1}C$ containing the cusp $\infty \in X_1(4)$. Thus $\mathcal{V}_2 = D_2$.

Now, suppose $p = 3$. Let

$$y_3 = \frac{E(1,0)/V(E(1,0)) - 1}{6}.$$

Then $y_3$ is a level 9 modular function giving an isomorphism $X_0(9) \to \mathbb{P}^1$. By Theorem 4.2 and the proof of Corollary 4.3 of [Ro09],

$$\mathcal{E}_3 \in \mathbb{Z}_3[[w_3/3, 3y_3]] \cap \mathbb{Z}_3[[w_3, y_3]].$$
By Lemma 2.4(3) of [Ro09], if $F_3 = \sqrt{V(\Delta)/\Delta}$ (a level 3 modular function yielding $X_1(3) = X_0(3) \cong \mathbb{P}^1$), then

$$F_3 = \frac{y_3(1 + 3y_3 + 9y_3^2)}{(1 - 3y_3)^2}.$$ 

Thus, if $v(y_3) > -1$, then $|F_3/y_3 - 1| < 1$.

As McMurdy pointed out, $j = \frac{(1 + 27F_3)(1 + 243F_3)^3}{F_3}$.

If $v(F_3) > -3$, then $v(F_3) = -v(j)$. Also, using Theorem 222 if $v(y_3) > -1$, we see that $v(y_3) = v(F_3) = -v(j) = -6h$. We can now argue as above to show $E_3$ extends to and is bounded by one on $V_3$. This proves (a).

To establish the rest of the theorem we will use

**Lemma 1.3.** If $f$ is an analytic function on the open unit disk $B(0,1)$ and $f$ analytically continues to a strictly larger connected rigid subspace of the closed unit disk $B^1$, then there exists an open affine $U$ in $\mathbb{A}_k^1$ containing 0 and an analytic function $F$ on the affinoid $Y = \text{red}^{-1}U$ such that $F|_{B(0,1)} = f$. Moreover, if $||f||_{B(0,1)} \leq 1$, then $||F||_Y \leq 1$ and $\mathbb{A}^o(Y) = \mathcal{O}_{\mathbb{A}_k^1}(U)$.

**Proof.** This follows from the fact that any connected affinoid in $B^1$ which properly intersects $B(0,1)$ must contain the complement of finitely many residue disks. □

Suppose $p = 2$ and suppose $\kappa \in B$ and $v(w_2(\kappa)) < 3$. Then the fiber above $\kappa$ of $V_2$ is isomorphic to $B(0,1)$ and we may regard $X =: X_\kappa = w_2(\kappa)y_2$ as a uniformizing parameter on it. Thus, as pointed out on page 614 of [BK05], there is a $g_\kappa(z) \in \mathcal{O}_\kappa[[z]]$ such that, after pullback, the restriction of $E_2$ to this fiber equals $g_\kappa(w_2(\kappa)y_2)$.

It follows that if we take $f = Xg_\kappa(X)$ and $f$ extends to an analytic function on a larger connected region in the fiber above $\kappa$ of $X_1(4) \times B_3$ than $V_2/\kappa$, then there must be an open affine $U$ in $\mathbb{A}_k^1$ containing 0 and an analytic function $F$ on $Y = \text{red}^{-1}U$ such that $F|_{B(0,1)} = f$. Then $F$ must be a rational function in $X$. It follows from the analysis on page 614 of [BK05] that

$$F(X)^2 + F(X) + X = 0.$$ 

But, this means $F$ is integral over $\mathbb{F}[X]$, which implies $F \in \mathbb{F}[X]$, but there are no solutions of (5) in $\mathbb{F}[X]$. This implies $E_2$ does not continue to a larger connected region in the fiber above $\kappa$.

Now suppose $p = 3$ and $\kappa \in B$. Then if $v(w_3(\kappa(4))) < 1$, by §5 of [Ro09] there is a $g_\kappa(z) \in \mathcal{O}_\kappa[[z]]$ such that the restriction of $E_3$ to the fiber above $\kappa$ equals $g_\kappa(w_3(\kappa)y_3)$. By the proof of Lemma 5.1 of [Ro09], if $G(X) = X^2g_\kappa(X)$, then

$$G(X)^3 + G(X)^2 + XG(X) - X^3 = 0.$$ 

This has no solution in $\mathbb{F}[X]$, so as above $E_3$ does not continue to a larger connected region in the fiber above $\kappa$. This completes the proof of (b).
2. Valuation of Hasse

We use the notation and formulas of [DT75]. Suppose \( R \) is a ring, \( p = 0 \) in \( R \) and \( a_1, \ldots, a_6 \in R \). Let

\[
E: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,
\]

\[
\pi = dx/(2y + a_1 x + a_3) = dy/(3x^2 + 2a_2 x + a_4 - a_1 y),
\]

\[
b_2 = a_1^2 + 4a_2, \quad -4b_8 = b_1^2 - b_2 b_6, \quad c_4 = b_2^2 - 24 b_4 \quad \text{and} \quad \Delta = b_2^3 - 27 b_6^2 + b_8(36b_4 - b_6^2).\]

Suppose \( \Delta \in R^* \). Then \( E \) is an elliptic curve over \( R \).[4] \( \pi \) is a non-vanishing differential and

\[
(6) \quad j(E) = c_4^3/\Delta.
\]

The following lemma should be well known (it follows easily from the definition).

**Lemma 2.1.** We have \( H(E, \pi) = a_1 \) if \( p = 2 \), and \( H(E, \pi) = b_2 \) if \( p = 3 \).

**Theorem 2.2.** Suppose \( P \) is a point on the \( j \)-line \( X(1) \) over \( \mathbb{R}_p \) corresponding to an elliptic curve \( E_P \) with good supersingular reduction and \( P_0 \in X(1)(\mathbb{W}) \) such that \( P_0 = P \) and \( j(P_0) = 0 \) if \( p \leq 3 \) or \( E_{p-1}(P_0) = 0 \) if \( p > 3 \). Then

\[
h(E_P) = \frac{v(j(P) - j(P_0))}{|\text{Aut}(E_P)|/2}
\]

if either expression is strictly less than one.

**Proof.** When \( p = 2 \) or 3, this follows from (6), Theorem 10.1 of chapter III of [Si86] (which implies \( |\text{Aut}(E_P)|/2 = 12 \) if \( p = 2 \) and 6 if \( p = 3 \)) and Lemma 2.1. When \( p > 3 \),

\[
H = E_{p-1} \bmod p,
\]

by §2.1 of [Ka73]. Suppose \( N \geq 3 \), \( (N, p) = 1 \), and let \( E(N) \) denote the universal elliptic curve over \( Y(N) \) (the open modular curve of level \( N \)) and \( G = \Gamma(1)/\Gamma(N) \). Let \( f: Y(N) \to Y(1) \) be the natural map. Then \( G \) acts on \( (E(N), Y(N)) \) and hence on \( \omega = f_* \Omega^1_E|_{E(1)} \). Now \( E_{p-1} \) may be considered a \( G \)-invariant section of \( \omega^{p-1} \). Let \( U \subset Y(1) \) be the residue disk of \( \bar{P} \) and \( V \) one of the residue classes (which are disks over \( \mathbb{W} \)) above \( f^{-1}(\bar{P}) \). Then \( f: V \to U \) is finite, surjective and of degree \( d := |\text{Aut}(E_P)|/2 \). Moreover, if \( d > 1 \), \( U \) will contain a point \( R \) with \( j(R) \) equal to 0 or 1728 and \( f|_U \) is totally ramified at the unique point \( Q \) above \( R \). As is well-known, \( E_{p-1}(R) = 0 \) (as \( P \) is supersingular)[6] so \( R = P_0 \). If \( \eta \) is a basis for \( \omega(V) \) on the formal scheme attached to \( V \), then \( E_{p-1}|_U = s \eta^{p-1} \), where \( s: V \to B(0, 1) \) is an isomorphism which vanishes at \( Q \). Moreover,

\[
f^*_U(j|_U) = j(P_0) + s^d g, \quad \text{where} \quad g \in A^0(V)^*.
\]

Now, suppose \( A \in V(R_p) \) and \( f(A) = P \). Then

\[
j(P) - j(P_0) = s(A)^d g(A).
\]

---

6If \( p > 2 \), \( E \) also has the equation \( y^2 = 4x^3 + b_2 x^2 + 2b_4 x + b_6 \). (See chapter III, §1, of [Si86].)
7It is known that if \( p > 3 \), then \( E_{p-1} \) has unique zero in each supersingular disk.
8If \( \tau \) is in the upper half-plane, \( E_k(\tau) = \frac{1}{2} \sum_{(a, b)=1} (a \tau + b)^{-k} \), so if \( \tau \) is a quadratic root of unity, then \( \tau^{-k} E_k(\tau) = E_k(\tau) \).
Since \( v(g(A)) = 0 \),
\[
v(j(P) - j(P_0)) = dv(s(A)) = (|\Aut(\tilde{E}_P)|/2) \cdot h(E_P)
\]
if \( v(j(P) - j(P_0))/(|\Aut(E_P)|/2) \) or \( h(E_P) \) is strictly less than one.

**Remarks** 1. (i) One can show that \( h(E) = v(E) \), where \( v(E) \) is as defined on page 36 of [Bu03] when \( v(E) < p/(p+1) \).

(ii) Proposition 1 of [BC06] follows from Theorem 2.2.

**References**


**Department of Mathematics, University of California, Berkeley, California 94720-3840**

**E-mail address:** coleman@math.berkeley.edu